Towards a Rigorous Proof of Haldane's Photonic Bulk-edge Correspondence

joint work with Giuseppe De Nittis

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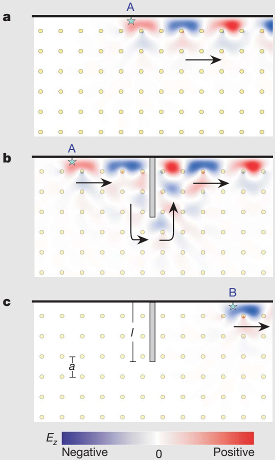
Quantum Hall Effect for Light

Predicted theoretically by Raghu & **Haldane** (2005) ...

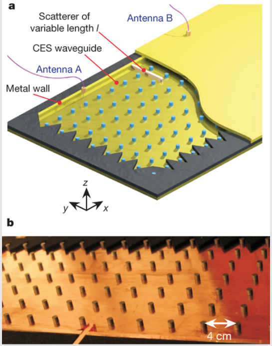
$$\left.\begin{matrix} \begin{pmatrix} \bar{\varepsilon} & 0 \\ 0 & \bar{\mu} \end{pmatrix} \neq \begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix} \\ \text{symmetry breaking} \end{matrix}\right\} \implies$$



Joannopoulos, Soljačić et al (2009)

Quantum Hall Effect for Light

... and realized experimentally by Joannopoulos et al (2009)



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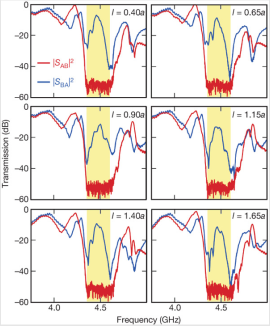

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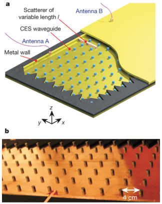
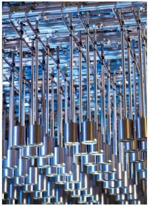
Haldane's Insight

Topological effects are **wave**,
not quantum **phenomena**!

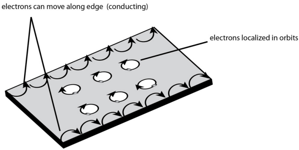
Topological Effects: Phenomenological Similarities



Light



Coupled Oscillators



Quantum

- Periodic structure ⇝ **bulk band gap**
- **Breaking** of time-reversal **symmetries**
- Unidirectional edge modes
- Robust under perturbations

Different manifestations of the same underlying physical principles!

Haldane's Photonic Bulk-Edge Correspondence

Conjecture

In a **two-dimensional** photonic crystals with boundary the **difference of the number of left- and right-moving boundary modes**

$$\mathrm{Ch}_{\mathrm{bulk}} = T_{\mathrm{edge}} = \text{net} \sharp \text{ of edge modes}$$

in bulk band gaps is a **topologically protected quantity and equals the Chern number** associated to the frequency bands below the bulk band gap.

Haldane's Photonic Bulk-Edge Correspondence

Conjecture

$$T_{\mathrm{bulk}} = T_{\mathrm{edge}} = \text{net } \sharp \text{ of edge modes}$$

My Main Goal

Make the statement mathematically precise and provide a proof.

Proof of Haldane's Photonic Bulk-Edge Correspondence

Seems simple enough …

1. Rewrite Maxwell's equations in the form of a Schrödinger equation.

   De Nittis & L., Comm. Math. Phys. **332**, pp. 221–260, 2014

2. Classify electromagnetic media using the Cartan-Altland-Zirnbauer scheme.

   De Nittis & L., Annals of Physics **350**, pp. 568–587, 2014

3. Adapt existing techniques to prove bulk-boundary correspondences

   (relying on e. g. Hatsugai, Graf & Porta, Hayashi; Kellendonk & Schulz-Baldes)

Easy! **… No!**

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Easy! … No!

Main Messages of This Talk

- Explain why things are not so simple.
- Explain how to deal with the complications in Steps 1 & 2.
- Explain the obstacles to be overcome in Step 3.

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   De Nittis & L., Annals of Physics **396**, pp. 221–260, 2018

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The Real-Valuedness Condition

Classical waves such as $(\mathbf{E}, \mathbf{H}) = (\overline{\mathbf{E}}, \overline{\mathbf{H}})$ are **real-valued**!

- Our earlier works start with the standard equations used in the physics community.
- These equations violate real-valuedness condition.
- We were aware of this problem and discussed it in one of our earlier works.
  (Section 6 of De Nittis & L., Annals of Physics **350**, pp. 568–587, 2014)
- Clarified thanks to discussions with Duncan Haldane and Kostya Bliokh.

$\Longrightarrow$ Mathematically correct results about unphysical equations.

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Quantum vs. Classical Equations: a Single Spin

Idea

Start with the same equations of motion,

$$\mathrm{i}\frac{\partial}{\partial t}\begin{pmatrix}\psi_1(t)\\ \psi_2(t)\end{pmatrix} = \begin{pmatrix}0 & -\mathrm{i}\,\omega_0\\ +\mathrm{i}\,\omega_0 & 0\end{pmatrix}\begin{pmatrix}\psi_1(t)\\ \psi_2(t)\end{pmatrix},$$

once in the quantum and then in the classical context.

- Math is trivial, everything is explicit
- Immediately transfers to many other hamiltonian equations as $J = \mathrm{i}\sigma_2$ is the **canonical symplectic form**
- Conceptually applies to **all** classical wave equations

Symmetries of Classical and Quantum Spin Equations

Purpose
Anticipate symmetry classification of electromagnetic media

$$\mathrm{i}\frac{\partial}{\partial t}\begin{pmatrix}\psi_1(t)\\ \psi_2(t)\end{pmatrix} = \begin{pmatrix}0 & -\mathrm{i}\,\omega_0\\ +\mathrm{i}\,\omega_0 & 0\end{pmatrix}\begin{pmatrix}\psi_1(t)\\ \psi_2(t)\end{pmatrix},$$

- What is the difference between the quantum and classical equations when it comes to symmetries?
- What types of symmetries does the classical equation possess (in the context of the Cartan-Altland-Zirnbauer classification)?

$\Longrightarrow$ Requires us to work with complex Hilbert spaces

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Symmetries of Classical and Quantum Spin Systems

Quantum

Fundamental equation

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Building blocks

Hamiltonian: $H = \omega_0\,\sigma_2 = H^*$

States: $\begin{pmatrix}\psi_1(t)\\ \psi_2(t)\end{pmatrix} \in \mathcal{H}_{\mathbb{C}} = \mathbb{C}^2$

Symmetries

$\sigma_{1,3}\,H\,\sigma_{1,3}^{-1} = -H$ (chiral)

$\sigma_2\,H\,\sigma_2^{-1} = +H$ (ordin.)

$C\,H\,C = \overline{H} = -H$ (+PH)

$(\sigma_{1,3}\,C)\,H\,(\sigma_{1,3}\,C)^{-1} = +H$ (+TR)

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Classical

Fundamental equation

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Symmetries

$\sigma_{1,3}\,H\,\sigma_{1,3}^{-1} = -H$ (???)
$(\mathrm{i}\sigma_2)\,H\,(\mathrm{i}\sigma_2)^{-1} = +H$ (???)
~~$C\,H\,C = \overline{H} = -H$ (+PH)~~
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Building blocks

Hamiltonian: $H = \omega_0\,\sigma_2$

States: $\begin{pmatrix}M_x(t)\\ M_y(t)\end{pmatrix} \in \mathcal{H}_{\mathbb{R}} = \mathbb{R}^2$

Symmetries

C not defined on $\mathcal{H}_{\mathbb{R}} = \mathbb{R}^2$

$\sigma_1, \mathrm{i}\sigma_2$ and σ_3 are real matrices

$\rightsquigarrow$ classical spin transformations

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States: $\begin{pmatrix}\psi_1(t)\\ \psi_2(t)\end{pmatrix} \in \mathcal{H}_{\mathbb{C}} = \mathbb{C}^2$

Symmetries

$U_1 = \sigma_1$	(chiral)	$T_1 = \sigma_1\,C$	(+TR)
$U_2 = \sigma_2$	(ordin.)	$T_2 = \sigma_2\,C$	(-PH)
$U_3 = \sigma_3$	(chiral)	$T_3 = \sigma_3\,C$	(+TR)
		C	(+PH)

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Symmetries

$V_1^{\mathbb{R}} = \sigma_1$ (???)

$V_2^{\mathbb{R}} = \mathrm{i}\sigma_2$ (???)

$V_3^{\mathbb{R}} = \sigma_3$ (???)

Consider Classical Equation on Complex Hilbert Space

Cartan-Altland-Zirnbauer classification scheme for Topological Insulators (and many other techniques from quantum mechanics) only work for operators acting on *complex* Hilbert spaces

Two Ways to Work With Complex Hilbert Spaces

1. Complexify classical equations
   (introduces unphysical degrees of freedom)
2. Work with **complex** Ψ which *represent* real states $M = 2\mathrm{Re}\,\Psi$
   (establish 1-to-1 correspondence $\mathcal{H}_{\mathbb{C}} \leftrightarrow \mathcal{H}_{\mathbb{R}}$)

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Complexifying the Classical Equations

Complexification

Fundamental equation

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Building blocks

Hamiltonian: $H = \omega_0\, \sigma_2 = H^*$

States: $M = \Psi_+ + \Psi_- \in \mathcal{H}_{\mathbb{R}} \subset \mathcal{H}_{\mathbb{C}}$

Symmetries

$V_1^{\mathbb{C}} = ???$
$V_2^{\mathbb{C}} = ???$
$V_3^{\mathbb{C}} = ???$

Classical

Fundamental equation

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Hamiltonian: $H = \omega_0\, \sigma_2$

States: $\begin{pmatrix} M_x(t) \\ M_y(t) \end{pmatrix} \in \mathcal{H}_{\mathbb{R}} = \mathbb{R}^2$

Symmetries

$V_1^{\mathbb{R}} = \sigma_1$
$V_2^{\mathbb{R}} = \mathrm{i}\sigma_2$
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Symmetries

$\mathbb{1}\big|_{\mathcal{H}_{\mathbb{R}}} = C\big|_{\mathcal{H}_{\mathbb{R}}}$ none vs. +PH

$\sigma_{1,3}\big|_{\mathcal{H}_{\mathbb{R}}} = \sigma_{1,3}\,C\big|_{\mathcal{H}_{\mathbb{R}}}$ chiral vs. +TR

$\mathrm{i}\sigma_2\big|_{\mathcal{H}_{\mathbb{R}}} = \mathrm{i}\sigma_2\,C\big|_{\mathcal{H}_{\mathbb{R}}}$ ordin. vs. -PH

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Symmetries

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Complexifying the Classical Equations

Complexification

Fundamental equation

$$\mathrm{i}\frac{\partial}{\partial t}\begin{pmatrix} M_x(t) \\ M_y(t) \end{pmatrix} = \begin{pmatrix} 0 & -\mathrm{i}\,\omega_0 \\ +\mathrm{i}\,\omega_0 & 0 \end{pmatrix} \begin{pmatrix} M_x(t) \\ M_y(t) \end{pmatrix}$$

Building blocks

Hamiltonian: $H = \omega_0\,\sigma_2 = H^*$
States: $M = \Psi_+ + \Psi_- \in \mathcal{H}_{\mathbb{R}} \subset \mathcal{H}_{\mathbb{C}}$

Symmetries

- Redundant symmetry operations
- Different choices $\Rightarrow$ different topological classifications!?

Classical

Fundamental equation

$$\mathrm{i}\frac{\partial}{\partial t}\begin{pmatrix} M_x(t) \\ M_y(t) \end{pmatrix} = \begin{pmatrix} 0 & -\mathrm{i}\,\omega_0 \\ +\mathrm{i}\,\omega_0 & 0 \end{pmatrix} \begin{pmatrix} M_x(t) \\ M_y(t) \end{pmatrix}$$

Building blocks

Hamiltonian: $H = \omega_0\,\sigma_2$
States: $\begin{pmatrix} M_x(t) \\ M_y(t) \end{pmatrix} \in \mathcal{H}_{\mathbb{R}} = \mathbb{R}^2$

Symmetries

$V_1^{\mathbb{R}} = \sigma_1$
$V_2^{\mathbb{R}} = \mathrm{i}\sigma_2$
$V_3^{\mathbb{R}} = \sigma_3$

Consider Classical Equation on Complex Hilbert Space

Cartan-Altland-Zirnbauer classification scheme for Topological Insulators (and many other techniques from quantum mechanics) only work for operators acting on *complex* Hilbert spaces

Two Ways to Work With Complex Hilbert Spaces

1. Complexify classical equations
   (introduces unphysical degrees of freedom)
2. Work with **complex** Ψ which *represent* real states $M = 2\mathrm{Re}\,\Psi$
   (establish 1-to-1 correspondence $\mathcal{H}_{\mathbb{C}} \leftrightarrow \mathcal{H}_{\mathbb{R}}$)

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Schrödinger Formalism of Classical Spin Waves

Complexification

Fundamental equation

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Building blocks

Hamiltonian: $H = \omega_0\,\sigma_2 = H^*$

States: $M = \Psi_+ + \Psi_- \in \mathcal{H}_{\mathbb{R}} \subset \mathcal{H}_{\mathbb{C}}$

Symmetries

$\mathbb{1}\big|_{\mathcal{H}_{\mathbb{R}}} = C\big|_{\mathcal{H}_{\mathbb{R}}}$

$\sigma_{1,3}\big|_{\mathcal{H}_{\mathbb{R}}} = \sigma_{1,3}\,C\big|_{\mathcal{H}_{\mathbb{R}}}$

$\mathrm{i}\sigma_2\big|_{\mathcal{H}_{\mathbb{R}}} = \mathrm{i}\sigma_2\,C\big|_{\mathcal{H}_{\mathbb{R}}}$

Classical

Fundamental equation

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Building blocks

Hamiltonian: $H = \omega_0\,\sigma_2$

States: $\begin{pmatrix} M_x(t) \\ M_y(t) \end{pmatrix} \in \mathcal{H}_{\mathbb{R}} = \mathbb{R}^2$

Symmetries

$V_1^{\mathbb{R}} = \sigma_1$

$V_2^{\mathbb{R}} = \mathrm{i}\sigma_2$

$V_3^{\mathbb{R}} = \sigma_3$

Schrödinger Formalism of Classical Spin Waves

$\omega > 0$ Representation

Fundamental equation

$$\mathrm{i}\frac{\partial}{\partial t}\begin{pmatrix}\psi_{+,1}(t)\\ \psi_{+,2}(t)\end{pmatrix} = \begin{pmatrix}0 & -\mathrm{i}\,\omega_0\\ +\mathrm{i}\,\omega_0 & 0\end{pmatrix}\begin{pmatrix}\psi_{+,1}(t)\\ \psi_{+,2}(t)\end{pmatrix}$$

Building blocks

Hamiltonian: $H = \omega_0\,\sigma_2 = H^*$

States: $M = 2\mathrm{Re}\,\Psi_+ \in \mathcal{H}_{\mathbb{R}}$

Symmetries

$V_1^{\mathbb{C}} = ???$

$V_2^{\mathbb{C}} = ???$

$V_3^{\mathbb{C}} = ???$

Classical

Fundamental equation

$$\mathrm{i}\frac{\partial}{\partial t}\begin{pmatrix}M_x(t)\\ M_y(t)\end{pmatrix} = \begin{pmatrix}0 & -\mathrm{i}\,\omega_0\\ +\mathrm{i}\,\omega_0 & 0\end{pmatrix}\begin{pmatrix}M_x(t)\\ M_y(t)\end{pmatrix}$$

Building blocks

Hamiltonian: $H = \omega_0\,\sigma_2$

States: $\begin{pmatrix}M_x(t)\\ M_y(t)\end{pmatrix} \in \mathcal{H}_{\mathbb{R}} = \mathbb{R}^2$

Symmetries

$V_1^{\mathbb{R}} = \sigma_1$

$V_2^{\mathbb{R}} = \mathrm{i}\sigma_2$

$V_3^{\mathbb{R}} = \sigma_3$

Schrödinger Formalism of Classical Spin Waves

Eliminate superfluous degree of freedom in complexified equations
$\rightsquigarrow$ Systematically identify $\mathbb{R}^2 \cong \mathbb{C}$

$$\underbrace{M(t)}_{\text{real wave}} = 2\mathrm{Re}\,\underbrace{\Psi(t)}_{\text{complex } \omega > 0 \text{ wave}}$$

1-to-1 correspondence

Schrödinger Formalism of Classical Spin Waves

Eliminate superfluous degree of freedom in complexified equations
$\rightsquigarrow$ Systematically identify $\mathbb{R}^2 \cong \mathbb{C}$

$$
\begin{aligned}
M(t) &= \begin{pmatrix} \cos \omega_0 t & -\sin \omega_0 t \\ \sin \omega_0 t & \cos \omega_0 t \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \\
&= 2\mathrm{Re}\, \Psi(t) = 2\mathrm{Re} \Big((a - \mathrm{i}b)\, \mathrm{e}^{-\mathrm{i}\omega_0 t} \Psi_+ \Big)
\end{aligned}
$$

where $\Psi_+ = \left(\begin{smallmatrix} 1 \\ +\mathrm{i} \end{smallmatrix} \right)$ is the eigenvector of $H = \omega_0 \, \sigma_2$ to $+\omega_0 > 0$.

1-to-1 correspondence

Schrödinger Formalism of Classical Spin Waves

$\omega > 0$ Representation

Fundamental equation

$$\mathrm{i}\tfrac{\partial}{\partial t}\Psi(t) = \omega_0\,\sigma_2\,\Psi(t)$$

Building blocks

Hamiltonian: $H = \omega_0\,\sigma_2 = H^*$

States: $\Psi(t) \in \mathcal{H}_+ = \mathrm{span}_{\mathbb{C}}\left\{\left(\begin{smallmatrix} 1 \\ +\mathrm{i} \end{smallmatrix}\right)\right\}$

Symmetries

$V_1^{\mathbb{C}} = ???$
$V_2^{\mathbb{C}} = ???$
$V_3^{\mathbb{C}} = ???$

Classical

Fundamental equation

$$\mathrm{i}\tfrac{\partial}{\partial t}M(t) = \omega_0\,\sigma_2\,M(t)$$

Building blocks

Hamiltonian: $H = \omega_0\,\sigma_2$

States: $M(t) = 2\mathrm{Re}\,\Psi(t) \in \mathbb{R}^2$

Symmetries

$V_1^{\mathbb{R}} = \sigma_1$
$V_2^{\mathbb{R}} = \mathrm{i}\sigma_2$
$V_3^{\mathbb{R}} = \sigma_3$

Translating Real Symmetries to $\omega > 0$ Representation

Requirements

1. $M = 2\mathrm{Re}\,\Psi$, then
   $V_j^{\mathbb{R}} M = 2\mathrm{Re}\,(V_j^{\mathbb{C}} \Psi)$
2. $V_j^{\mathbb{C}}$ is a (anti)unitary on $\mathcal{H}_+$, i. e. it maps $\omega > 0$ waves onto $\omega > 0$ waves.

Consequences

1. $V_j^{\mathbb{C}} = \begin{cases} V_j^{\mathbb{R}} & \text{(unitary)} \\ V_j^{\mathbb{R}}\, C & \text{(antiunitary)} \end{cases}$
2. $V_j^{\mathbb{C}}$ must commute with $H = \omega_0\, \sigma_2$

Translating Real Symmetries to $\omega > 0$ Representation

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2. $V_j^{\mathbb{C}}$ must commute with $H = \omega_0\,\sigma_2$

Translating Real Symmetries to $\omega > 0$ Representation

Real Symmetry	Complex Representative	TI Classification
$V_1^{\mathbb{R}} = \sigma_1$	$V_1^{\mathbb{C}} = \sigma_1 \, C$	+TR
$V_2^{\mathbb{R}} = \mathrm{i}\sigma_2$	$V_2^{\mathbb{C}} = \mathrm{i}\sigma_2$	ordinary
$V_3^{\mathbb{R}} = \sigma_3$	$V_3^{\mathbb{C}} = \sigma_3 \, C$	+TR

Translating Real Symmetries to $\omega > 0$ Representation

$\omega > 0$ Representation

Fundamental equation

$$\mathrm{i}\tfrac{\partial}{\partial t}\Psi(t) = \omega_0\,\sigma_2\,\Psi(t)$$

Building blocks

Hamiltonian: $H = \omega_0\,\sigma_2 = H^*$

States: $\Psi(t) \in \mathcal{H}_+ = \mathrm{span}_{\mathbb{C}}\left\{\left(\begin{smallmatrix} 1 \\ +\mathrm{i} \end{smallmatrix}\right)\right\}$

Symmetries

$V_1^{\mathbb{C}} = \sigma_1\,C$ (+TR)

$V_2^{\mathbb{C}} = \mathrm{i}\sigma_2$ (ordinary)

$V_3^{\mathbb{C}} = \sigma_3\,C$ (+TR)

Classical

Fundamental equation

$$\mathrm{i}\tfrac{\partial}{\partial t}M(t) = \omega_0\,\sigma_2\,M(t)$$

Building blocks

Hamiltonian: $H = \omega_0\,\sigma_2$

States: $M(t) = 2\mathrm{Re}\,\Psi(t) \in \mathbb{R}^2$

Symmetries

$V_1^{\mathbb{R}} = \sigma_1$

$V_2^{\mathbb{R}} = \mathrm{i}\sigma_2$

$V_3^{\mathbb{R}} = \sigma_3$

Translating Real Symmetries to $\omega > 0$ Representation

Moral of the Story

- Not all "quantum" symmetries are symmetries of the classical equations
  $\leadsto$ Incompatible with the real-valuedness of classical waves
- **"Schrödinger" form** of classical equations necessary to **identify the nature of** these **symmetries** in the context of TIs
- C is **not a meaningful symmetry** of the **"Schrödinger" form** of the classical equations!
- **No fermionic time-reversal symmetry**

Ideas apply to all classical wave equations!

Applies directly to vacuum Maxwell equations

Spin $\longrightarrow$ *In Vacuo* Maxwell Equations

$$H = \omega_0 \, \sigma_2 \quad \longrightarrow \quad \mathrm{Rot} = -\sigma_2 \otimes \nabla^{\times}$$

$$V_{1,3}^{\mathbb{R}} = \sigma_{1,3} \quad \longrightarrow \quad V_{1,3}^{\mathbb{R}} = \sigma_{1,3} \otimes \mathbb{1}$$

$$V_2^{\mathbb{R}} = \mathrm{i}\sigma_2 \quad \longrightarrow \quad V_2^{\mathbb{R}} = \mathrm{i}\sigma_2 \otimes \mathbb{1}$$

Same Strategy

1. Complexify classical equations
2. Eliminate superfluous states in complex Hilbert space
3. Identify complex implementation of the three symmetries

Schrödinger Formalism of *In Vacuo* Maxwell Equations

Complexification

Fundamental equation

$$\mathrm{i}\frac{\partial}{\partial t}\begin{pmatrix}\psi^E(t)\\ \psi^H(t)\end{pmatrix} = \begin{pmatrix}0 & +\mathrm{i}\,\nabla^\times\\ -\mathrm{i}\,\nabla^\times & 0\end{pmatrix}\begin{pmatrix}\psi^E(t)\\ \psi^H(t)\end{pmatrix}$$

Building blocks

Hamiltonian: $M = -\sigma_2 \otimes \nabla^\times = M^*$

States: $\Psi(t) \in L^2(\mathbb{R}^3, \mathbb{C}^6)$

Symmetries

$V_1^{\mathbb{C}} = (\sigma_1 \otimes \mathbb{1})\, C$ (+TR)

$V_2^{\mathbb{C}} = \mathrm{i}\sigma_2 \otimes \mathbb{1}$ (ordinary)

$V_3^{\mathbb{C}} = (\sigma_3 \otimes \mathbb{1})\, C$ (+TR)

Classical

Fundamental equation

$$\mathrm{i}\frac{\partial}{\partial t}\begin{pmatrix}\mathbf{E}(t)\\ \mathbf{H}(t)\end{pmatrix} = \begin{pmatrix}0 & +\mathrm{i}\,\nabla^\times\\ -\mathrm{i}\,\nabla^\times & 0\end{pmatrix}\begin{pmatrix}\mathbf{E}(t)\\ \mathbf{H}(t)\end{pmatrix}$$

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$V_3^{\mathbb{R}} = \sigma_3 \otimes \mathbb{1}$

Schrödinger Formalism of *In Vacuo* Maxwell Equations

Representing **real, transversal** EM Fields as complex $\omega > 0$ waves

$$\underbrace{\big(\mathbf{E}(t), \mathbf{H}(t)\big)}_{\text{real wave}} = 2\mathrm{Re} \underbrace{\Psi(t)}_{\text{complex } \omega > 0 \text{ wave}}$$

$\Longrightarrow \Psi \in \mathcal{H}_+ = \{\text{complex } \omega > 0 \text{ waves}\}.$

Schrödinger Formalism of *In Vacuo* Maxwell Equations

$\omega > 0$ Representation

Fundamental equation

$$\mathrm{i}\frac{\partial}{\partial t}\begin{pmatrix}\psi^E(t)\\ \psi^H(t)\end{pmatrix} = \begin{pmatrix}0 & +\mathrm{i}\,\nabla^\times \\ -\mathrm{i}\,\nabla^\times & 0\end{pmatrix}\begin{pmatrix}\psi^E(t)\\ \psi^H(t)\end{pmatrix}$$

Building blocks

Hamiltonian: $M = -\sigma_2 \otimes \nabla^\times = M^*$

States: $\Psi(t) \in \{$compl. $\omega > 0$ waves$\}$

Symmetries

$V_1^{\mathbb{C}} = (\sigma_1 \otimes \mathbb{1})\, C$ (+TR)

$V_2^{\mathbb{C}} = \mathrm{i}\sigma_2 \otimes \mathbb{1}$ (ordinary)

$V_3^{\mathbb{C}} = (\sigma_3 \otimes \mathbb{1})\, C$ (+TR)

Classical

Fundamental equation

$$\mathrm{i}\frac{\partial}{\partial t}\begin{pmatrix}\mathbf{E}(t)\\ \mathbf{H}(t)\end{pmatrix} = \begin{pmatrix}0 & +\mathrm{i}\,\nabla^\times \\ -\mathrm{i}\,\nabla^\times & 0\end{pmatrix}\begin{pmatrix}\mathbf{E}(t)\\ \mathbf{H}(t)\end{pmatrix}$$

Building blocks

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$V_3^{\mathbb{R}} = \sigma_3 \otimes \mathbb{1}$

Schrödinger Formalism of *In Vacuo* Maxwell Equations

Real Symmetry	Complex Representative	TI Classification	Meaning
$V_1^{\mathbb{R}} = \sigma_1 \otimes \mathbb{1}$	$V_1^{\mathbb{C}} = (\sigma_1 \otimes \mathbb{1})\, C$	+TR	Flips helicity *and* arrow of time
$V_2^{\mathbb{R}} = \mathrm{i}\sigma_2 \otimes \mathbb{1}$	$V_2^{\mathbb{C}} = \mathrm{i}\sigma_2 \otimes \mathbb{1}$	ordinary	Dual symmetry
$V_3^{\mathbb{R}} = \sigma_3 \otimes \mathbb{1}$	$V_3^{\mathbb{C}} = (\sigma_3 \otimes \mathbb{1})\, C$	+TR	Ordinary EM time-reversal

Media selectively break or preserve these symmetries!

Schrödinger Formalism of *In Vacuo* Maxwell Equations

$\omega > 0$ Representation

Fundamental equation

$$\mathrm{i}\frac{\partial}{\partial t}\begin{pmatrix}\psi^E(t)\\ \psi^H(t)\end{pmatrix} = \begin{pmatrix}0 & +\mathrm{i}\,\nabla^\times \\ -\mathrm{i}\,\nabla^\times & 0\end{pmatrix}\begin{pmatrix}\psi^E(t)\\ \psi^H(t)\end{pmatrix}$$

Building blocks

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States: $\Psi(t) \in \{\text{compl. } \omega > 0 \text{ waves}\}$

Symmetries

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Classical

Fundamental equation

$$\mathrm{i}\frac{\partial}{\partial t}\begin{pmatrix}\mathbf{E}(t)\\ \mathbf{H}(t)\end{pmatrix} = \begin{pmatrix}0 & +\mathrm{i}\,\nabla^\times \\ -\mathrm{i}\,\nabla^\times & 0\end{pmatrix}\begin{pmatrix}\mathbf{E}(t)\\ \mathbf{H}(t)\end{pmatrix}$$

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Symmetries

$V_1^{\mathbb{R}} = \sigma_1 \otimes \mathbb{1}$

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Main Messages of This Talk

1. Rewrite Maxwell's equations in the form of a Schrödinger equation.

   De Nittis & L., Annals of Physics **396**, pp. 221–260, 2018

2. Classify electromagnetic media using the Cartan-Altland-Zirnbauer scheme.

   De Nittis & L., arXiv:1710.08104, 2017; De Nittis & L., arXiv:1806.07783, 2018

3. Adapt existing techniques to prove bulk-boundary correspondences

   … in progress

Maxwell's Equations in Linear, Dispersionless Media

$$\begin{pmatrix} \varepsilon & \chi^{EH} \\ \chi^{HE} & \mu \end{pmatrix} \frac{\partial}{\partial t} \begin{pmatrix} \mathbf{E}(t) \\ \mathbf{H}(t) \end{pmatrix} = \begin{pmatrix} +\nabla \times \mathbf{H}(t) \\ -\nabla \times \mathbf{E}(t) \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \text{(dynamical)}$$

$$\begin{pmatrix} \nabla \cdot \\ \nabla \cdot \end{pmatrix} \begin{pmatrix} \varepsilon & \chi^{EH} \\ \chi^{HE} & \mu \end{pmatrix} \begin{pmatrix} \mathbf{E}(t) \\ \mathbf{H}(t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \text{(constraint)}$$

Constituent Parts

- Material weights phenomenologically describe properties of the medium
- Absence of sources

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Abbreviations and Notation

- Multiply both sides of dynamical Maxwell equations by i
- $W(x) = \begin{pmatrix} \varepsilon(x) & \chi^{EH}(x) \\ \chi^{HE}(x) & \mu(x) \end{pmatrix}$
- Introduce $\mathrm{Rot} := \begin{pmatrix} 0 & +\mathrm{i}\nabla^{\times} \\ -\mathrm{i}\nabla^{\times} & 0 \end{pmatrix}$ and $\mathrm{Div} := \begin{pmatrix} \nabla \cdot \\ \nabla \cdot \end{pmatrix}$

Maxwell's Equations in Linear, Dispersionless Media

$$\begin{pmatrix} \varepsilon & \chi^{EH} \\ \chi^{HE} & \mu \end{pmatrix} \mathrm{i} \frac{\partial}{\partial t} \begin{pmatrix} \mathbf{E}(t) \\ \mathbf{H}(t) \end{pmatrix} = \begin{pmatrix} +\mathrm{i}\nabla \times \mathbf{H}(t) \\ -\mathrm{i}\nabla \times \mathbf{E}(t) \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \text{(dynamical)}$$

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Maxwell's Equations in Linear, Dispersionless Media

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Commonly Used, But Unphysical Maxwell's Equations

$$W \, \mathrm{i} \frac{\partial}{\partial t} \begin{pmatrix} \mathbf{E}(t) \\ \mathbf{H}(t) \end{pmatrix} = \mathrm{Rot} \begin{pmatrix} \mathbf{E}(t) \\ \mathbf{H}(t) \end{pmatrix} \qquad \text{(dynamical)}$$

$$\mathrm{Div} \, W \begin{pmatrix} \mathbf{E}(t) \\ \mathbf{H}(t) \end{pmatrix} = 0 \qquad \text{(constraint)}$$

Usually material weights are $W \neq \overline{W}$ complex!

$\rightsquigarrow$ e. g. gyrotropic media (QHE of Light!)

Immediate Consequences

- Equations must be considered on subspaces **complex** Banach space $L^2(\mathbb{R}^3, \mathbb{C}^6)$
- Even if initial conditions are real, solutions $\big(\mathbf{E}(t)\,,\,\mathbf{H}(t)\big) \neq \big(\overline{\mathbf{E}(t)}\,,\,\overline{\mathbf{H}(t)}\big)$ acquire imaginary part over time!

Commonly Used, But Unphysical Maxwell's Equations

$$W \, \mathrm{i} \frac{\partial}{\partial t} \begin{pmatrix} \mathbf{E}(t) \\ \mathbf{H}(t) \end{pmatrix} = \mathrm{Rot} \begin{pmatrix} \mathbf{E}(t) \\ \mathbf{H}(t) \end{pmatrix} \qquad \text{(dynamical)}$$

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Immediate Consequences

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- Even if initial conditions are real, solutions $\big(\mathbf{E}(t), \mathbf{H}(t)\big) \neq \big(\overline{\mathbf{E}(t)}, \overline{\mathbf{H}(t)}\big)$ acquire imaginary part over time!

Commonly Used, But Unphysical Maxwell's Equations

$$W \, \mathrm{i} \frac{\partial}{\partial t} \begin{pmatrix} \mathbf{E}(t) \\ \mathbf{H}(t) \end{pmatrix} = \mathrm{Rot} \begin{pmatrix} \mathbf{E}(t) \\ \mathbf{H}(t) \end{pmatrix} \qquad \text{(dynamical)}$$

$$\mathrm{Div} \, W \begin{pmatrix} \mathbf{E}(t) \\ \mathbf{H}(t) \end{pmatrix} = 0 \qquad \text{(constraint)}$$

Usually material weights are $W \neq \overline{W}$ complex!
$\leadsto$ e. g. gyrotropic media (QHE of Light!)

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Three Options

1. Take the real part of the complex wave
   $\Longrightarrow$ Breaks conservation of energy!
2. Give up on real-valuedness of electromagnetic fields.
   $\rightsquigarrow$ Inconsistent interpretation (complex Lorentz force!?!)
3. Modify equations of motion. $\rightsquigarrow$ Correct choice!

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Maxwell's Equations for Gyrotropic Media

Real solutions linear combination of complex $\pm\omega$ waves:

$$(\mathbf{E},\mathbf{H}) = \Psi_+ + \Psi_- = 2\mathrm{Re}\,\Psi_\pm$$

Pair of equations
(*derived* from Maxwell's equations for linear, **dispersive** media!)

$$\omega > 0 : \quad \begin{cases} W_+ \, \mathrm{i}\partial_t \Psi_+ = \mathrm{Rot}\,\Psi_+ \\ \mathrm{Div}\, W_+ \, \Psi_+ = 0 \end{cases}$$

$$\omega < 0 : \quad \begin{cases} W_- \, \mathrm{i}\partial_t \Psi_- = \mathrm{Rot}\,\Psi_- \\ \mathrm{Div}\, W_- \, \Psi_- = 0 \end{cases}$$

$$W(t,x) = \overline{W(t,x)} \iff W_-(x) = \overline{W_+(x)}$$

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Physically Meaningful Equations

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- Compatibility with reality-condition baked in!
- Difference between **physical** and **unphysical** equations: **defined on different subspaces** of Banach space $L^2(\mathbb{R}^3, \mathbb{C}^6)$

$$\mathcal{H}_+ = \{\text{complex } \omega > 0 \text{ states}\} \subsetneq \mathcal{H}_{\mathbb{C},\perp} = \ker(\mathrm{Div}\, W_+)$$

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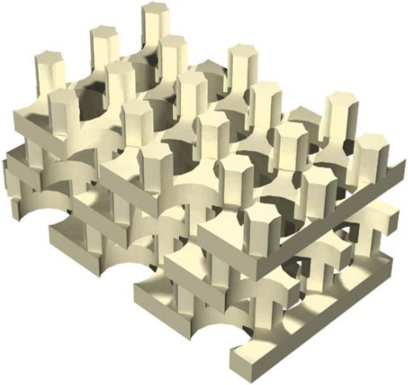
Schrödinger formalism for Maxwell's equations in non-dispersive media

Relevant Electromagnetic Media



Assumption (Material weights)

$$W_+(x) = \begin{pmatrix} \varepsilon(x) & \chi(x) \\ \chi(x)^* & \mu(x) \end{pmatrix}$$

1. The medium is **lossless.** $(W_+^* = W_+)$
2. W_+ describes a **positive index medium.** $(0 < c\,\mathbb{1} \leq W_+ \leq C\,\mathbb{1})$

Schrödinger Formalism of Maxwell's Equations

Theorem (De Nittis & L. (2018))

Real transversal states

$$(\mathbf{E}, \mathbf{H}) = 2\mathrm{Re}\,\Psi$$

$$\begin{pmatrix} \varepsilon & \chi \\ \chi^* & \mu \end{pmatrix} \frac{\partial}{\partial t} \begin{pmatrix} \psi^E \\ \psi^H \end{pmatrix} = \begin{pmatrix} +\nabla \times \psi^E \\ -\nabla \times \psi^H \end{pmatrix}$$

$\longleftrightarrow$

Complex states with $\omega > 0$

$$\Psi = P_+ (\mathbf{E}, \mathbf{H})$$

$$M = W^{-1} \,\mathrm{Rot}\, \big|_{\omega > 0} = M^{*w}$$

$$\mathrm{i}\,\partial_t \Psi = M \Psi$$

$$\mathcal{H} = \Big\{ \Psi \in L^2(\mathbb{R}^3, \mathbb{C}^6) \;\Big|\; \Psi \text{ is } \omega > 0 \text{ state} \Big\}$$

$$\langle \Phi, \Psi \rangle_W = \int_{\mathbb{R}^3} \mathrm{d}x \, \Phi(x) \cdot W(x) \Psi(x)$$

Energy scalar product

(All subscripts $_+$ dropped to simplify notation.)

(De Nittis & L., Annals of Physics **396**, pp. 221–260, 2018)

Main Messages of This Talk

1. Rewrite Maxwell's equations in the form of a Schrödinger equation.

   De Nittis & L., Annals of Physics **396**, pp. 221–260, 2018

2. Classify electromagnetic media using the Cartan-Altland-Zirnbauer scheme.

   De Nittis & L., arXiv:1710.08104, 2017; De Nittis & L., arXiv:1806.07783, 2018

3. Adapt existing techniques to prove bulk-boundary correspondences

   … in progress

Symmetries of the *In Vacuo* Maxwell Equations

$$\omega > 0 : \qquad \begin{cases} \mathrm{i}\partial_t \Psi = \mathrm{Rot}\,\Psi \\ \mathrm{Div}\,\Psi = 0 \end{cases}$$

Real Symmetry	**Complex Representative**	**TI Classification**	**Meaning**
$V_1^{\mathbb{R}} = \sigma_1 \otimes \mathbb{1}$	$V_1^{\mathbb{C}} = (\sigma_1 \otimes \mathbb{1})\, C$	+TR	Flips helicity *and* arrow of time
$V_2^{\mathbb{R}} = \mathrm{i}\sigma_2 \otimes \mathbb{1}$	$V_2^{\mathbb{C}} = \mathrm{i}\sigma_2 \otimes \mathbb{1}$	ordinary	Dual symmetry
$V_3^{\mathbb{R}} = \sigma_3 \otimes \mathbb{1}$	$V_3^{\mathbb{C}} = (\sigma_3 \otimes \mathbb{1})\, C$	+TR	Ordinary EM time-reversal

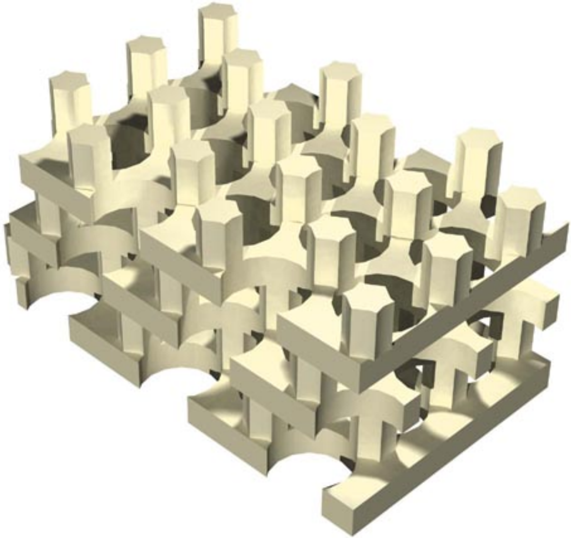
Media Breaking/Preserving Symmetries

Medium has symmetry $V^{\mathbb{C}}$ $\Longleftrightarrow$ $\begin{cases} [\mathrm{Rot}\,, V^{\mathbb{C}}] = 0 \text{ (vac. symm.)} \\ V^{\mathbb{C}} \text{ (anti)unitary on } \mathcal{H} \end{cases}$

Media Breaking/Preserving Symmetries

Medium has symmetry $V^{\mathbb{C}} \iff [W, V^{\mathbb{C}}] = 0$

Photonic Crystals: Periodic Electromagnetic Media



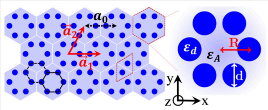
Material vs. Crystallographic Symmetries

Material

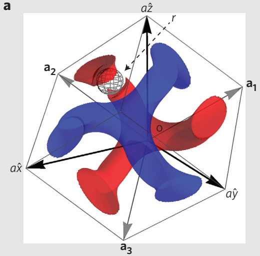
$$W = \begin{pmatrix} \varepsilon & \chi \\ \chi^* & \mu \end{pmatrix}$$

- Properties of and relations between ε, μ and χ
- $V_1^{\mathbb{C}}$, $V_2^{\mathbb{C}}$ and $V_3^{\mathbb{C}}$

Crystallographic



Wu & Hu (2015)



Lu et al (2013)

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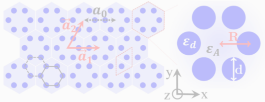

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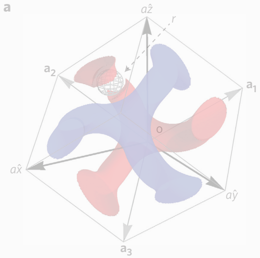

Lu et al (2013)

Topological Classification of EM Media

Assumption

W has no crystallographic symmetries.

Topological Classification of EM Media

Theorem (De Nittis & L. (2017))

Non-gyrotropic

$$W = \left(\begin{smallmatrix} \varepsilon & 0 \\ 0 & \mu \end{smallmatrix}\right) = \left(\begin{smallmatrix} \overline{\varepsilon} & 0 \\ 0 & \overline{\mu} \end{smallmatrix}\right)$$

$$V_3^{\mathbb{C}} = (\sigma_3 \otimes \mathbb{1})\, C$$

Gyrotropic

$$W = \left(\begin{smallmatrix} \varepsilon & 0 \\ 0 & \mu \end{smallmatrix}\right) \neq \left(\begin{smallmatrix} \overline{\varepsilon} & 0 \\ 0 & \overline{\mu} \end{smallmatrix}\right)$$

No symmetries

Dual-symmetric, non-gyrotr.

$$W = \begin{pmatrix} \varepsilon & -\mathrm{i}\chi \\ +\mathrm{i}\chi & \varepsilon \end{pmatrix} = \begin{pmatrix} \overline{\varepsilon} & -\mathrm{i}\,\overline{\chi} \\ +\mathrm{i}\,\overline{\chi} & \overline{\varepsilon} \end{pmatrix}$$

$$V_1^{\mathbb{C}} = (\sigma_1 \otimes \mathbb{1})\, C, \;\; V_3^{\mathbb{C}} = (\sigma_3 \otimes \mathbb{1})\, C$$

Magneto-electric

$$W = \left(\begin{smallmatrix} \varepsilon & \chi \\ \chi & \varepsilon \end{smallmatrix}\right) = \begin{pmatrix} \overline{\varepsilon} & \overline{\chi} \\ \overline{\chi} & \overline{\varepsilon} \end{pmatrix}$$

$$V_1^{\mathbb{C}} = (\sigma_1 \otimes \mathbb{1})\, C$$

(De Nittis & L., arxiv:1710.08104 (2017))

Topological Classification of EM Media

Theorem (De Nittis & L. (2017))

Non-gyrotropic	Gyrotropic
Class AI	Class A (Quantum Hall Class)
Realized, e. g. dielectrics	Realized, e. g. YIG for microwaves

Dual-symmetric, non-gyrotr.	Magneto-electric
Two +TR $\Longrightarrow$ 2 $\times$ Class AI	Class AI
Realized, e. g. vacuum and YIG	Realized, e. g. Tellegen media

4 different topological classes of EM media

(De Nittis & L., arxiv:1710.08104 (2017))

Topological Classification of EM Media

Theorem (De Nittis & L. (2017))

Non-gyrotropic	Gyrotropic
Class AI	Class A **(Quantum Hall Class)**
Realized, e. g. dielectrics	Realized, e. g. YIG for microwaves

Dual-symmetric, non-gyrotr.	Magneto-electric
Two +TR $\Longrightarrow$ $2 \times$ Class AI	Class AI
Realized, e. g. vacuum and YIG	Realized, e. g. Tellegen media

Only one is topologically non-trivial in $d \leq 3$

(De Nittis & L., arxiv:1710.08104 (2017))

Conclusions from Topological Classification

- Some works proposed to use unphysical symmetries (e. g. fermionic time-reversal symmetries $V_{\mathrm{f}} = (\sigma_2 \otimes \mathbb{1})\, C$)
- Class AII cannot occur via material symmetries alone $\rightsquigarrow$ No $\mathbb{Z}_2$-valued Kane-Mele-type topological invariants supported!
- Tight-binding operators cannot have incompatible symmetries!

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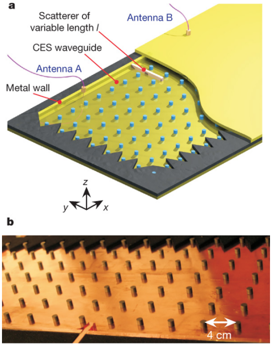
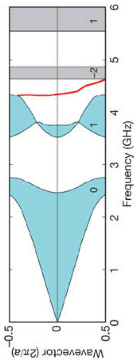
Physical Setting



Joannopoulos, Soljačić et al (2009)

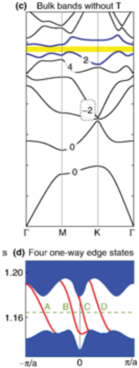
- Quasi-2d photonic crystal
- Topological photonic crystal of class A (i. e. W breaks $V_1^{\mathbb{C}}$ and $V_3^{\mathbb{C}}$)

A Physicist's POV of the Bulk-Edge Correspondence



Joannopoulos, Soljačić et al (2009)

$0 + 1 = 1 \Rightarrow 1$ edge mode



Skirlo et al, PRL 113, 113904, 2014

$0 + 0 - 2 + 4 + 2 = 4 \Rightarrow 4$ edge modes

Works as advertised!

Haldane's Photonic Bulk-Edge Correspondence

Conjecture

$$T_{\mathrm{bulk}} = T_{\mathrm{edge}} = \text{net } \sharp \text{ of edge modes}$$

Tasks

1. Define topological bulk invariant T_{bulk}
2. Define edge system
   ($\rightsquigarrow$ boundary conditions can break +TR symmetries!)
3. Proof of "mathematical" bulk-edge correspondence
4. Identify the **topological observable** $\rightsquigarrow$ Poynting vector?

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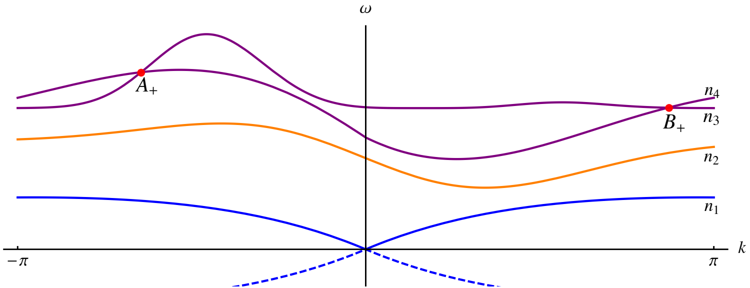
The Frequency Band Picture

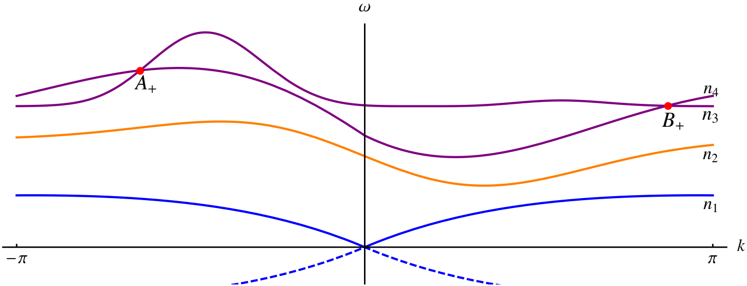


Theorem (De Nittis & L., 2014)

1. *Bloch bands and functions* **locally analytic** *away from crossings*
2. 2 **ground state bands** *with* $\approx$ *linear dispersion at* $k = 0$ *and* $\omega = 0$
3. $P_{\mathrm{gs}}(k)$ *discontinuous at* $k = 0$ *(jump in dimensionality!)*

(Theorem 1.4 and Lemma 3.7 in De Nittis & L., Documenta Math. **19**, pp. 63–101, 2014)

The Bloch ~~Vector~~ Bundle



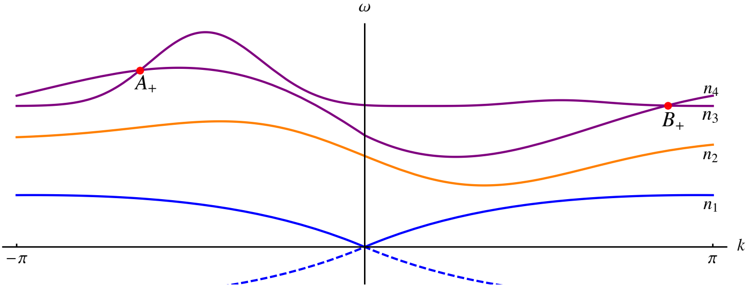
Proceed as Usual

1. Select bulk frequency band gap.
2. Define the "Fermi projection" $P(k) := \sum_{j=1}^{n} |\varphi_j(k)\rangle\langle\varphi_j(k)|$.
3. Define the **Bloch bundle**

$$\mathcal{E}_{\mathbb{T}^*}(P) : \bigsqcup_{k \in \mathbb{T}^*} \operatorname{ran} P(k) \xrightarrow{\pi} \mathbb{T}^*$$

In Bloch-Floquet representation.

The Bloch ~~Vector~~ Bundle



Proposition

1. $\mathcal{E}_{\mathbb{T}^*}(P)$ is only a bundle over the Brillouin torus $\mathbb{T}^*$, but **not** a *vector* bundle.
2. The restriction $\mathcal{E}_{\mathbb{T}^* \setminus \{0\}}(P) := \mathcal{E}_{\mathbb{T}^*}(P)\big|_{\mathbb{T}^* \setminus \{0\}}$ **is** a vector bundle.

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Origin of the Problem

- $k \mapsto P_{\mathrm{gs}}(k)$ is **not continuous** at $k = 0$.
- $k \mapsto P(k) - P_{\mathrm{gs}}(k)$ *is* analytic at $k = 0$.
  (The ground state bands are responsible for the bad behavior.)

$\Longrightarrow k \mapsto P(k)$ is continuous (in fact, analytic) **only on $\mathbb{T}^* \setminus \{0\}$**.

Classification of the Bloch Bundle

Idea 1

Classify the bundle over the entire Brillouin torus $\mathbb{T}^*$

$$\mathcal{E}_{\mathbb{T}^*}(P) : \bigsqcup_{k \in \mathbb{T}^*} \mathrm{ran}\, P(k) \xrightarrow{\pi} \mathbb{T}^*$$

- Only a bundle, **not** a vector bundle!
- Classification theory not well-developed

What is the topological invariant here?!?

Classification of the Bloch Bundle

Idea 2

Classify the bundle over $\mathbb{T}^* \setminus \{0\}$

$$\mathcal{E}_{\mathbb{T}^* \setminus \{0\}}(P) : \bigsqcup_{k \in \mathbb{T}^* \setminus \{0\}} \mathrm{ran}\, P(k) \xrightarrow{\pi} \mathbb{T}^* \setminus \{0\}$$

- *Bona fide* **vector** bundle
- $\mathbb{T}^* \setminus \{0\}$ deformation retracts to $\mathbb{S}^1 \Longrightarrow$

$$\mathrm{Vec}_k(\mathbb{T}^* \setminus \{0\}) \cong \mathrm{Vec}_k(\mathbb{S}^1) = 0$$

Chern number $\mathrm{Ch}_1\big(\mathcal{E}_{\mathbb{T}^* \setminus \{0\}}(P)\big) = 0$ *well-defined, but always* 0*!*

Classification of the Bloch Bundle

Idea 3

Extend the vector bundle over $\mathbb{T}^* \setminus B_\varepsilon$

$$\mathcal{E}_{\mathbb{T}^* \setminus B_\varepsilon}(P) : \bigsqcup_{k \in \mathbb{T}^* \setminus B_\varepsilon} \operatorname{ran} P(k) \xrightarrow{\pi} \mathbb{T}^* \setminus B_\varepsilon$$

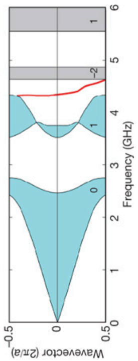
to a vector bundle over $\mathbb{T}^*$

- *Relative* cohomology theory? Might work, but we need to construct a “natural” extension
- It seems that the **Chern charge**

$$\int_{|k|=\varepsilon} \mathrm{d}k \, \mathrm{Tr}\big(\mathcal{A}_{\mathrm{gs}}(k)\big) = 0$$

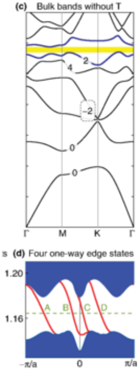
vanishes. $\rightsquigarrow$ Extension by vector bundle surgery possible?

Classification of the Bloch Bundle



Joannopoulos, Soljačić et al (2009)

$0 + 1 = 1 \Rightarrow 1$ edge mode



Skirlo et al, PRL 113, 113904, 2014

$0 + 0 - 2 + 4 + 2 = 4 \Rightarrow 4$ edge modes

Chern charges of ground state bands 0

Classification of the Bloch Bundle

Idea 3

Extend the vector bundle over $\mathbb{T}^* \setminus B_\varepsilon$

$$\mathcal{E}_{\mathbb{T}^* \setminus B_\varepsilon}(P) : \bigsqcup_{k \in \mathbb{T}^* \setminus B_\varepsilon} \operatorname{ran} P(k) \xrightarrow{\pi} \mathbb{T}^* \setminus B_\varepsilon$$

to a vector bundle over $\mathbb{T}^*$

- *Relative* cohomology theory? Might work, but we need to construct a "natural" extension
- It seems that the **Chern charge**

$$\int_{|k|=\varepsilon} \mathrm{d}k \, \mathrm{Tr}\big(\mathcal{A}_{\mathrm{gs}}(k)\big) = 0$$

vanishes. $\rightsquigarrow$ Extension by vector bundle surgery possible?

Thank you for your attention!

Interaction of Time-Reversal & Crystallographic Symmetries

Talk Based On

- De Nittis & L., *On the Role of Symmetries in Photonic Crystals*, Annals of Physics **350**, pp. 568–587, 2014
- De Nittis & L., *The Schrödinger Formalism of Electromagnetism and Other Classical Waves — How to Make Quantum-Wave Analogies Rigorous*, Annals of Physics **396**, pp. 221–260, 2018
- De Nittis & L., *Symmetry Classification of Topological Photonic Crystals*, arXiv 1710.08104, 1–49, 2017
- De Nittis & L., *Equivalence of Electric, Magnetic and Electromagnetic Chern Numbers for Topological Photonic Crystals*, arxiv 1806.07783, pp. 1–33, 2018
- L., *Taking Inspiration from Quantum-Wave Analogies — Recent Results for Photonic Crystals*, Macroscopic Limits of Quantum Systems — Munich, Germany, March 20–April 1, 2017, to appear in Springer Proceedings in Mathematics and Statistics, 2018

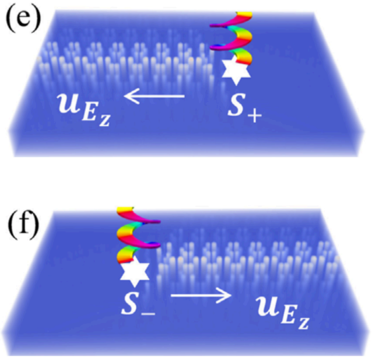
Interaction of Material and Crystallographic Symmetries

- Idea of real vs. complex implementation of symmetries works also for other symmetries (e. g. rotations, parity)
  $\Longrightarrow$ **Crystallographic symmetries can be handled** within the Schrödinger formalism of classical electromagnetism
- Recent works from condensed matter physics on crystallographic TIs (e. g. by Shiozaki, Sato & Gomi, arxiv:1802.06694 (2018))

Example: "Spin-Valley Hall Effect"

Wu & Hu (2015)

- **Edge modes topological**
- Pseudospin degree of freedom in a **time-reversal-symmetric** medium
- Time-reversal symmetry $T_3 \neq T_\uparrow \oplus T_\downarrow$ **not blockdiagonal** $\Longrightarrow M_{\uparrow/\downarrow}$ class A (no symmetry)
- Chern numbers $C_\uparrow = -C_\downarrow \neq 0$ possible
- *Not* in contradiction, edge modes come in $\uparrow / \downarrow$ **pairs**
- **Topologically protected** against perturbations which preserve T_3 symmetry and honeycomb structure



Wu & Hu (2015)

Derivation of Maxwell's Equations in Linear Media

Fundamental Equations

Maxwell's Equations in Media

1. *Maxwell's equations*

$$\mathrm{i}\frac{\partial}{\partial t}\begin{pmatrix}\mathbf{D}\\ \mathbf{B}\end{pmatrix} = \begin{pmatrix}0 & +\mathrm{i}\nabla^{\times}\\ -\mathrm{i}\nabla^{\times} & 0\end{pmatrix}\begin{pmatrix}\mathbf{H}\\ \mathbf{E}\end{pmatrix} - \mathrm{i}\begin{pmatrix}J^D\\ J^B\end{pmatrix} \quad \text{(dynamical)}$$

$$\begin{pmatrix}\nabla\cdot\mathbf{D}\\ \nabla\cdot\mathbf{B}\end{pmatrix} = \begin{pmatrix}\rho^D\\ \rho^B\end{pmatrix} \quad \text{(constraint)}$$

2. *Constitutive relations*

$$\begin{pmatrix}\mathbf{D}\\ \mathbf{B}\end{pmatrix} = W\begin{pmatrix}\mathbf{E}\\ \mathbf{H}\end{pmatrix}$$

3. *Conservation of charge*

$$\nabla\cdot J^{\sharp} + \rho^{\sharp} = 0, \quad \sharp = D, B$$

Fundamental Equations

Maxwell's Equations in Media

1. *Maxwell's equations*

$$\mathrm{i}\frac{\partial}{\partial t}\begin{pmatrix}\mathbf{D}\\ \mathbf{B}\end{pmatrix} = \begin{pmatrix}0 & +\mathrm{i}\nabla^{\times}\\ -\mathrm{i}\nabla^{\times} & 0\end{pmatrix}\begin{pmatrix}\mathbf{H}\\ \mathbf{E}\end{pmatrix} \qquad \text{(dynamical)}$$

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$$\begin{pmatrix}\mathbf{D}\\ \mathbf{B}\end{pmatrix} = W\begin{pmatrix}\mathbf{E}\\ \mathbf{H}\end{pmatrix}$$

3. *Conservation of charge* ⇝ neglect sources for simplicity

$$\nabla\cdot J^{\sharp} + \rho^{\sharp} = 0, \qquad \sharp = D, B$$

Constitutive Relations for a Linear, Dispersive Medium

For a **linear** medium the constitutive relations maps a **trajectory**

$$(-\infty, t] \ni s \mapsto \bigl(\mathbf{E}(s), \mathbf{H}(s)\bigr)$$

onto

$$\begin{pmatrix} \mathbf{D}(t,x) \\ \mathbf{B}(t,x) \end{pmatrix} := \int_{-\infty}^{t} \mathrm{d}s \, W(t-s,x) \, \begin{pmatrix} \mathbf{E}(s,x) \\ \mathbf{H}(s,x) \end{pmatrix}$$

$\rightsquigarrow$ reaction of medium to impinging em wave depends on the **past**

Constitutive Relations for a Linear, Dispersive Medium

$$\big(\mathbf{D}(t), \mathbf{B}(t)\big) := \int_{-\infty}^{t} \mathrm{d}s \, W(t-s) \, \big(\mathbf{E}(s), \mathbf{H}(s)\big)$$

Assumption (Constitutive relations)

We assume that $W(t,x) = \begin{pmatrix} \varepsilon(t,x) & \chi^{EH}(t,x) \\ \chi^{HE}(t,x) & \mu(t,x) \end{pmatrix} \in \mathrm{Mat}_{\mathbb{C}}(6)$

1. *is* ***real****,* $W = \overline{W}$*, and*
2. *satisfies the* **causality condition** $W(t) = 0$ *for all* $t > 0$*.*

Constitutive Relations for a Linear, Dispersive Medium

$$\big(\mathbf{D}(t), \mathbf{B}(t)\big) = \big(W * (\mathbf{E}, \mathbf{H})\big)(t)$$

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We assume that $W(t,x) = \begin{pmatrix} \varepsilon(t,x) & \chi^{EH}(t,x) \\ \chi^{HE}(t,x) & \mu(t,x) \end{pmatrix} \in \mathrm{Mat}_{\mathbb{C}}(6)$

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Fundamental Equations for Linear, Dispersive Media

1. *Maxwell's equations*

$$\mathrm{i}\tfrac{\partial}{\partial t}\big(W * (\mathbf{E},\mathbf{H})\big)(t) = \mathrm{Rot}\big(\mathbf{E}(t),\mathbf{H}(t)\big) \qquad \text{(dynamical)}$$
$$\mathrm{Div}\big(W * (\mathbf{E},\mathbf{H})\big)(t) = 0 \qquad \text{(constraint)}$$

2. *Constitutive relations*

$$\big(\mathbf{D}(t),\mathbf{B}(t)\big) = \big(W * (\mathbf{E},\mathbf{H})\big)(t)$$

3. *Conservation of charge* $\rightsquigarrow$ neglect sources for simplicity

$$\nabla \cdot J^{\sharp} + \rho^{\sharp} = 0, \quad \sharp = D, B$$

Heuristically Neglecting Dispersion in Maxwell's Equations

$$\mathrm{i}\,\tfrac{\partial}{\partial t} W * \Psi(t) = \mathrm{Rot}\,\Psi(t)$$

$$\Big\downarrow \mathcal{F}^{-1}$$

$$\omega\,\widehat{W}(\omega)\,\widehat{\Psi}(\omega) = \mathrm{Rot}\,\widehat{\Psi}(\omega)$$

$$\Big\downarrow \approx$$

$$\pm\,\omega > 0 : \;\omega\,\widehat{W}(\pm\omega_0)\,\widehat{\Psi}(\omega) = \mathrm{Rot}\,\widehat{\Psi}(\omega)$$

$$\Big\downarrow \mathcal{F}$$

$$\widehat{W}(\pm\omega_0)\,\mathrm{i}\,\tfrac{\partial}{\partial t}\Psi_{\pm}(t) = \mathrm{Rot}\,\Psi_{\pm}(t)$$

1. Apply **inverse Fourier transform** in time to go from time-dependent to frequency-dependent equations.
2. Approximate material weights $\widehat{W}(\pm\omega) \approx \widehat{W}(\pm\omega_0) = W_{\pm}$ for frequencies $\pm\omega \approx \pm\omega_0$.

   $+\omega_0$ *and* $-\omega_0$ contributions necessary to reconstruct **real** solutions.
3. **Undo Fourier transform** to obtain dynamical equations in the **absence of dispersion**.

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Heuristically Neglecting Dispersion in Maxwell's Equations

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