

Chirality induced interface currents in the Chalker-Coddington Model*

Alain Joye

* joint with **J. Asch** & **O. Bourget**

2D electronic motion

Classical electron in $\mathbb{R}^2$

($m=1$, $|q|=1$)

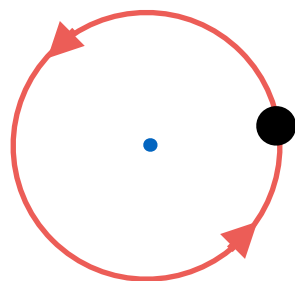
Potential $V(x, y)$ & magnetic field $B \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

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$$V \equiv 0$$



$$B \odot$$

$$r = \frac{\sqrt{2E}}{B}$$

$$\omega_c = B$$

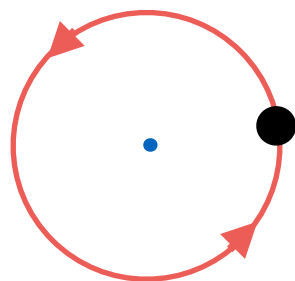
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Potential $V(x, y)$ & magnetic field $B \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, $B \gg 1$

Level lines: $\Gamma_E = \{(x, y) \mid V(x, y) = E\}$

$V \equiv 0$

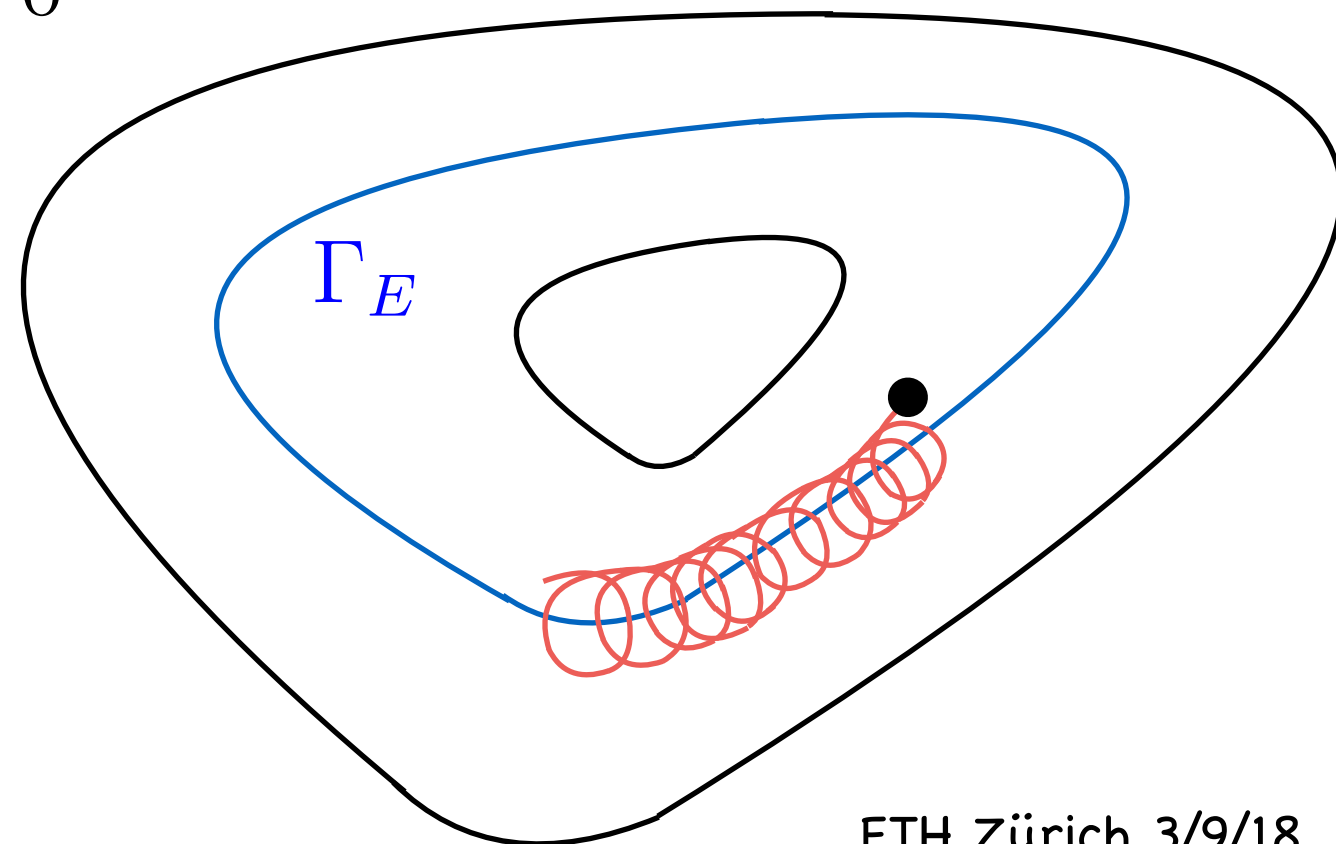


$V \neq 0$

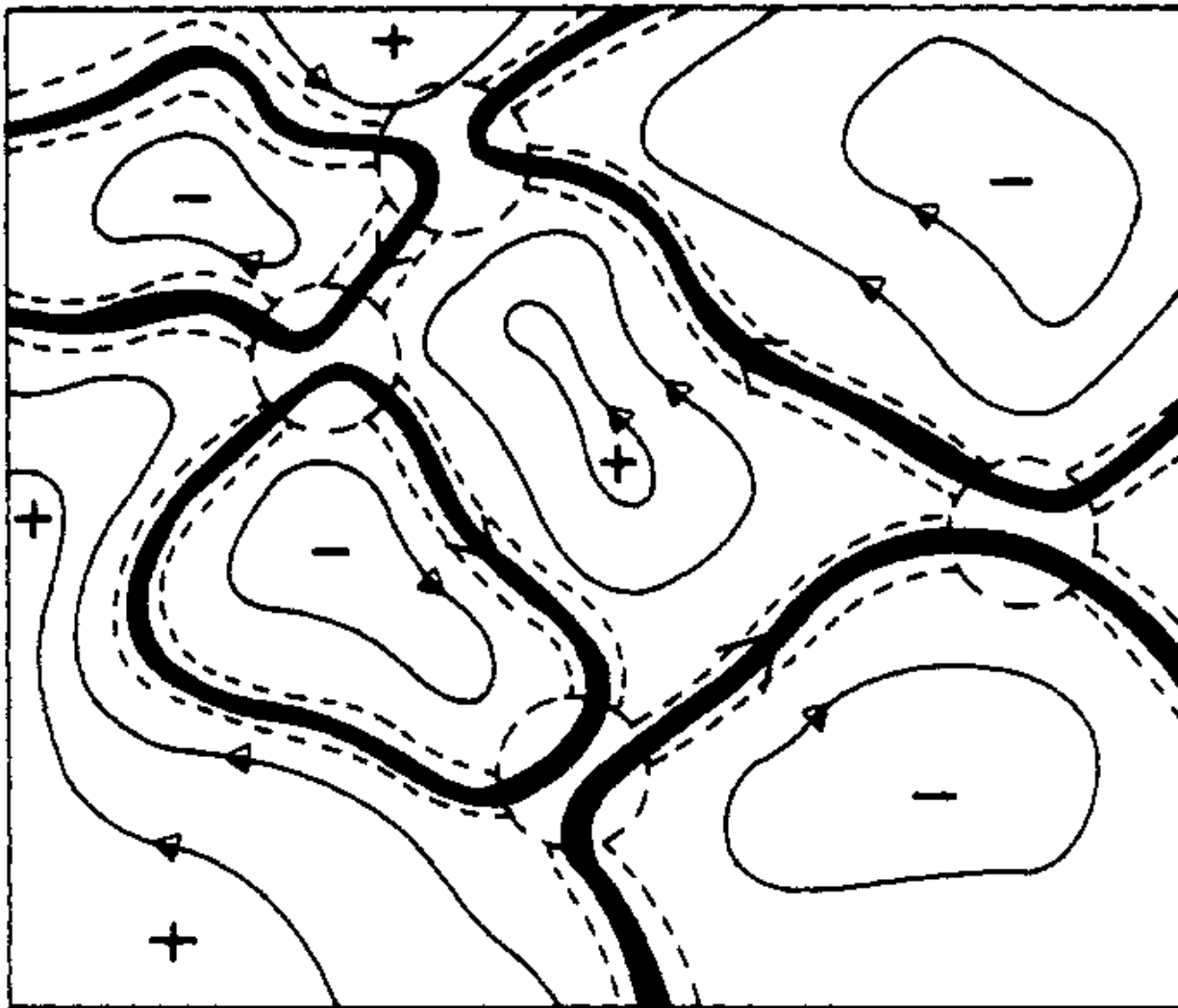
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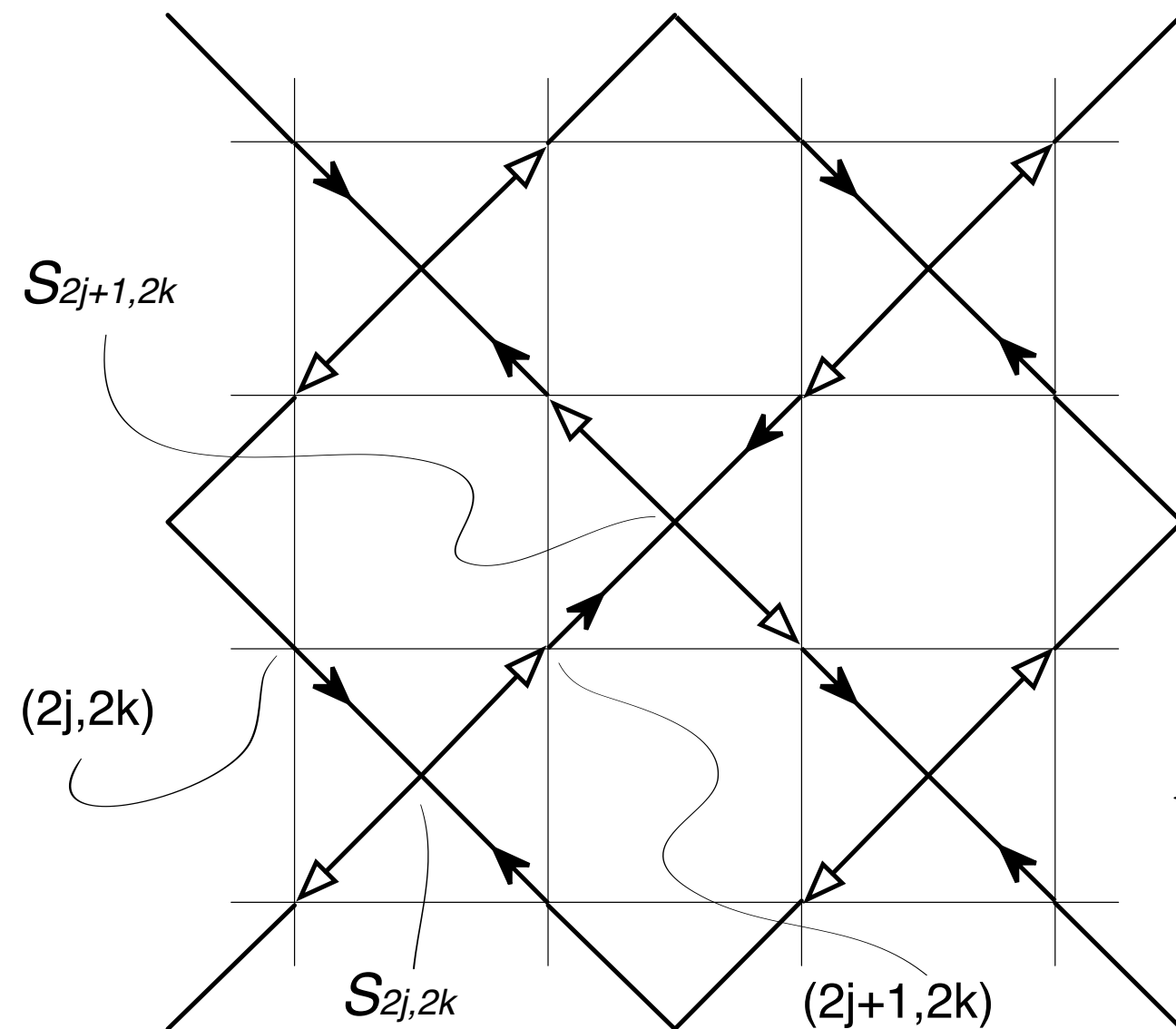
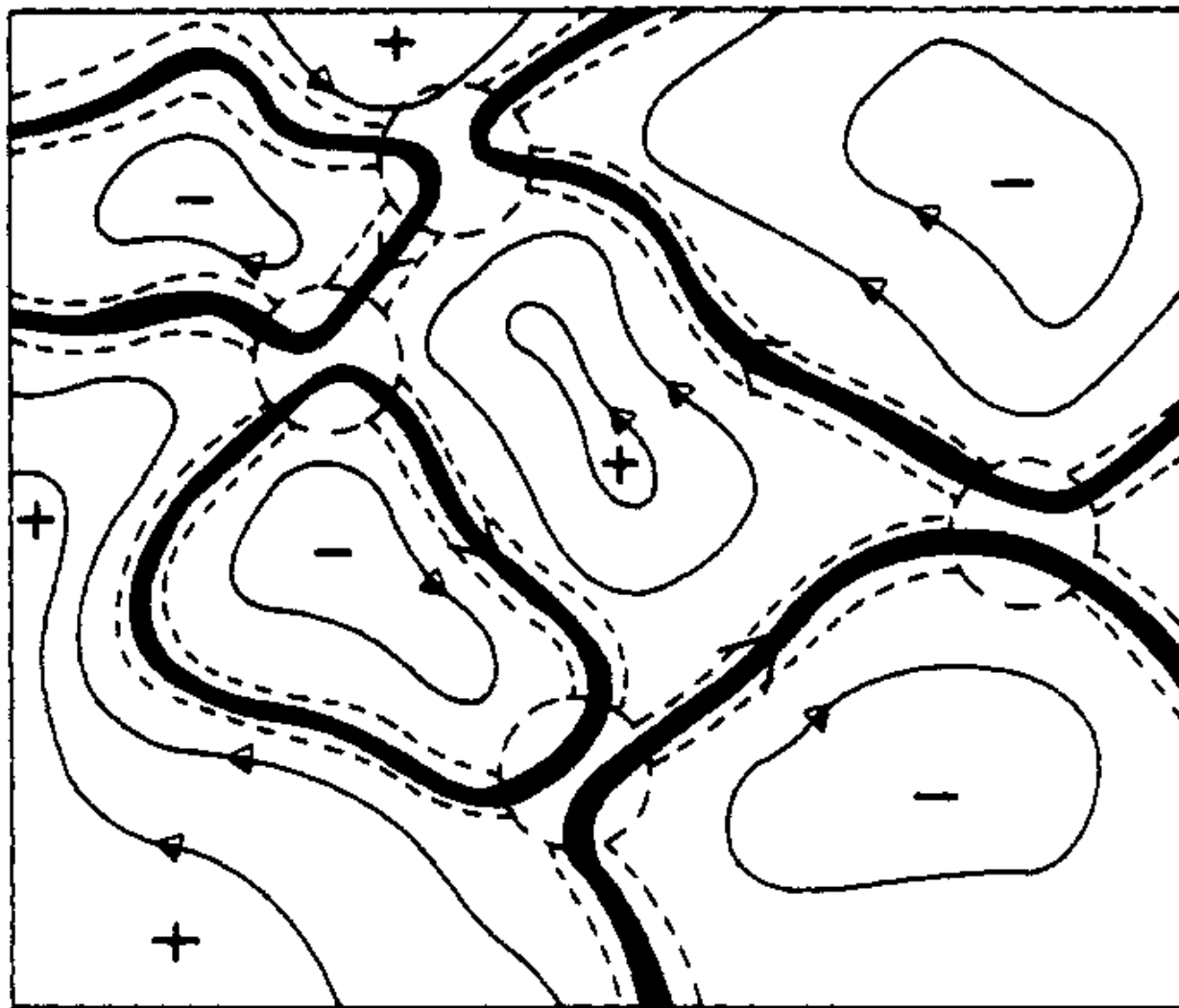
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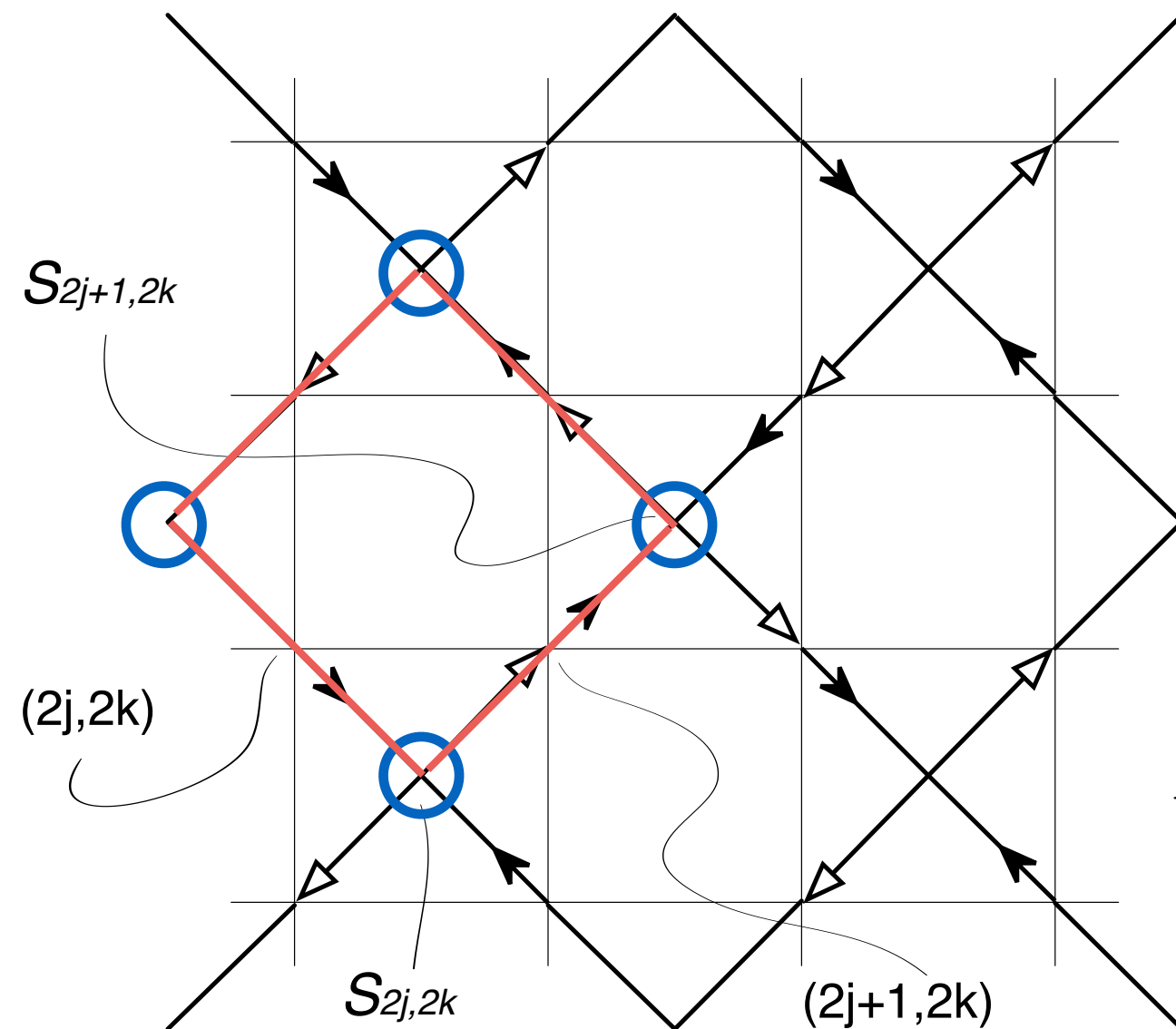
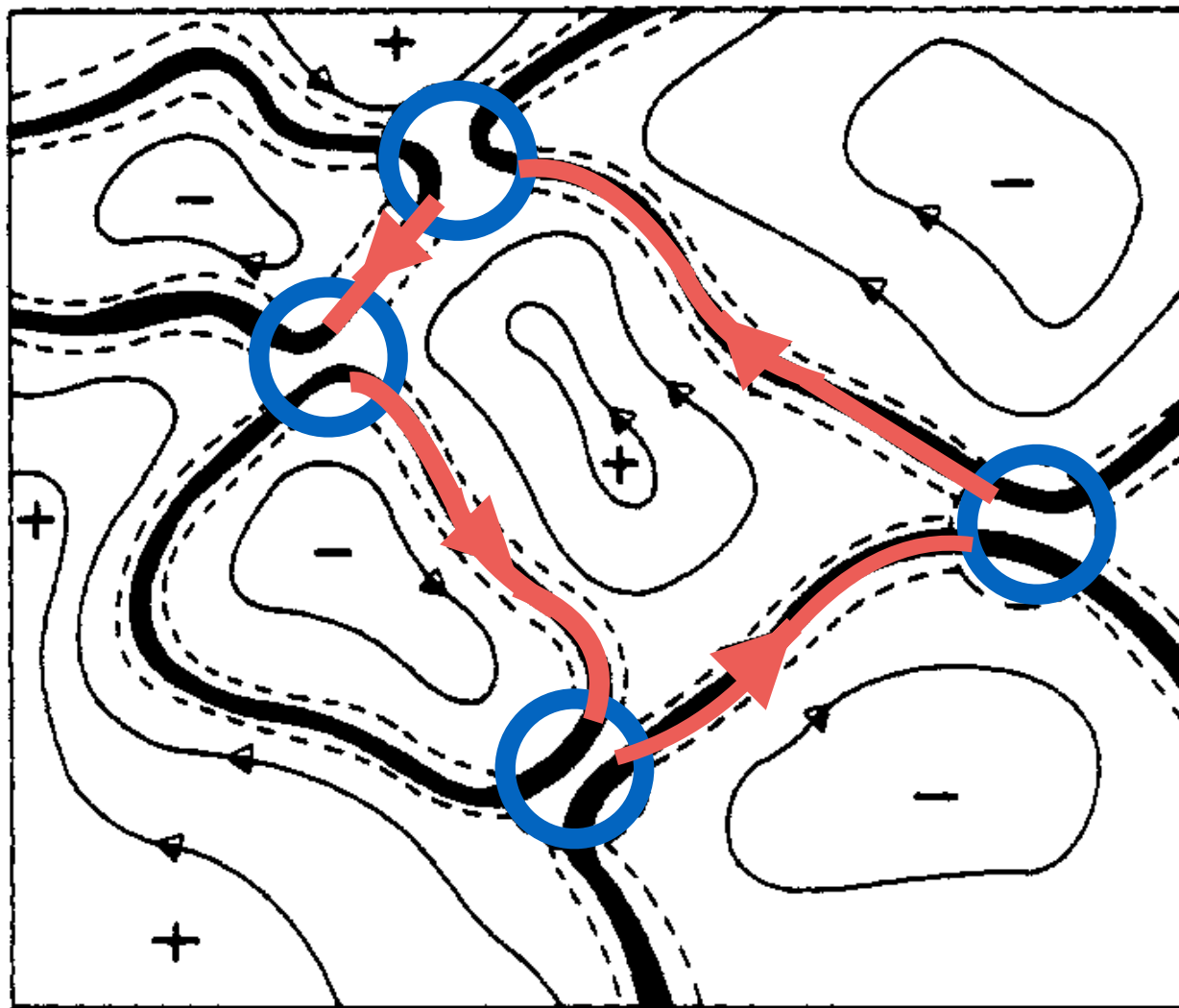
Simplified Quantum version in '88



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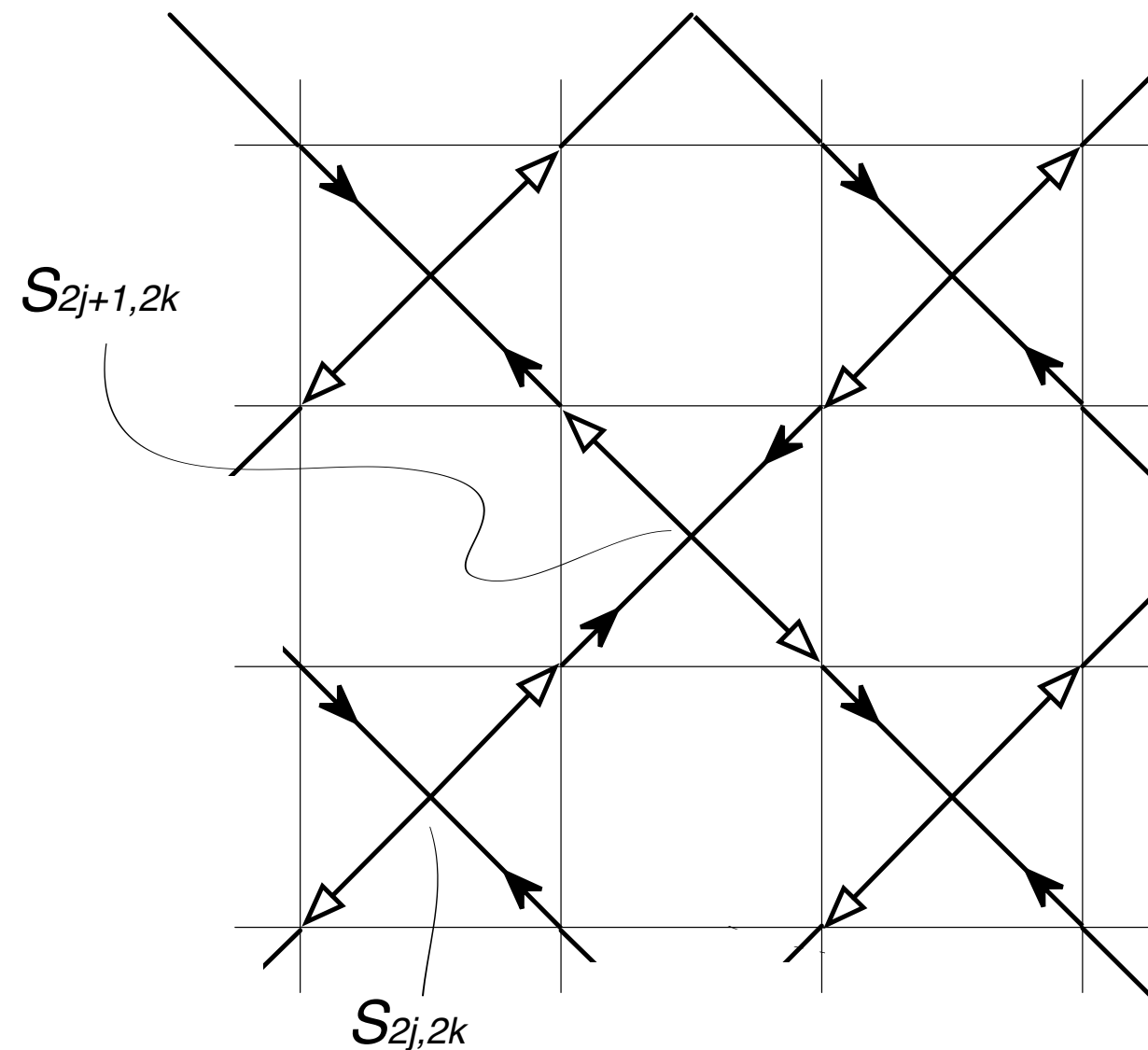


$S_{j,2k} \in U(2)$ "scattering matrices"

Chalker–Coddington Model

Config. $S : (j, k) \mapsto S_{j,2k} \in U(2)$

Hilbert space: $\ell^2(\mathbb{Z}^2; \mathbb{C})$

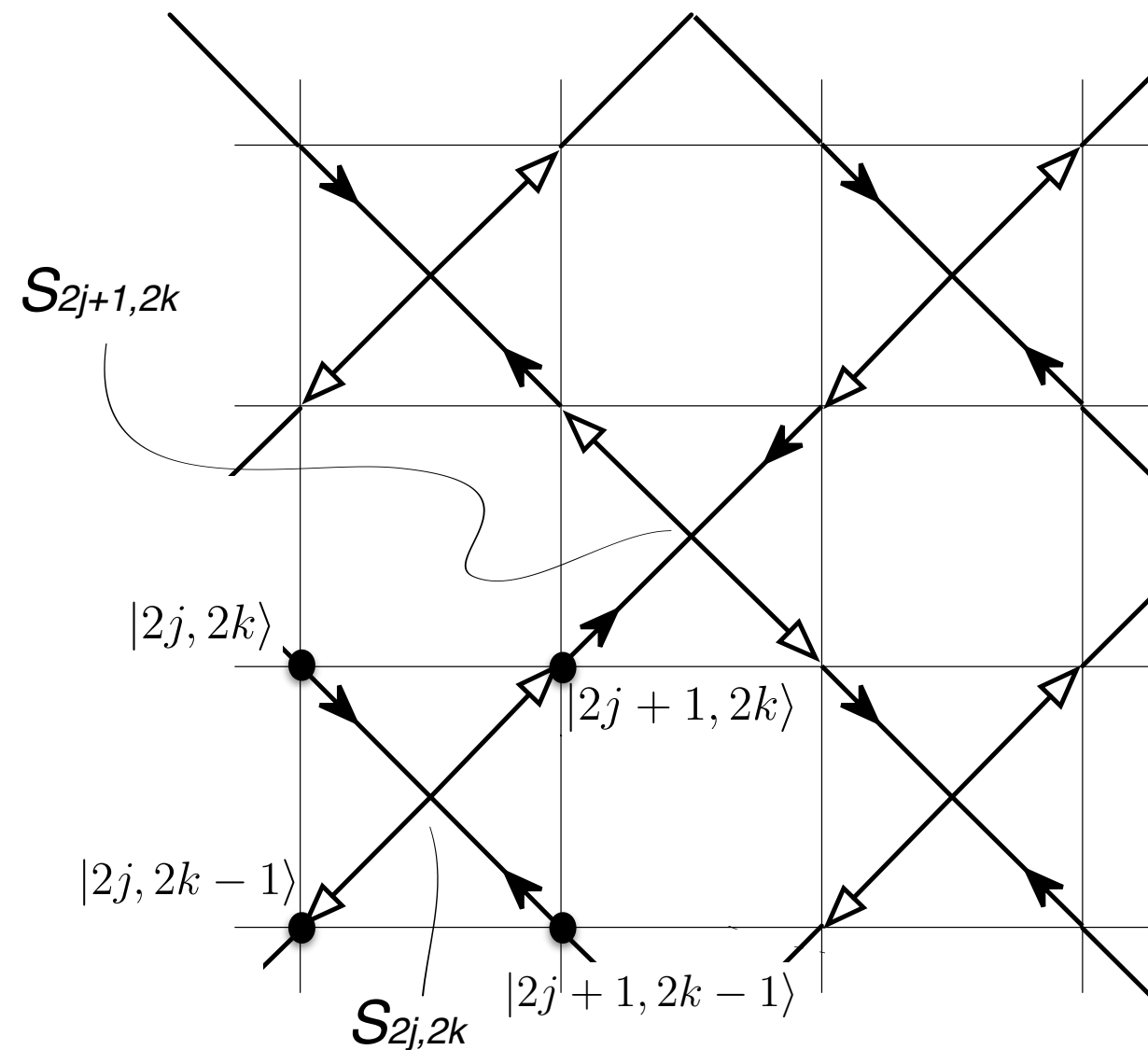


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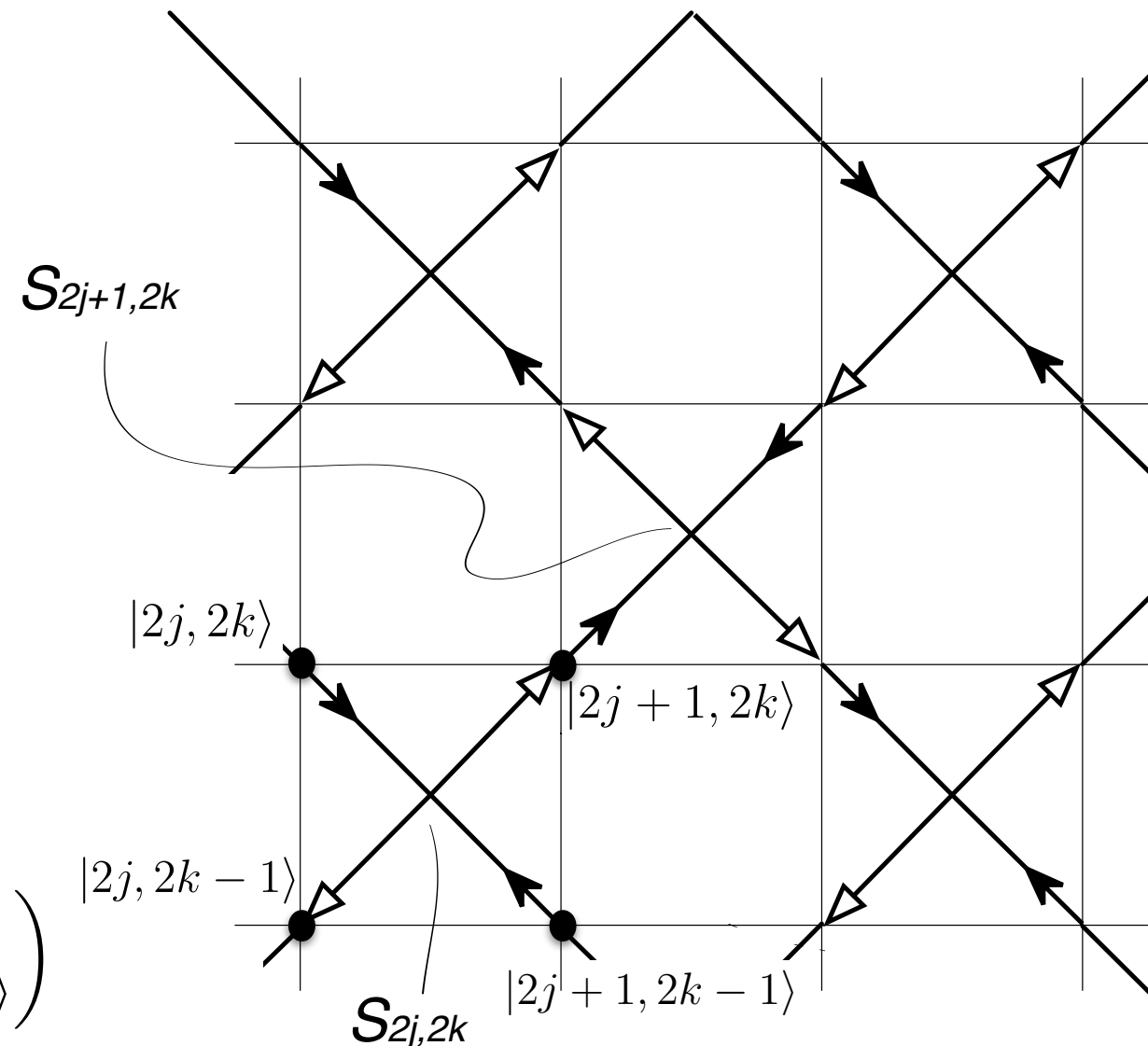
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Definition of U_{CC} :

$$\begin{pmatrix} U_{CC}|2j, 2k\rangle \\ U_{CC}|2j+1, 2k-1\rangle \end{pmatrix} := S_{2j,2k} \begin{pmatrix} |2j, 2k-1\rangle \\ |2j+1, 2k\rangle \end{pmatrix}$$

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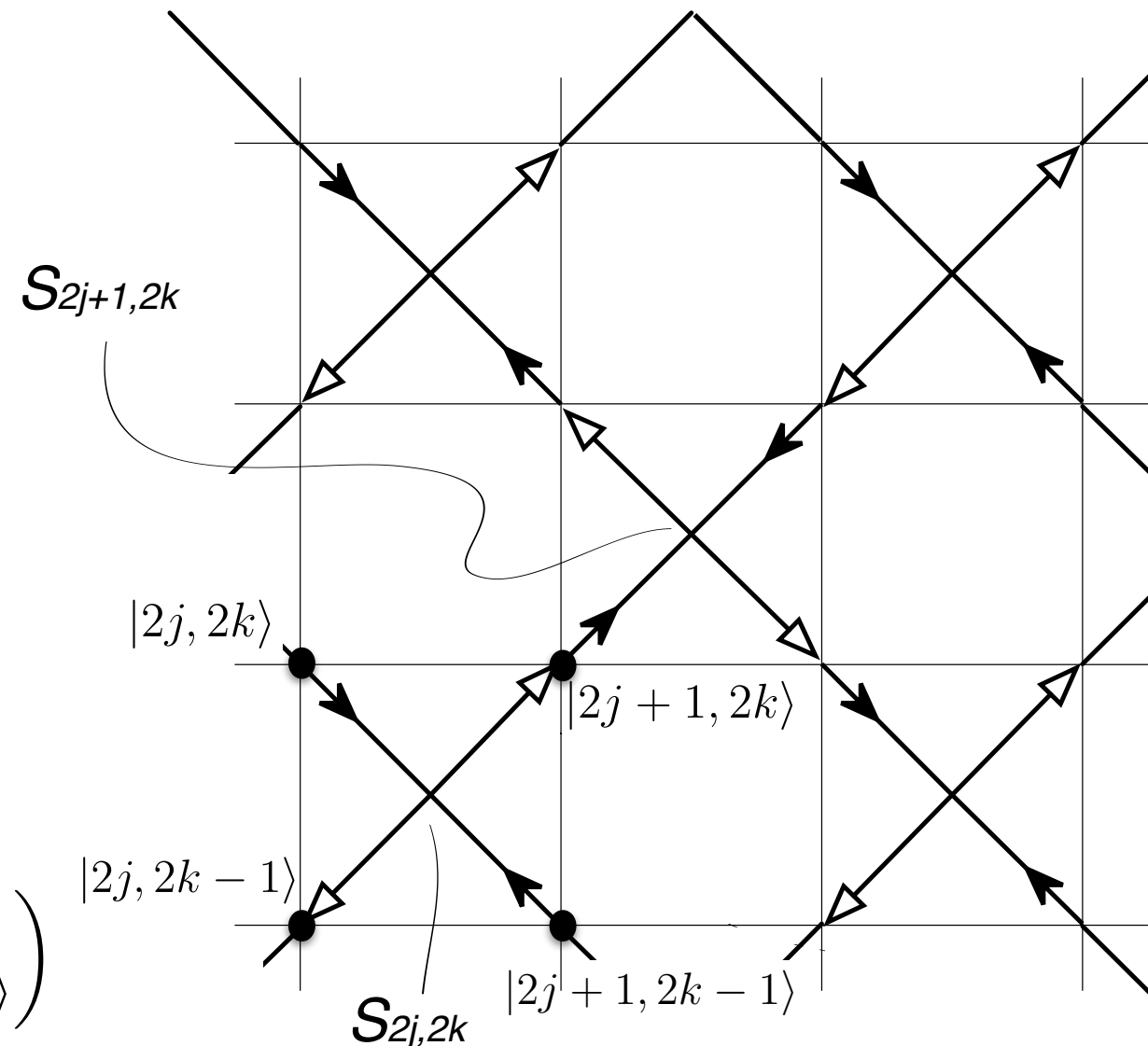
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Lemma: $U_{CC} \equiv U_{CC}(\mathbf{S})$ is unitary, $\forall \mathbf{S}$

$$\|U_{CC}(\mathbf{S}) - U_{CC}(\mathbf{S}')\| \leq c\|\mathbf{S} - \mathbf{S}'\|_\infty$$

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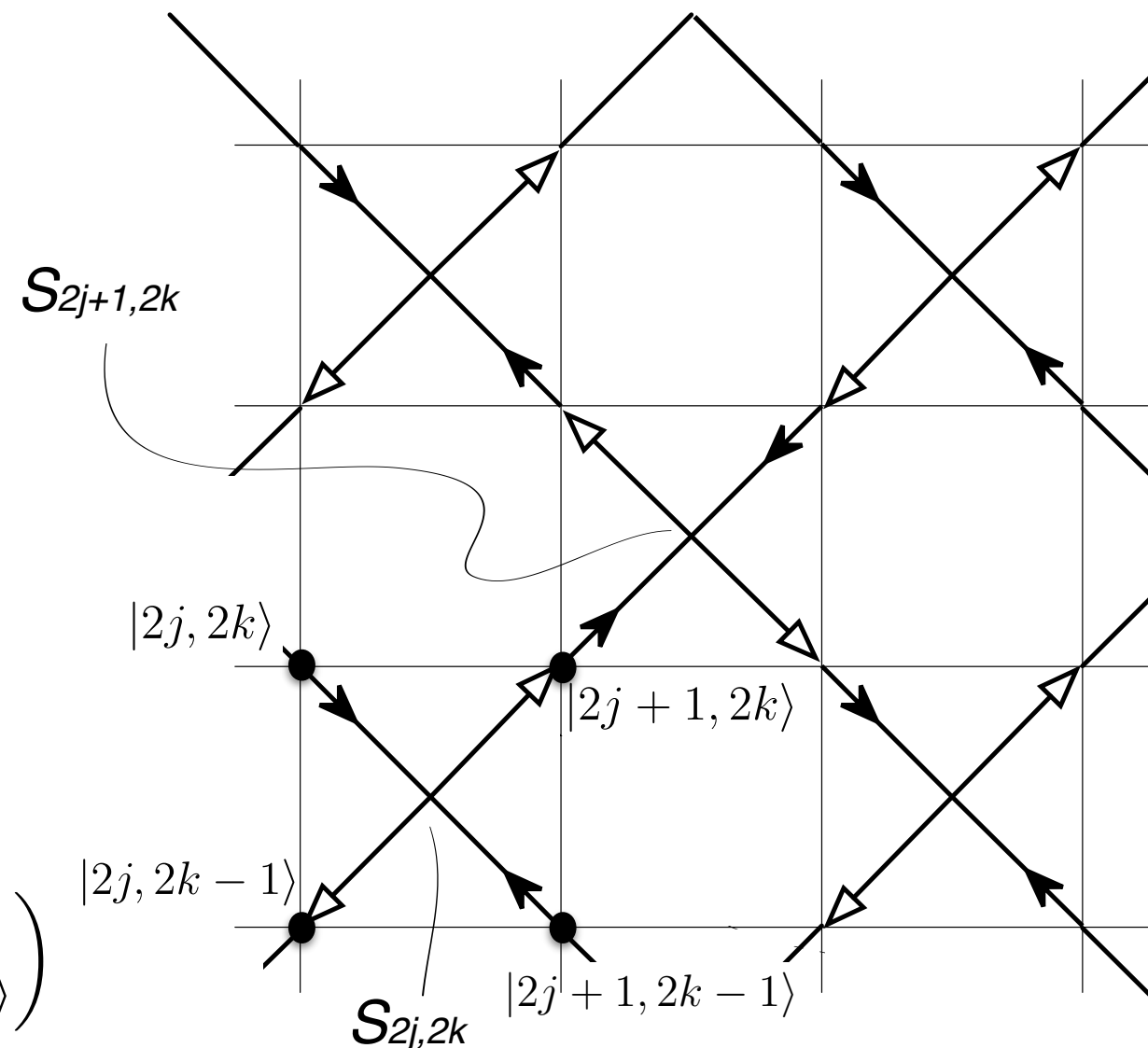
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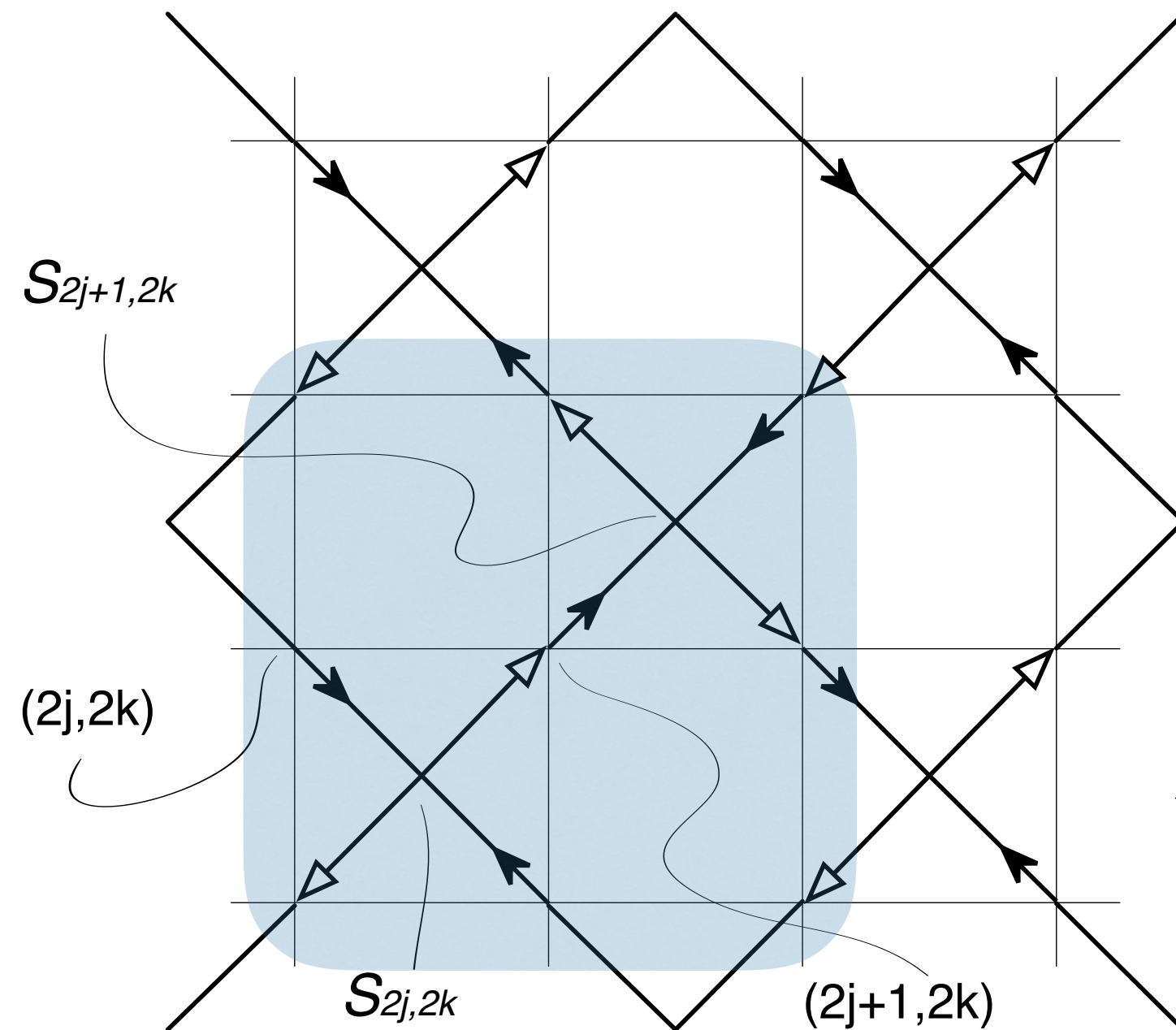
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Lemma: $U_{CC} \equiv U_{CC}(S)$ is unitary, $\forall S$

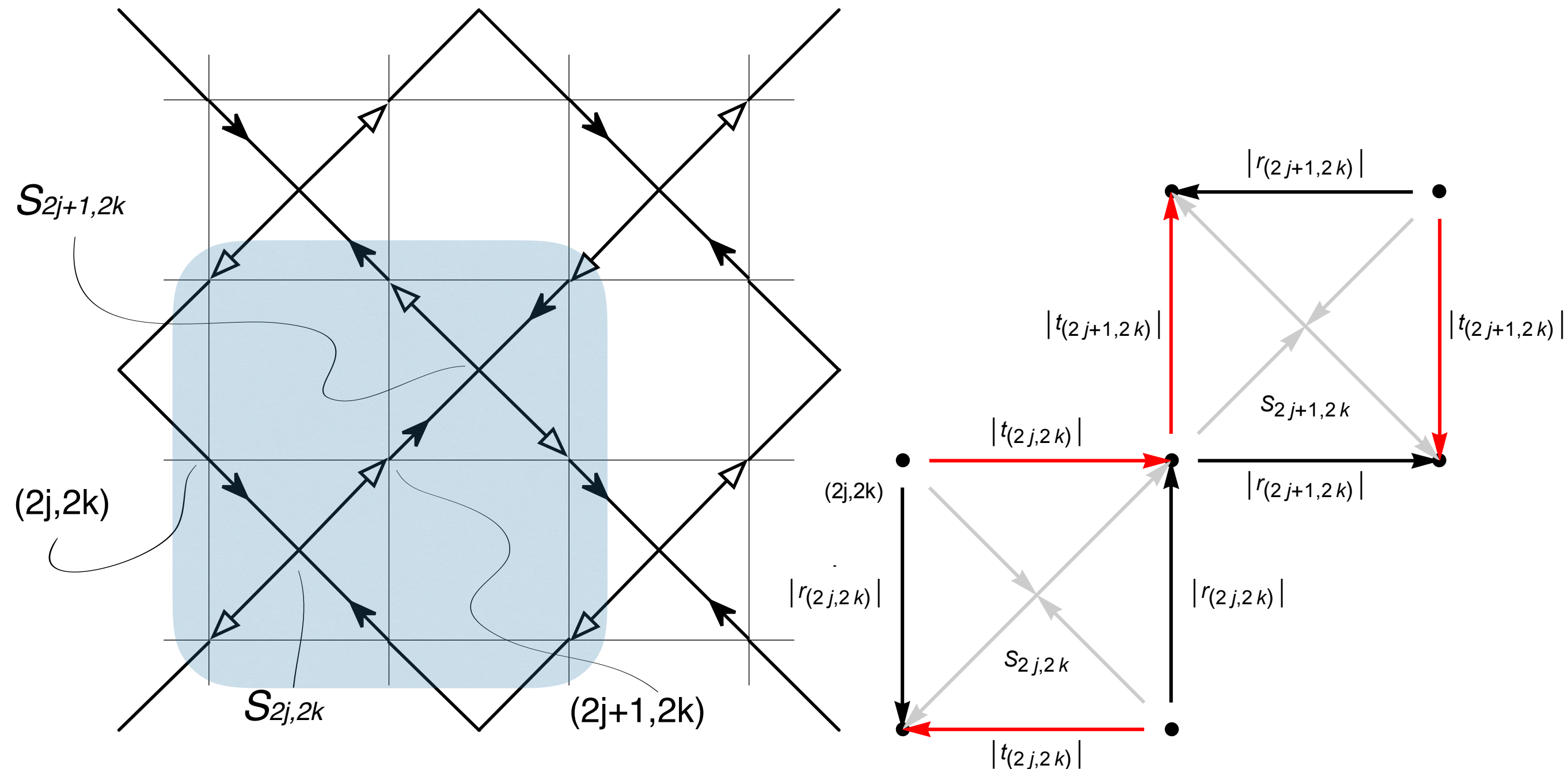
$$\|U_{CC}(S) - U_{CC}(S')\| \leq c\|S - S'\|_\infty$$

Questions: U_{CC}^n as $n \rightarrow \infty$, nature of $\sigma(U_{CC})$?



Parametrisation:

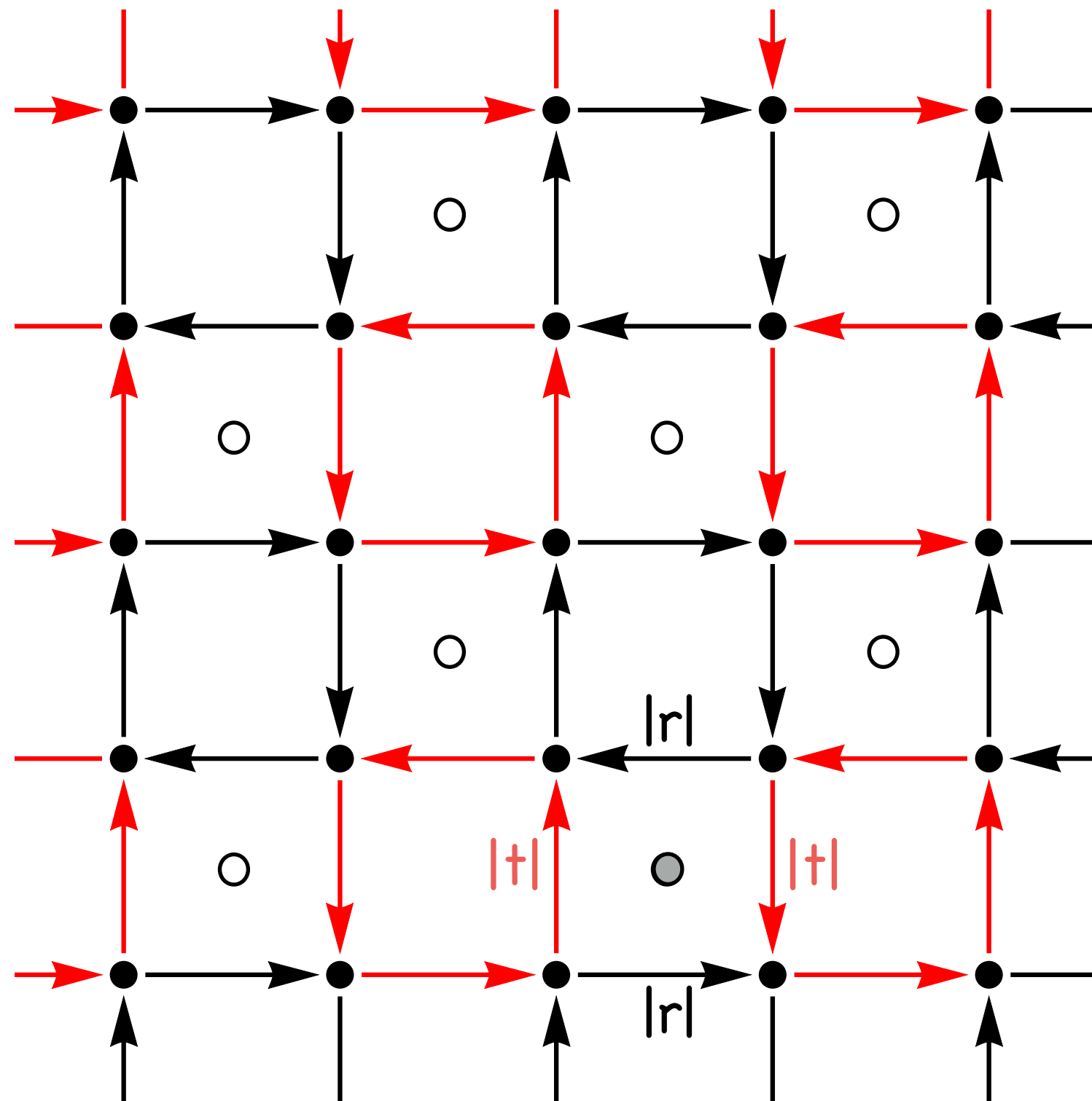
$$S = q \begin{pmatrix} r & -t \\ \bar{t} & \bar{r} \end{pmatrix} \quad \begin{cases} q \in \mathbb{S}^1 \\ |r|^2 + |t|^2 = 1 \end{cases}$$



Red $\leftrightarrow |t|$

Black $\leftrightarrow |r|$

○ $\leftrightarrow$ scat. mat.

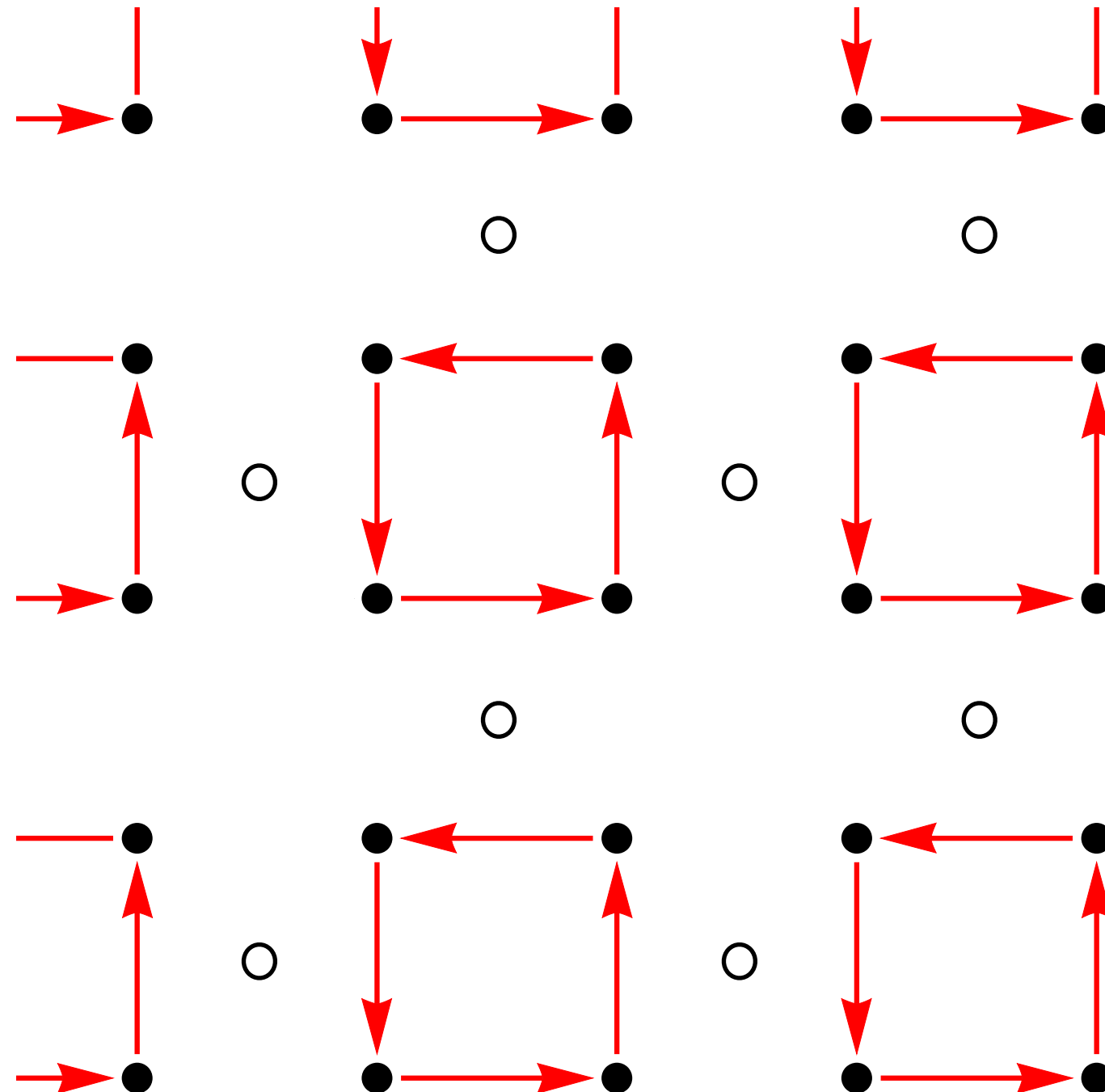


Localised phases

All j, k

$$r_{j,2k} = 0 \quad |t_{j,2k}| = 1$$

$S_{j,2k}$ off-diag.



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"Left movers"

$\sigma(U_{CC})$ pure point

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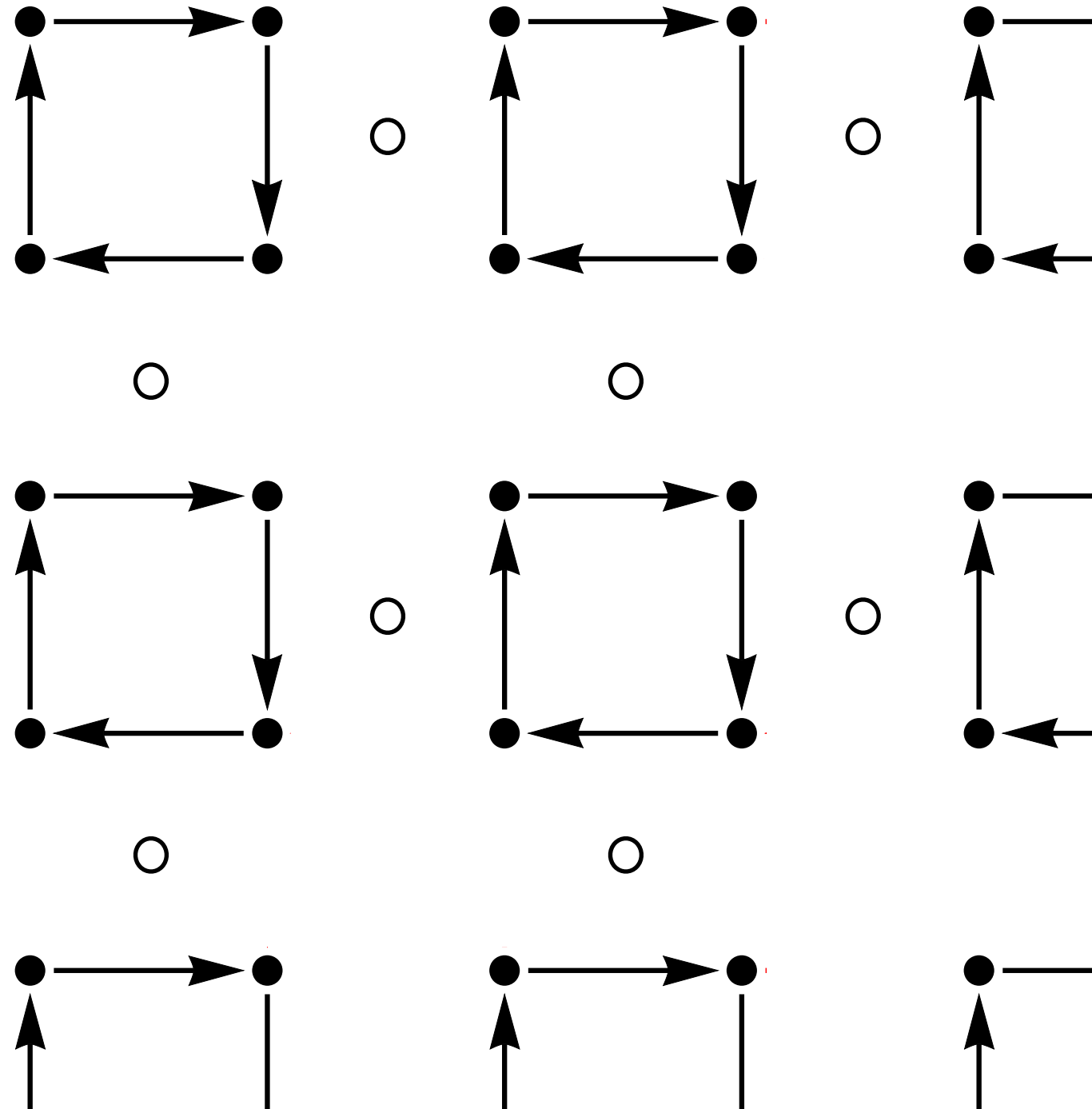
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Random iid phases, constant amplitudes $(|r|, |t|) \in \mathbb{S}^1$

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$$\forall r, t \Rightarrow \begin{cases} \sigma(U_{CC}(\omega)) = \sigma_{pp}(U_{CC}(\omega)) \text{ a.s.} \\ \text{finite localisation length} \end{cases}$$

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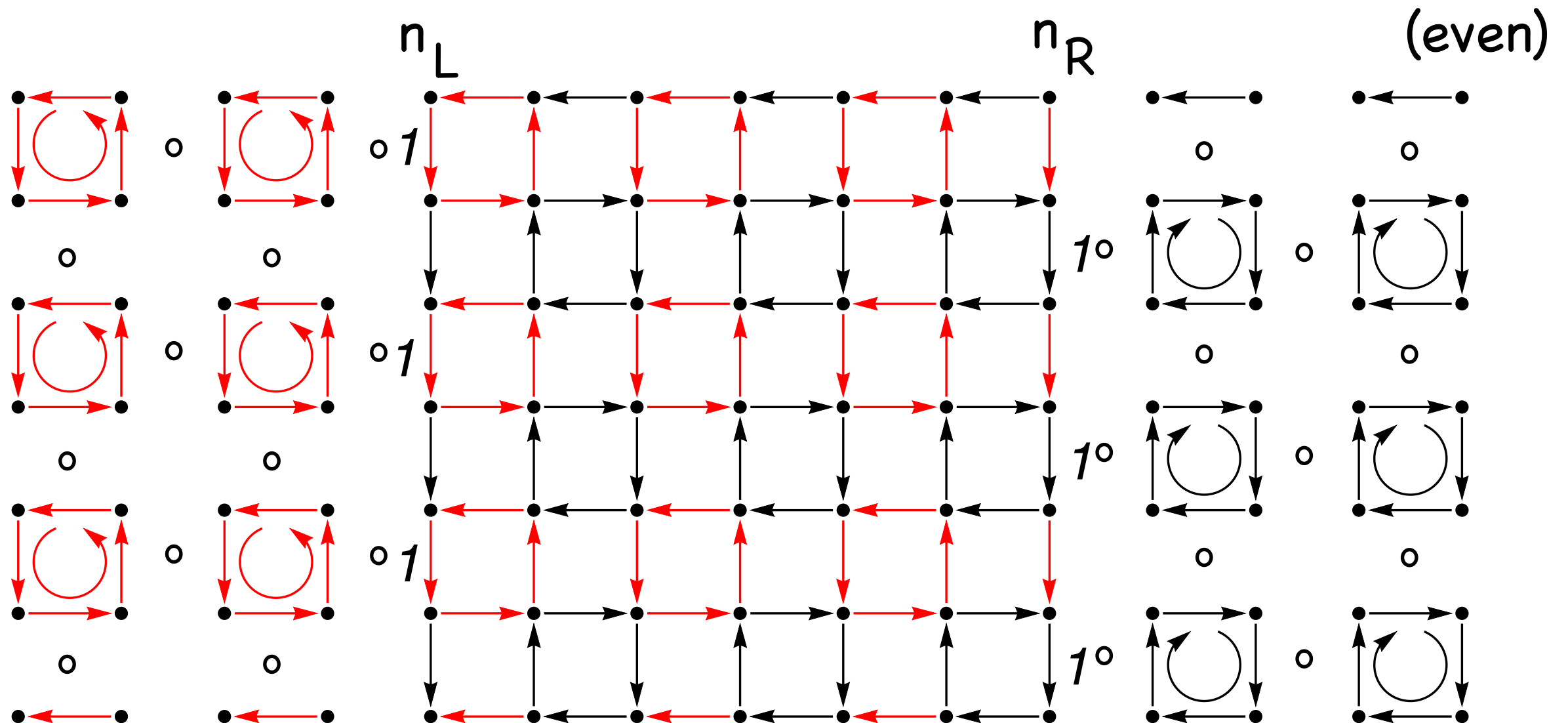
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- On $\mathbb{Z}^2$ ABJ '12

$$|r \cdot t| \ll 1 \Rightarrow \begin{cases} \sigma(U_{CC}(\omega)) = \sigma_{pp}(U_{CC}(\omega)) \text{ a.s.} \\ \text{dynamical localisation} \end{cases}$$

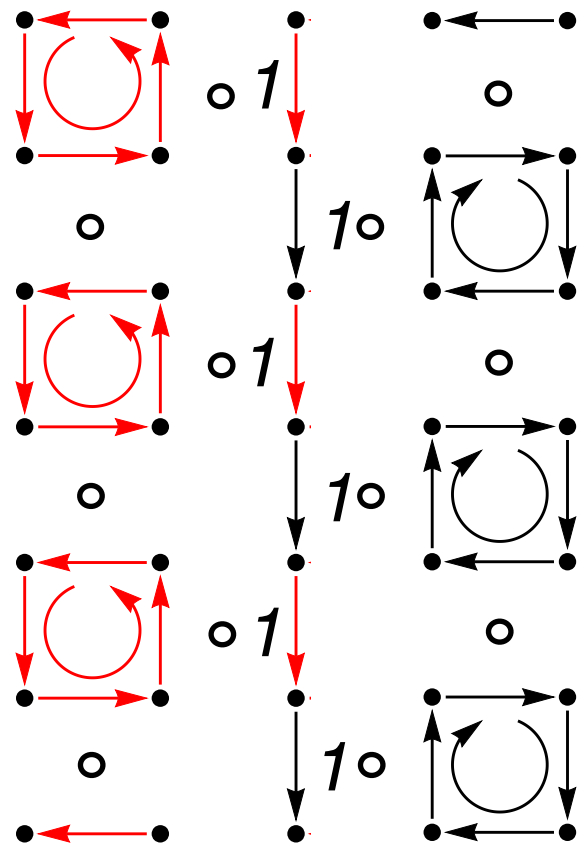
Let $n_L \leq n_R$ & $S_{j,2k} \begin{cases} \text{off-diagonal} & \text{if } j < n_L \\ \text{diagonal} & \text{if } j \geq n_R \end{cases}$

$\Rightarrow$ **focus** on $U_I := U_{CC}|_{I_{n_L, n_R}}$ $I_{n_L, n_R} := \ell^2(\{n_L, n_R\} \times \mathbb{Z})$



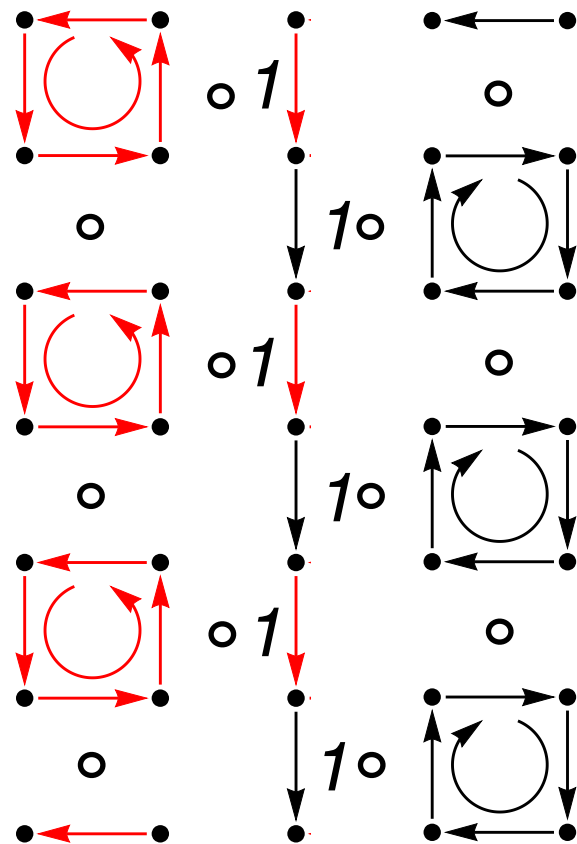
Special interfaces

$$n_L = n_R = 0$$



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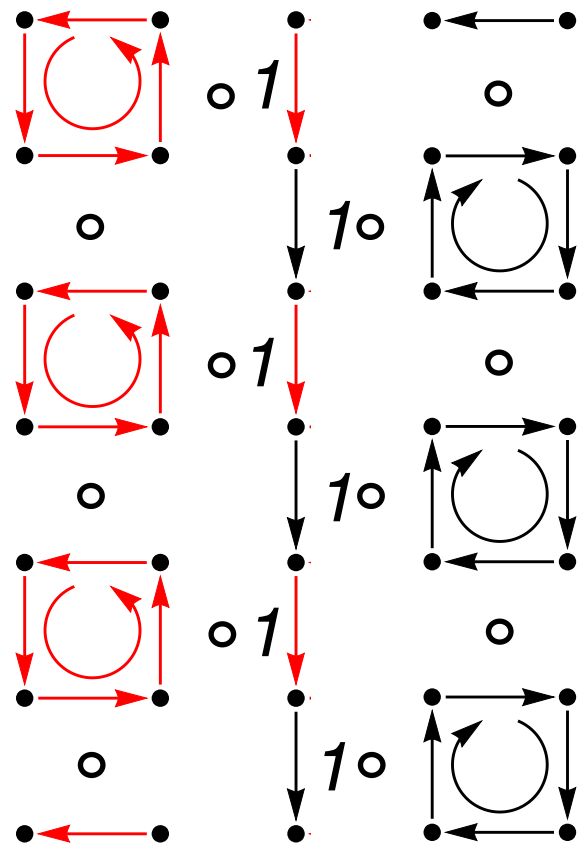
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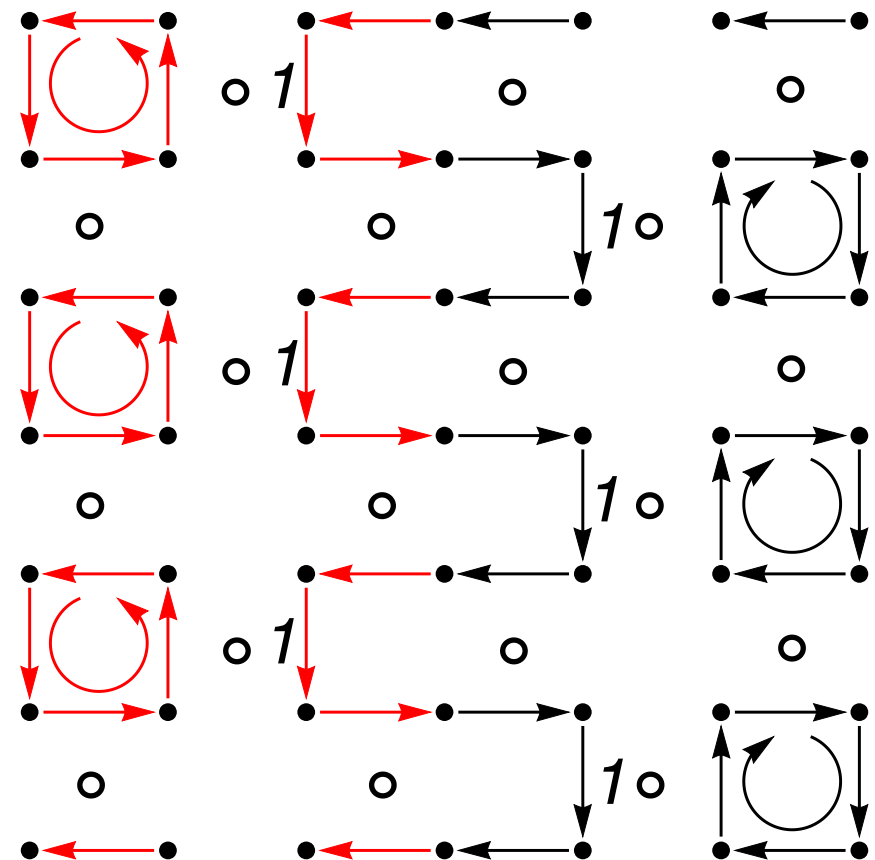
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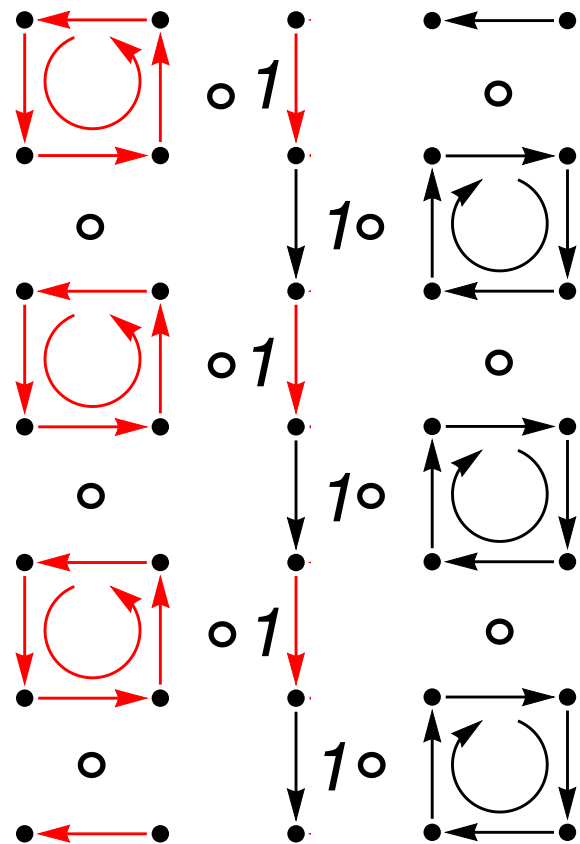
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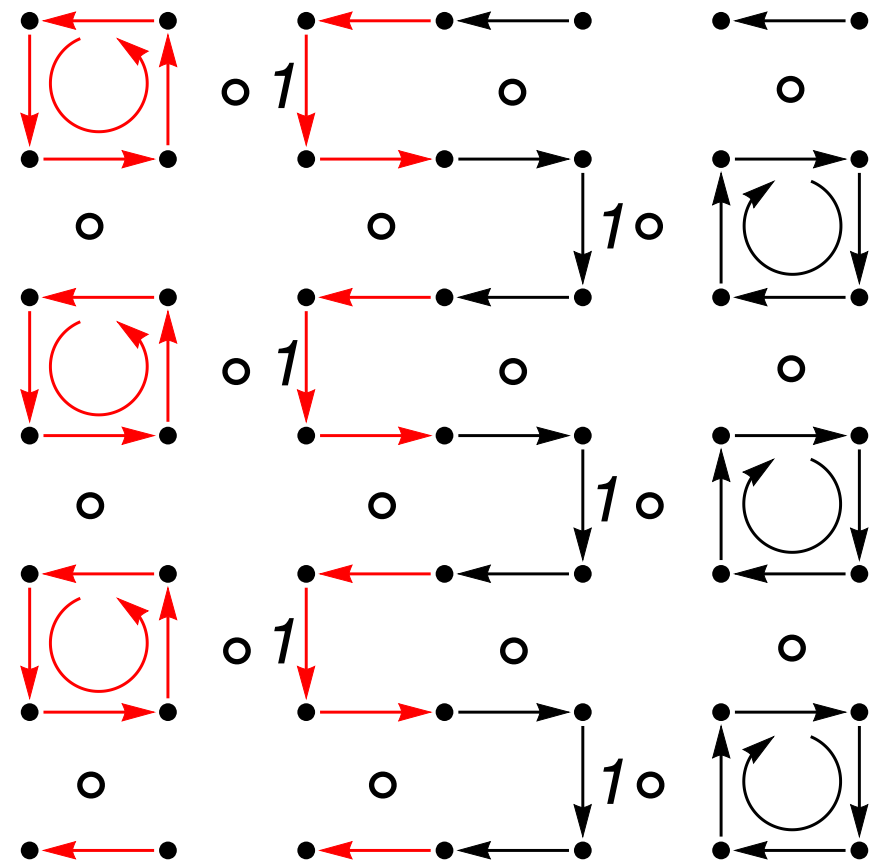
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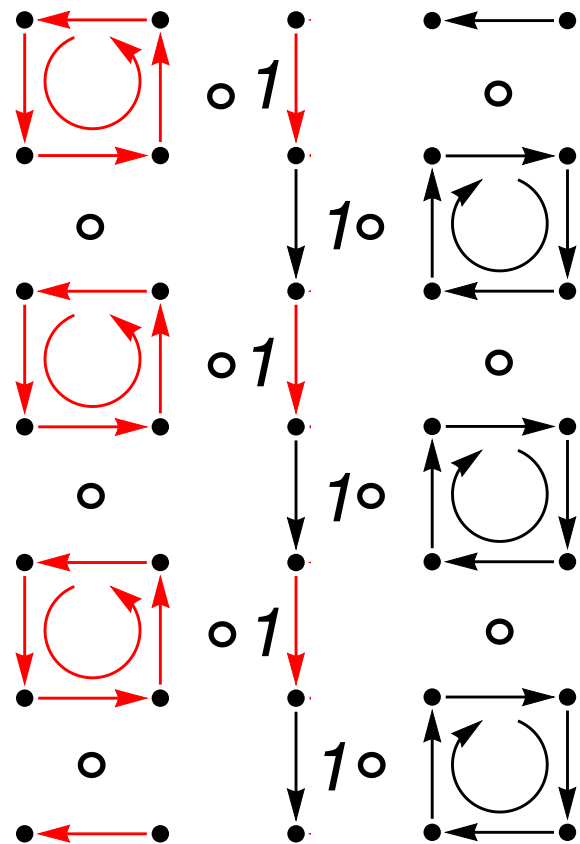
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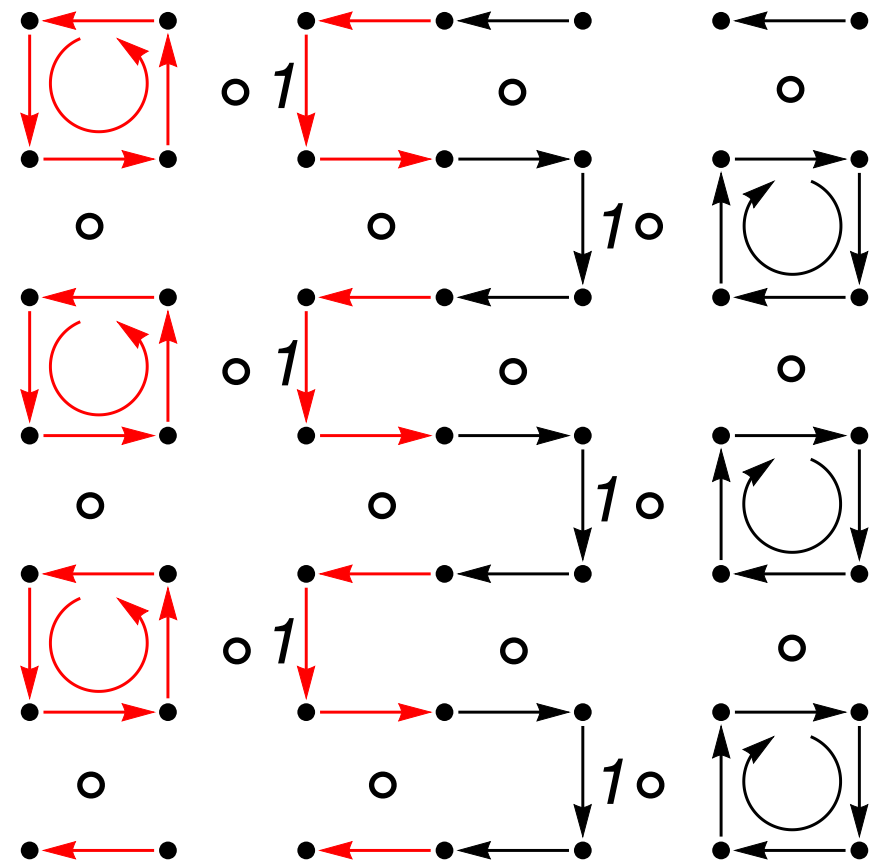
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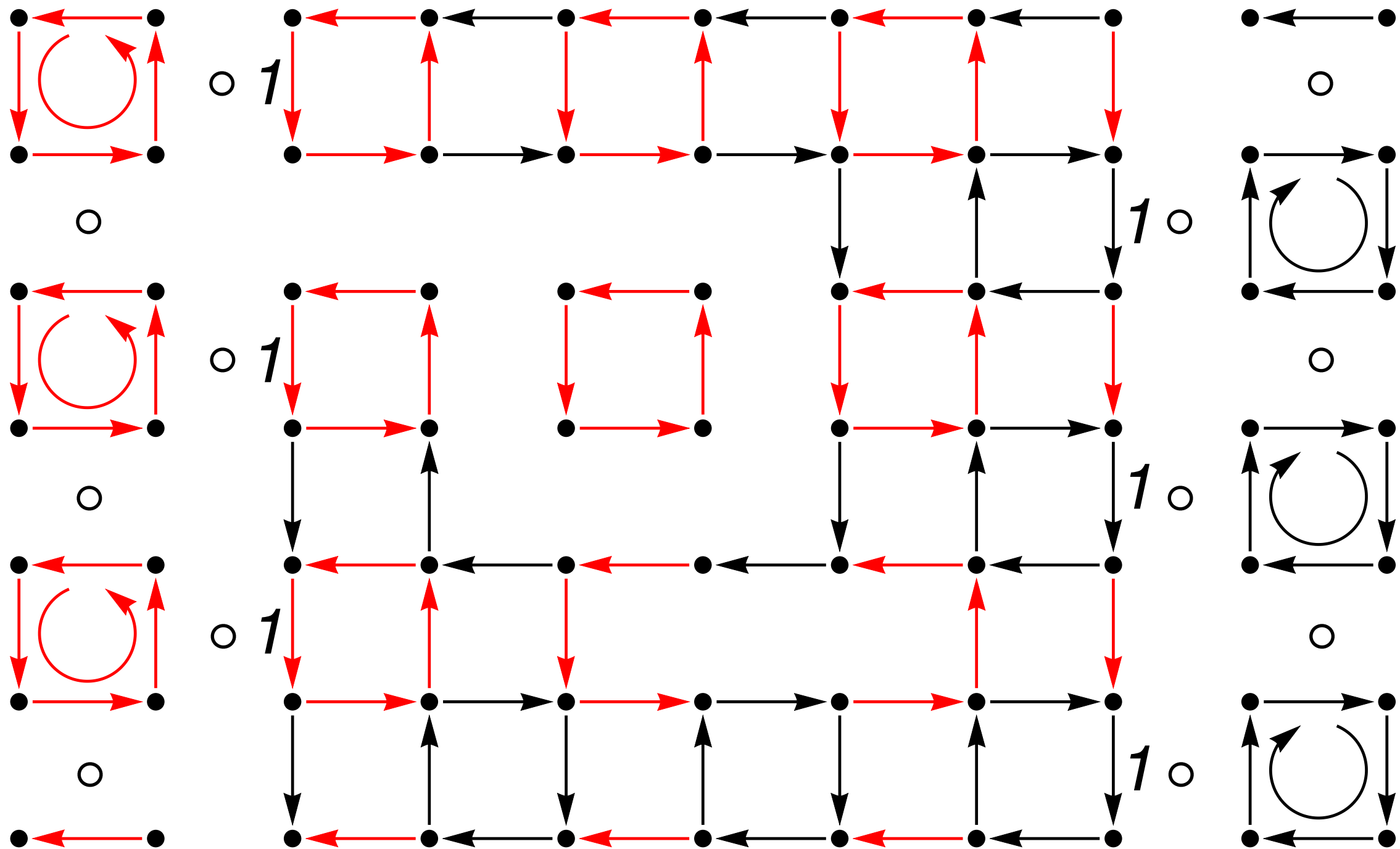
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Transport & $\sigma(U_I) = \sigma_{ac}(U_I) = \mathbb{S}^1$

Arbitrary interface

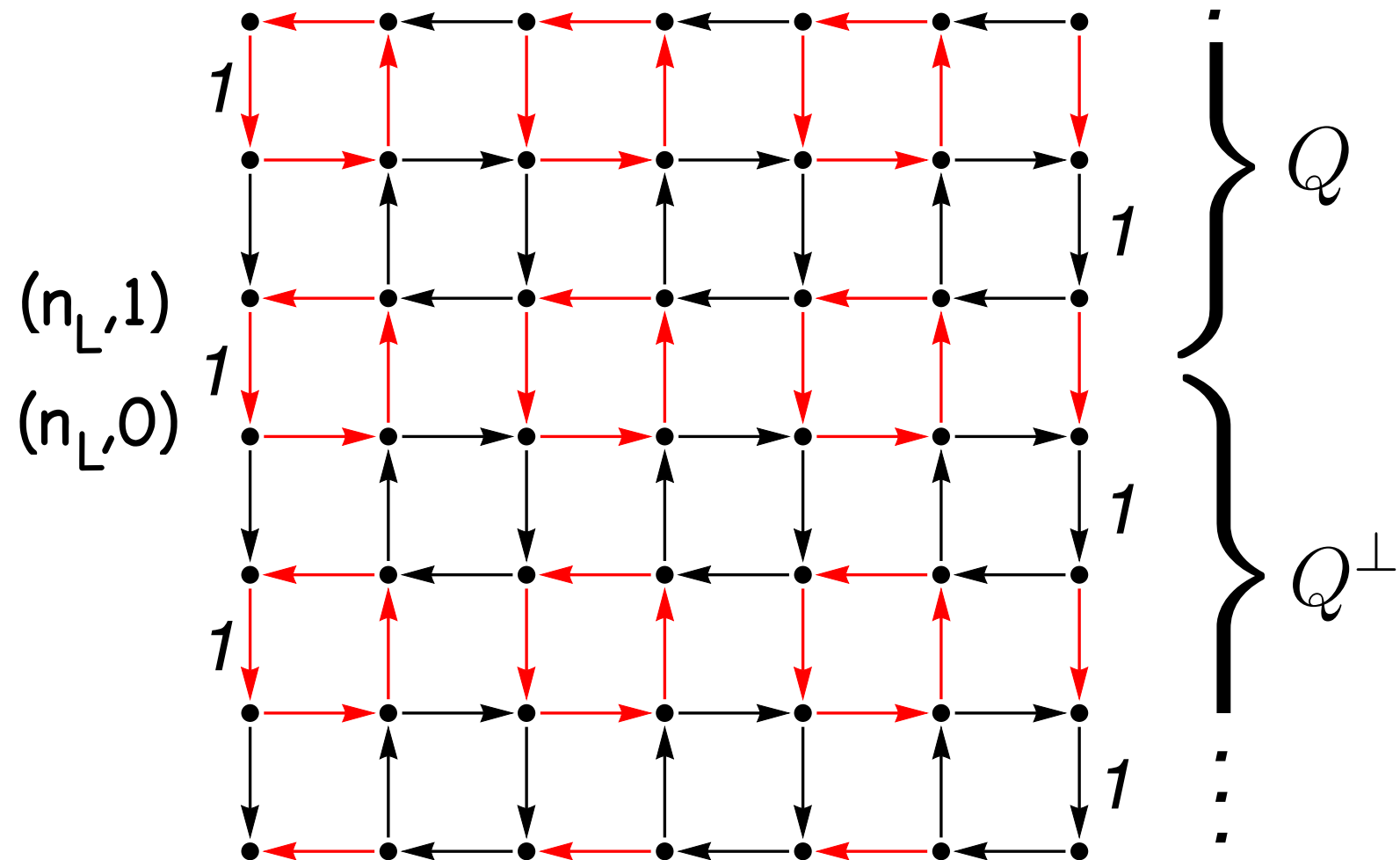


Flux observable

Let Q mult. by $\chi(k \geq 1)$

$$\Phi := U_I^* Q U_I - Q$$

flux through $k = 1$



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Prop. • Φ is finite rank

• $\text{Tr}(\Phi) = -1$

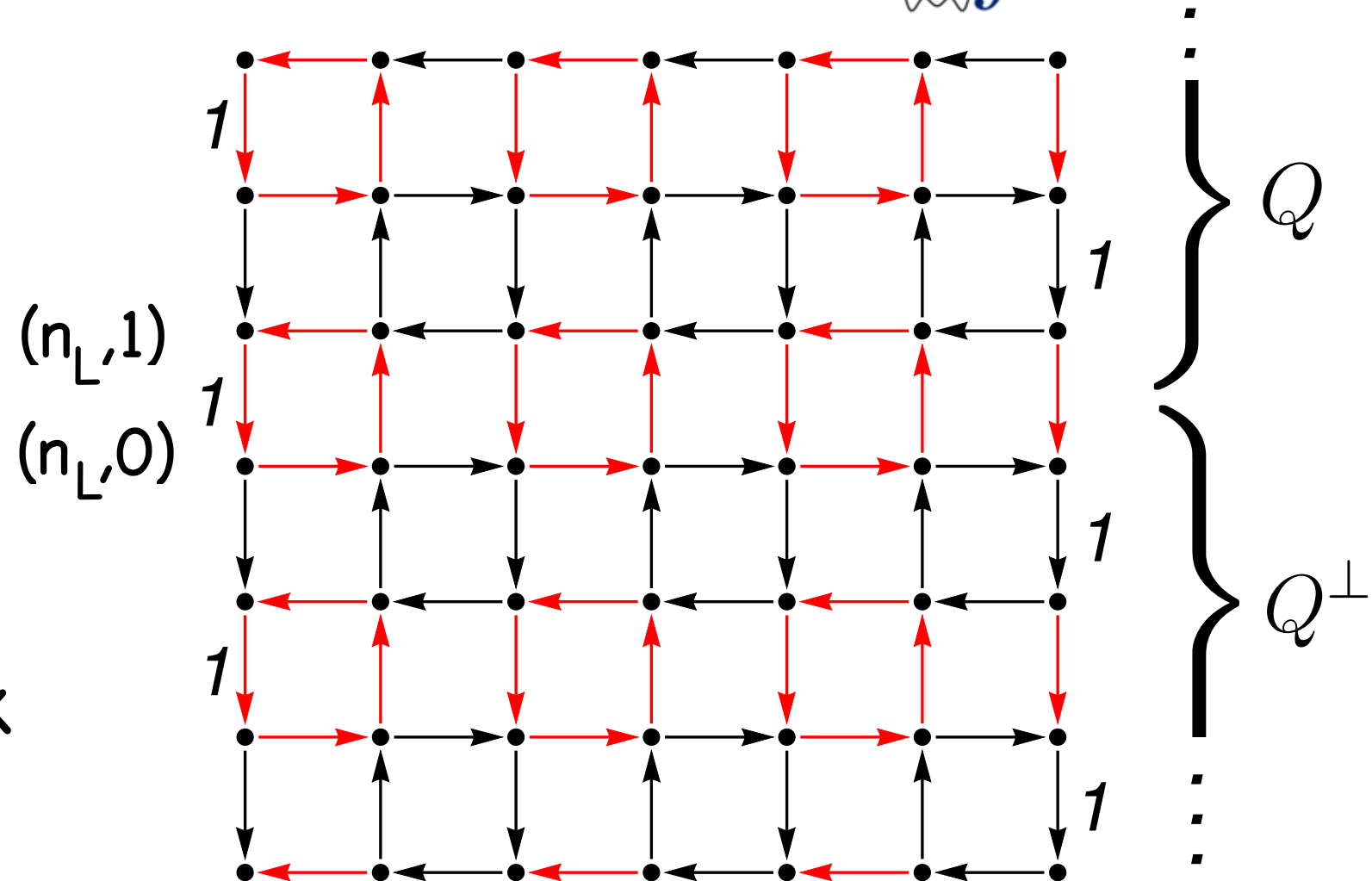
• $\text{Tr}(\Phi) = \text{index}(U_I^* Q U_I, Q)$

where

Avron, Seiler, Simon '94

$$\text{index}(P, Q) := \dim \text{Ker}(P - Q - 1) - \dim \text{Ker}(P - Q + 1) \quad \text{if } P - Q \text{ cpct.}$$

$$\& \text{index}(U_1^* Q U_1, Q) = \text{index}(U_0^* Q U_0, Q) \quad \text{if } U_1 - U_0 \text{ cpct.}$$

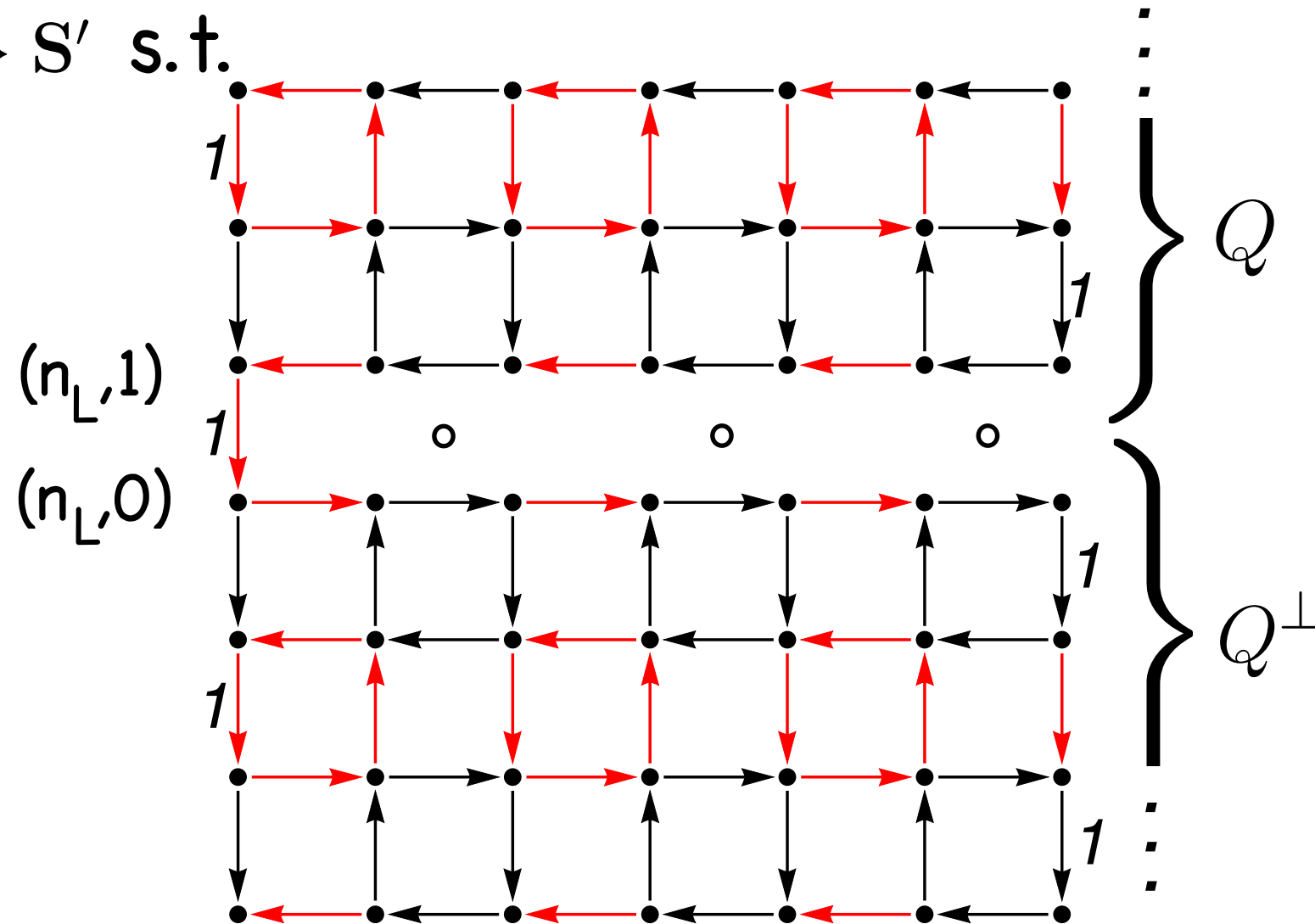


Deformation $U'_I = U_I + F$

- Given $U_I \leftrightarrow S$, let $U'_I \leftrightarrow S'$ s.t.

$$\begin{cases} S'_{j,0} := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, j \text{ odd} \\ S'_{j,2k} = S_{j,2k}, \text{ elsewhere} \end{cases}$$

$\Rightarrow F$ **finite rank**



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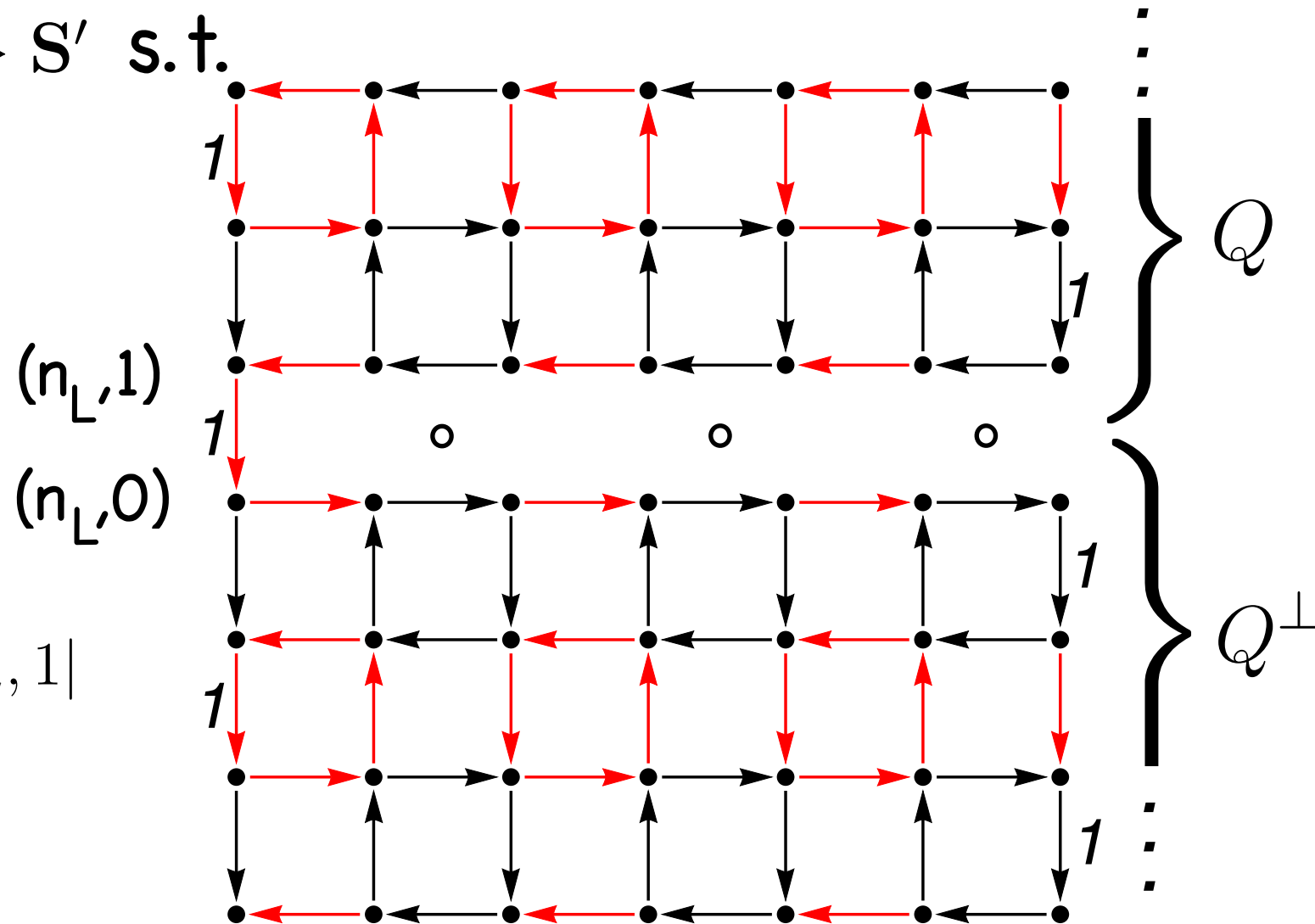
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$$\Phi' = U_I'^* Q U_I' - Q = -|n_L, 1\rangle \langle n_L, 1|$$

$$\Rightarrow \text{Tr}(\Phi') = -1 = \text{Tr}(\Phi)$$



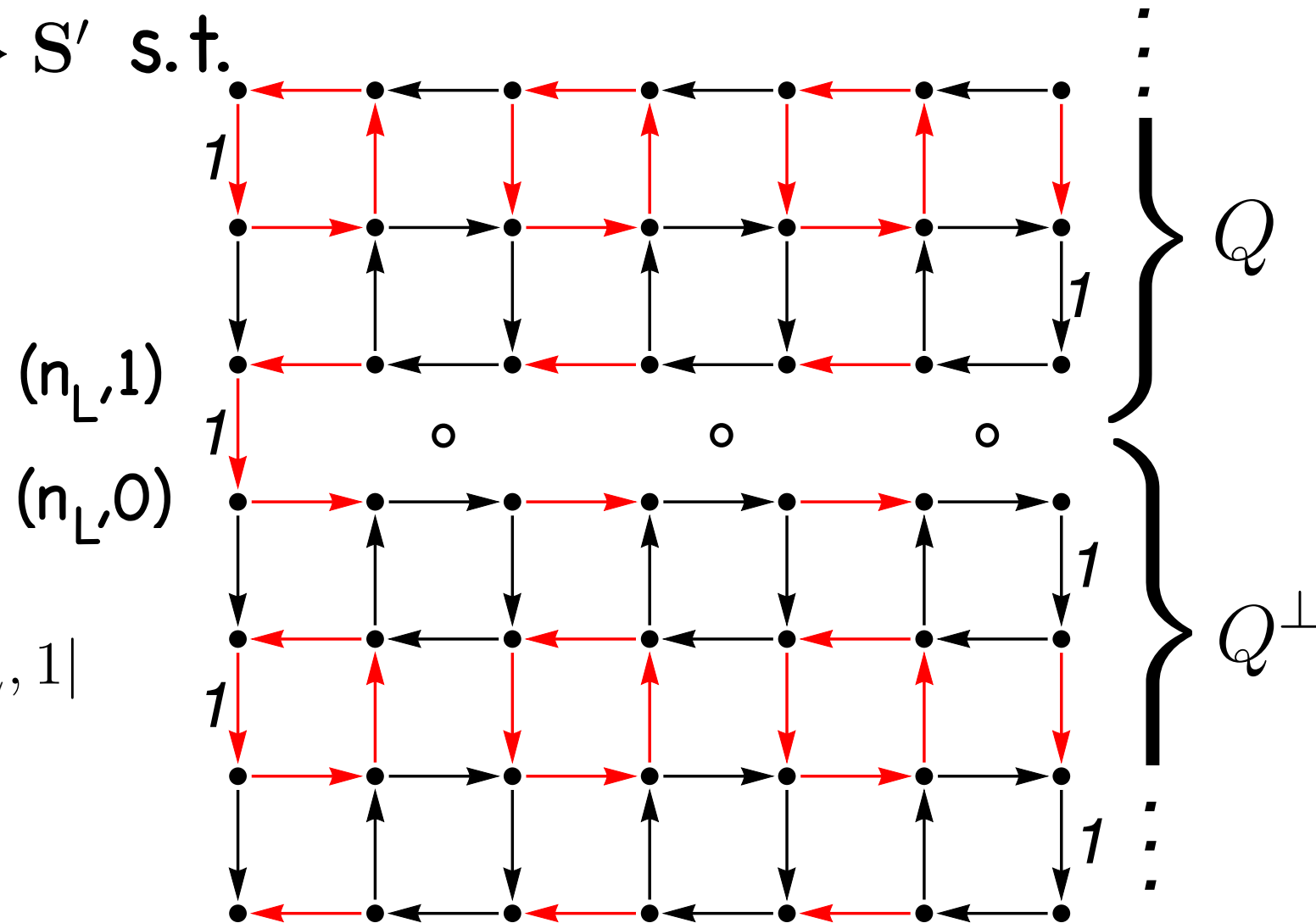
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- Moreover:**

$$\begin{cases} Q_c \text{ **mult.** by } \chi(k \geq c) \\ \Phi_c = U'^*_I Q_c U_I - Q_c \end{cases} \Rightarrow \text{Tr}(\Phi_c) = \text{Tr}(\Phi), \forall c \in \mathbb{Z}$$

Spectral consequences

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Proof: φ e.v. of $U \Rightarrow \langle \varphi, (U^*QU - Q)\varphi \rangle = 0 \Rightarrow \text{Tr}(P_{pp}(U)\Phi P_{pp}(U)) = 0$

$$\text{Tr}(\Phi) = \text{Tr}((P_{pp}(U) + P_{\text{cont}}(U))\Phi((P_{pp}(U) + P_{\text{cont}}(U)))) = \text{Tr}(P_{\text{cont}}(U)\Phi P_{\text{cont}}(U)) \neq 0$$

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- **Theorem I:** $\text{Tr}(\Phi) = -1$
 $\Rightarrow I_{n_L, n_R} = \mathcal{H}_s \oplus \mathcal{H}_s^\perp \quad \& \quad U_I = S \oplus V + F, \quad \text{with} \quad \begin{cases} S & \text{shift} \\ V & \text{unitary} \\ F & \text{finite rank} \end{cases}$

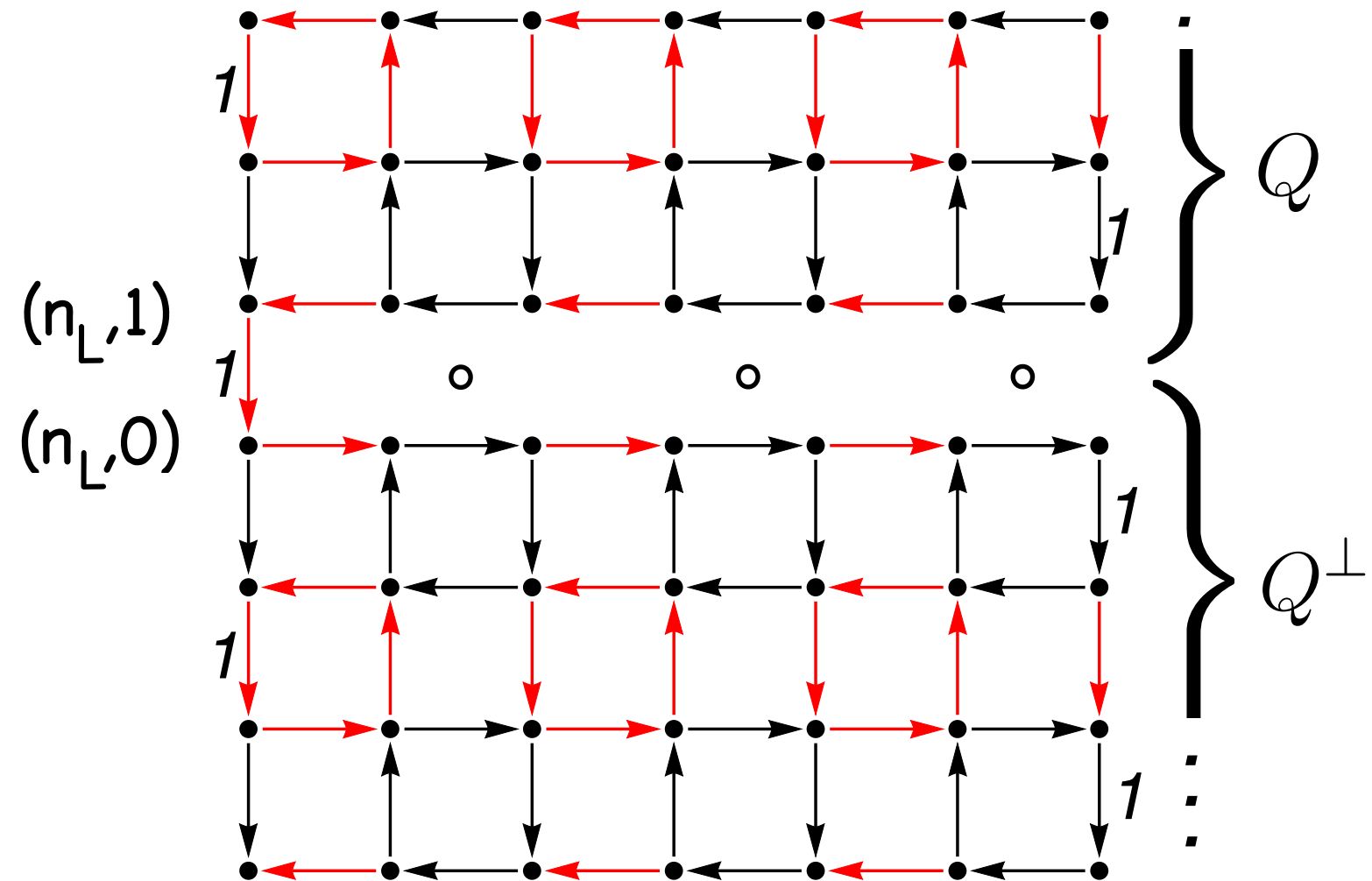
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- **Corollary:** $\sigma_{ac}(U_{CC}) = \mathbb{S}^1$
- **Theorem II:** $S_{j, 2k} = S_j, \quad \forall (j, k) \in \mathbb{Z}^2$ Transl. invar.
 $\Rightarrow \sigma_{sc}(U_{CC}) = \emptyset \quad \& \quad \sigma_{ac}(U_{CC}) = \mathbb{S}^1$

Proof of Thm I

- Consider $U'_I = U_I + F$:



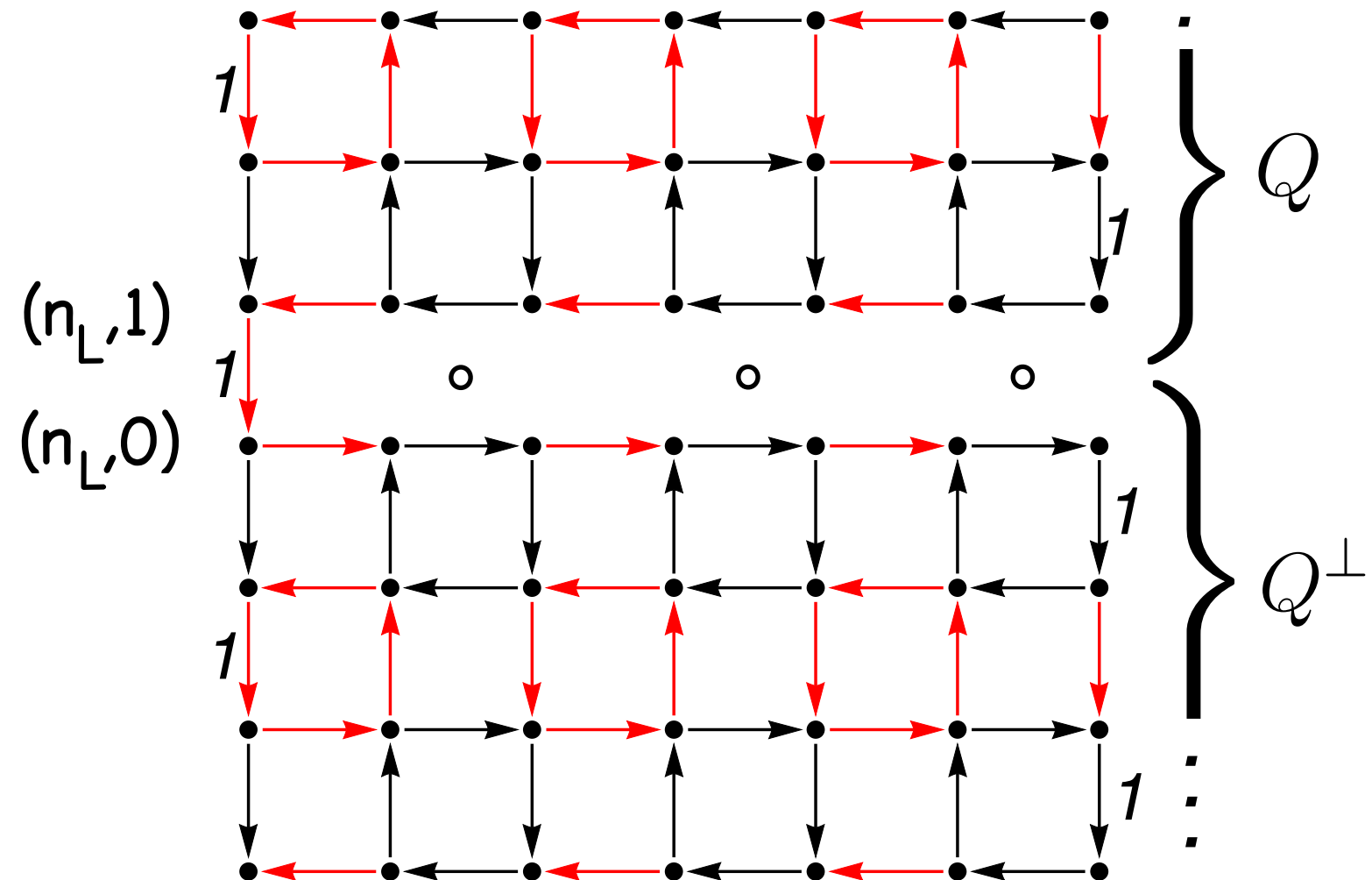
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$$Q^\perp U'_I Q = p |n_L, 0\rangle \langle n_L, 1|$$

$p \in \mathbb{S}^1$



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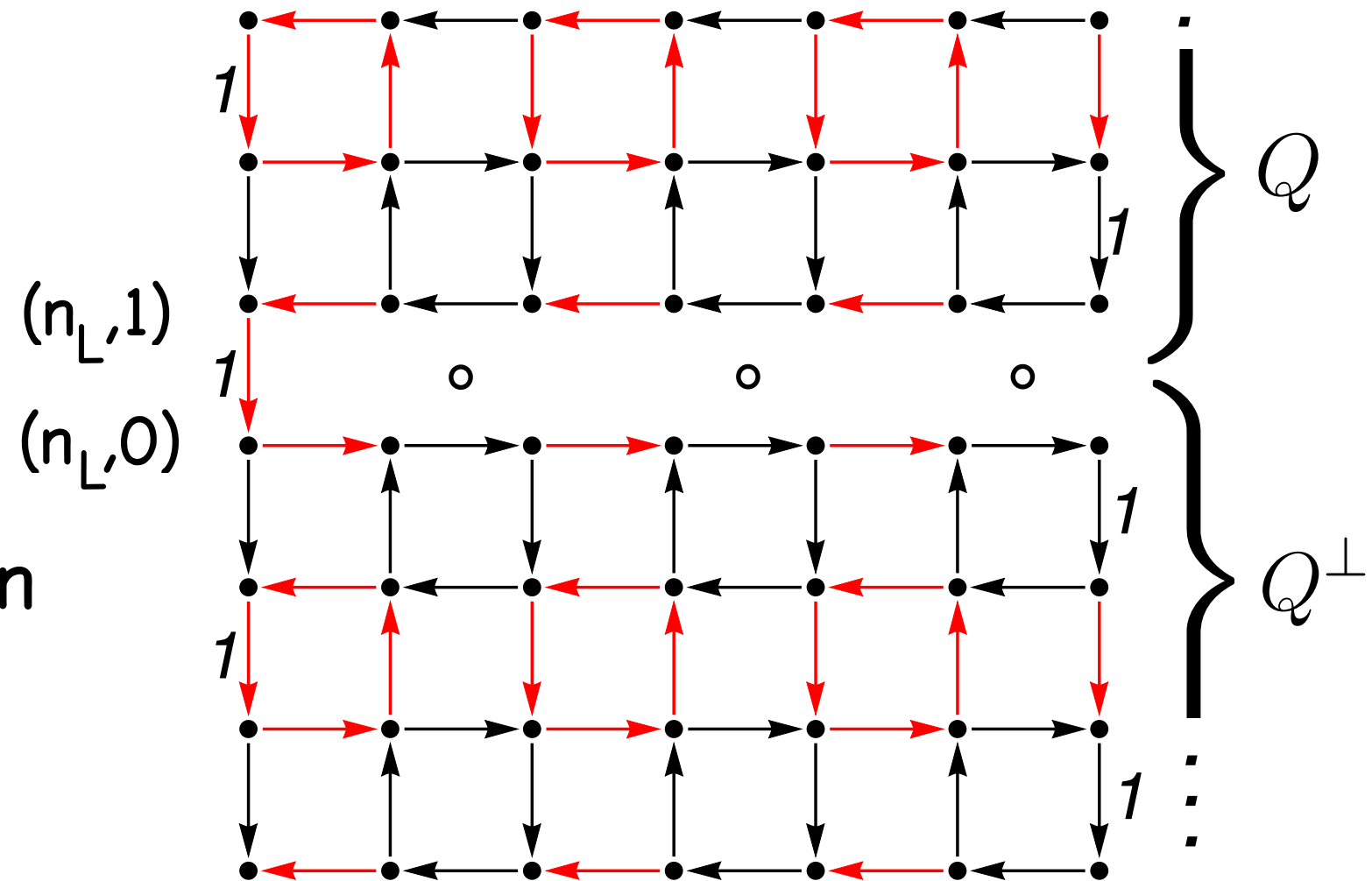
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- Let $\mathcal{L} := \text{span} \{|\mathbf{n}_L, 1\rangle\}$ then

$$U_I'^n \mathcal{L} \perp \mathcal{L} \quad \forall n \geq 1$$



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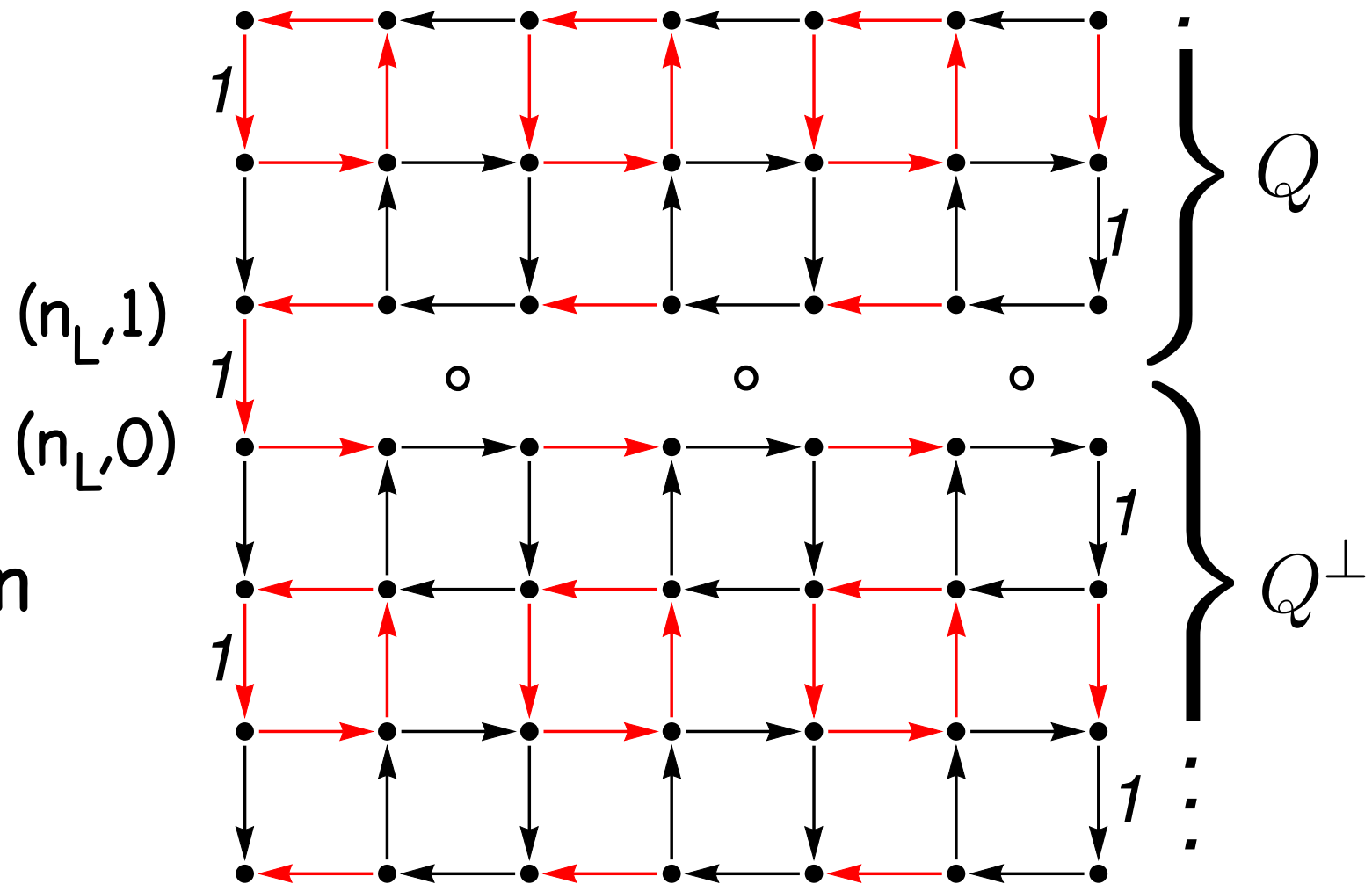
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$$U_I'^n \mathcal{L} \perp U_I'^m \mathcal{L}, \quad \forall n \neq m$$

i.e. $\mathcal{L}$ wandering subspace



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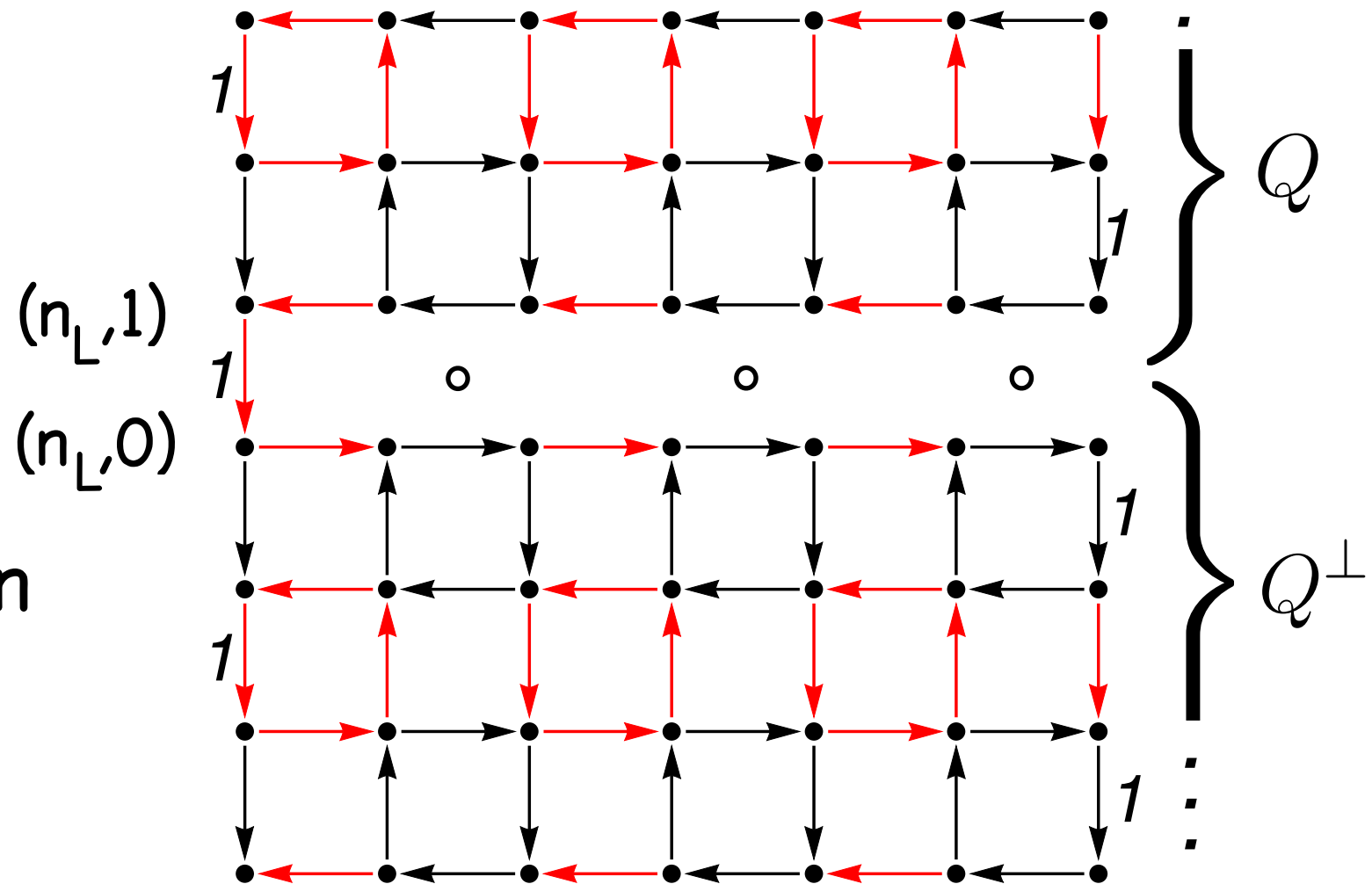
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$$U_I'^n \mathcal{L} \perp U_I'^m \mathcal{L}, \quad \forall n \neq m$$

i.e. $\mathcal{L}$ **wandering subspace**

- Set $\mathcal{H}_s := \bigoplus_{n \in \mathbb{Z}} U_I'^n \mathcal{L}$, **invariant**

$$U_I'|_{\mathcal{H}_s} \simeq S \text{ **shift** \& } U_I'|_{\mathcal{H}_s^\perp} = V \text{ **unitary**}$$



Translation invariant case

- Assume

$$S_{j,2k} = S_j, \quad \forall (j, k) \in \mathbb{Z}^2$$

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Fourier

$$S_{j,2k} = S_j, \quad \forall (j, k) \in \mathbb{Z}^2$$

$$\mathcal{F} : \ell^2(\mathbb{Z}^2; \mathbb{C}) \rightarrow L^2(\mathbb{S}^1) \otimes \ell^2(\mathbb{Z}; \mathbb{C}) \otimes \mathbb{C}^2$$

$$\begin{cases} \mathcal{F} |j, 2k\rangle := e^{iky} \otimes |j\rangle \otimes |+\rangle \\ \mathcal{F} |j, 2k+1\rangle := e^{iky} \otimes |j-1\rangle \otimes |-\rangle \end{cases}$$

Translation invariant case

- Assume
Fourier

$$S_{j,2k} = S_j, \quad \forall (j, k) \in \mathbb{Z}^2$$

$$\mathcal{F} : \ell^2(\mathbb{Z}^2; \mathbb{C}) \rightarrow L^2(\mathbb{S}^1) \otimes \ell^2(\mathbb{Z}; \mathbb{C}) \otimes \mathbb{C}^2$$

$$\begin{cases} \mathcal{F} |j, 2k\rangle := e^{iky} \otimes |j\rangle \otimes |+\rangle \\ \mathcal{F} |j, 2k+1\rangle := e^{iky} \otimes |j-1\rangle \otimes |-\rangle \end{cases}$$

- Prop: $\mathcal{F} U_{CC} \mathcal{F}^{-1} = \int_{\mathbb{S}^1}^{\oplus} U_{QW}(y) \frac{dy}{2\pi}$ on $L^2(\mathbb{S}^1; \ell^2(\mathbb{Z}) \otimes \mathbb{C}^2)$
where $U_{QW}(y) := \mathcal{SC}(y)$ is a 1D Quantum Walk

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$$\mathcal{S} |j\rangle \otimes |\pm\rangle := |j \pm 1\rangle \otimes |\pm\rangle$$

$$\mathbf{C}(y) |j\rangle \otimes v := |j\rangle \otimes C_j(y)v \quad \forall v \in \mathbb{C}^2$$

$$C_{2j}(y) := q_{2j} \begin{pmatrix} -t_{2j} & \overline{r_{2j}} e^{iy} \\ r_{2j} e^{-iy} & \overline{t_{2j}} \end{pmatrix}, \quad C_{2j+1}(y) := q_{2j+1} \begin{pmatrix} r_{2j+1} & \overline{t_{2j+1}} \\ -t_{2j+1} & \overline{r_{2j+1}} \end{pmatrix}.$$

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- Interface $[0, 2\pi) \ni y \mapsto U_{QW}(y)$ finite matrix, analytic

$\Rightarrow$ Thm II:

$$\sigma_{sc}(U_{CC}) = \emptyset \quad \& \quad \sigma_{ac}(U_{CC}) = \mathbb{S}^1$$

- Kitaev's Spectral flow formula: on $\ell^2(\mathbb{Z}; \mathbb{C}^d)$

$$U\psi(x) = \sum_{y \in \mathbb{Z}} U(x, y)\psi(y) \quad \text{s.t.} \quad \|U(x, y)\|_{HS} \leq \frac{c}{|x - y|^\alpha} \quad \alpha > 2$$

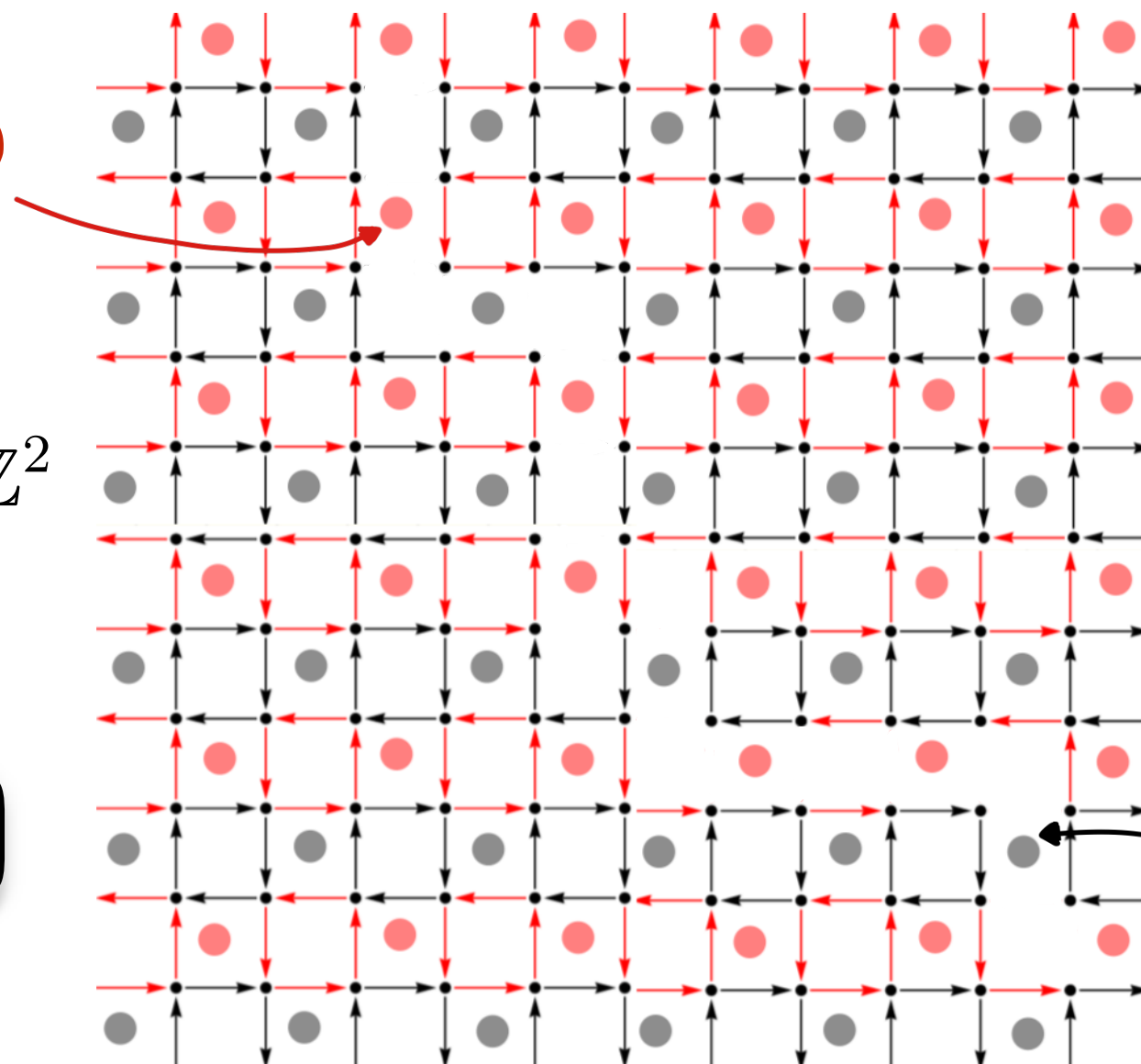
$$\text{index}(U^*QU, Q) = \text{Tr}(U^*QU - Q) = \sum_{z \geq 1} \sum_{y < 1} (\|U(z, y)\|_{HS}^2 - \|U(y, z)\|_{HS}^2)$$

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$r=0$



- Extensions to $\mathbb{Z}^2$

perturb. of:

$$\Rightarrow \sigma_{ac}(U_{CC}) = \mathbb{S}^1$$

Remarks:

- No **conjugate** operator for **Mourre** estimate
- Extensions to other **Quantum Walks** in $\mathbb{Z}^d$
- Reminiscent of **Iwatsuka** type situations
- **Bulk-edge** correspondence for Floquet operators:
Carpentier et al '15, Graf-Tauber '17, Sadel-Schulz-Baldes '17, Delplace et al '17,
Shapiro-Tauber '18.
- Inspiration **Kitaev** and **Werner's** group work on 1D QW's