

The Localization-Topology Correspondence
in gapped (periodic) quantum systems:
ordinary, Chern, and $\mathbb{Z}_2$ insulators

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based on joint work with D. Fiorenza, G. Marcelli,
D. Monaco, M. Moscolari, A. Pisante, S. Teufel

RECENT PROGRESS IN THE MATHEMATICS
OF TOPOLOGICAL INSULATORS

ETH Zürich, September 3, 2018

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- ◇ Intro: from the **Transport-Topology** Correspondence [TKNN] to the **Localization-Topology** Correspondence

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- III. TRS ordinary vs $\mathbb{Z}_2$ topological insulators: some results for **periodic** case
based on joint work with D. Fiorenza and D. Monaco
(Commun. Math. Phys. **343** (2016)) and further developments

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periodic* Hamiltonian $\xrightarrow{BF}$ Fermi projector $\{P(k)\}_{k \in \mathbb{T}^d} \longrightarrow$ Bloch bundle $\mathcal{E}_P$

$$\underbrace{\sigma_{xy}^{(\text{Kubo})}}_{\text{Transport}} = -\frac{i}{(2\pi)^2} \int_{\mathbb{T}^2} \text{Tr} \left(P(k) [\partial_{k_1} P(k), \partial_{k_2} P(k)] \right) dk = \frac{1}{2\pi} \underbrace{C_1(\mathcal{E}_P)}_{\text{Topology}}$$

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See also [AW] for the role of the **Linear Response Ansatz**.

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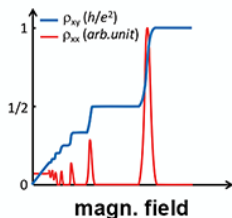
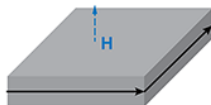
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- ▶ Recent generalizations to interacting electrons [GMP; BRF, MT].
- ▶ Other complementary explanations of the QHE are possible, but not covered here [Laughlin, Fröhlich, Halperin]. Some can be adapted to topological insulators.

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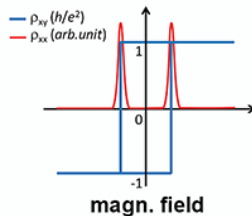
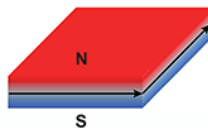
The predecessor: Transport-Topology Correspondence

Chern insulators and the Quantum Anomalous Hall effect

QHE



QAHE



*Left panel: transverse and **direct** resistivity in a Quantum Hall experiment*
*Right panel: transverse and **direct** resistivity in a Chern insulator (hysteresis cycle)*

Picture: © A. J. Bestwick (2015)

High-precision realization of robust quantum anomalous Hall state in a hard ferromagnetic topological insulator

Cui-Zu Chang^{1*}, Weiwei Zhao^{2*}, Duk Y. Kim², Haijun Zhang³, Badi H. Assaf⁴, Don Heiman⁴, Shou-Cheng Zhang³, Chaoxing Liu², Moses H. W. Chan² and Jagadeesh S. Moodera^{1,5*}

 Selected for a **Viewpoint** in *Physics*
PHYSICAL REVIEW LETTERS

PRL 114, 187201 (2015)

week ending
8 MAY 2015



Precise Quantization of the Anomalous Hall Effect near Zero Magnetic Field

A. J. Bestwick,^{1,2} E. J. Fox,^{1,2} Xufeng Kou,³ Lei Pan,³ Kang L. Wang,³ and D. Goldhaber-Gordon^{1,2,*}

¹*Department of Physics, Stanford University, Stanford, California 94305, USA*

²*Stanford Institute for Materials and Energy Sciences, SLAC National Accelerator Laboratory,
2575 Sand Hill Road, Menlo Park, California 94025, USA*

³*Department of Electrical Engineering, University of California, Los Angeles, California 90095, USA*
(Received 16 January 2015; revised manuscript received 16 March 2015; published 4 May 2015)

We report a nearly ideal quantum anomalous Hall effect in a three-dimensional topological insulator thin film with ferromagnetic doping. Near zero applied magnetic field we measure exact quantization in the Hall resistance to within a part per 10 000 and a longitudinal resistivity under 1 Ω per square, with chiral edge transport explicitly confirmed by nonlocal measurements. Deviations from this behavior are found to be caused by thermally activated carriers, as indicated by an Arrhenius law temperature dependence. Using the deviations as a thermometer, we demonstrate an unexpected magnetocaloric effect and use it to reach near-perfect quantization by cooling the sample below the dilution refrigerator base temperature in a process approximating adiabatic demagnetization refrigeration.

Part I

The Localization-Topology Correspondence: the periodic case

Joint work with D. Monaco, A. Pisante and S. Teufel

How to measure localization of extended states?

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What is the relevant notion of **localization** for **periodic** quantum systems?

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- ▶ Our question: Relation between **dissipationless transport**, **topology of the Bloch bundle** and **localization** of electronic states.
- ▶ **Spectral type??** For ergodic random Schrödinger operators localization is measured by the spectral type $(\sigma_{\text{pp}}, \sigma_{\text{ac}}, \sigma_{\text{sc}})$ [AW]. However, for **periodic** systems the spectrum is generically **purely absolutely continuous**.

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- ▶ **Kernel of the Fermi projector??** For **gapped** periodic 1-body Hamiltonians one has

$$|P_\mu(x, y)| \simeq e^{-\lambda_{\text{gap}}|x-y|}$$

as a consequence of Combes-Thomas theory [AS₂, NN].

-
- [AW] M. AIZENMAN, S. WARZEL: *Random operators*, AMS (2015).
[AS₂] J. AVRON, R. SEILER, B. SIMON: *Commun. Math. Phys.* **159** (1994).
[NN] A. NENCIU, G. NENCIU: *Phys. Rev B* **47** (1993).

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Notice that crucially

$$|P_\mu(x, y)| \simeq e^{-\lambda_{\text{gap}}|x-y|} \not\Rightarrow \begin{cases} \text{exist CWFs such that} \\ |w_{a,\gamma}(x)| \simeq e^{-c|x-\gamma|} \end{cases}$$

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(GAP CONDITION)

$\not\Rightarrow$

(TOPOLOGICAL TRIVIALITY)

Theorem [MPPT]: The Localization Dichotomy for periodic systems

Under assumptions specified later, the following holds:

- ▶ **either** there exists $\alpha > 0$ and a choice of CWFs $\tilde{w} = (\tilde{w}_1, \dots, \tilde{w}_m)$ satisfying

$$\sum_{a=1}^m \int_{\mathbb{R}^d} e^{2\beta|x|} |\tilde{w}_a(x)|^2 dx < +\infty \quad \text{for every } \beta \in [0, \alpha);$$

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Intermediate regimes are
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The result is largely
model-independent: it
holds for tight-binding as
well as continuum models

Setting: Magnetic periodic Schrödinger operators

Chern insulators: For $d \in \{2, 3\}$, consider the magnetic Schrödinger operator

$$H_\Gamma = \frac{1}{2} (-i\nabla_x - A_\Gamma(x))^2 + V_\Gamma(x) \quad \text{acting in } L^2(\mathbb{R}^d)$$

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Hence the **modified Bloch-Floquet transform**

$$(\mathcal{U}\psi)(k, y) := \sum_{\gamma \in \Gamma} e^{-ik \cdot (y - \gamma)} (T_\gamma \psi)(y), \quad y \in \mathbb{R}^d, k \in \mathbb{R}^d$$

provides a simultaneous decomposition of H_Γ and T_γ :

$$\mathcal{U} H_\Gamma \mathcal{U}^{-1} = \int_{\mathbb{B}}^{\oplus} dk H(k) \quad \text{with} \quad H(k) = \frac{1}{2} (-i\nabla_y - A_\Gamma(y) + k)^2 + V_\Gamma(y).$$

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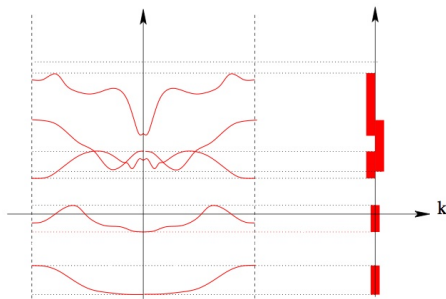
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The operator $H(k)$ acts in $L^2_{\text{per}}(\mathbb{R}^d) := \{\psi \in L^2_{\text{loc}}(\mathbb{R}^d) : T_\gamma \psi = \psi \text{ for all } \gamma \in \Gamma\}$.

The Bloch bundle and its topology

For each fixed $k \in \mathbb{R}^d$, the operator $H(k)$ has **compact resolvent**, so pure point spectrum accumulating at $+\infty$.



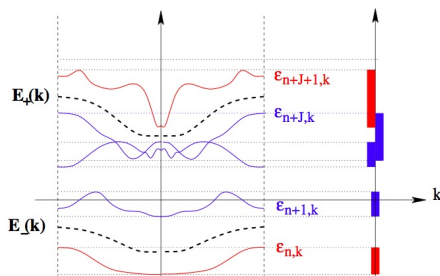
Question: how can I read from the picture whether **TR-symmetry is broken**?

Spectrum of $H(k)$ as a function of k_j (left).

Spectrum of $H_\Gamma = \int^\oplus H(k) dk$ (right).

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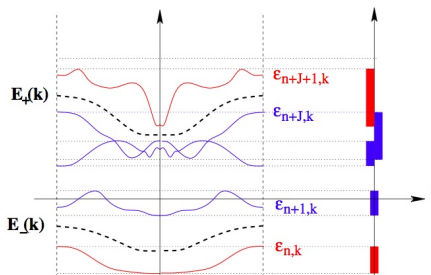


An isolated family of J Bloch bands.

Notice that the spectral bands may overlap.

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Topology is encoded in the orthogonal projections on an **isolated family of bands**

$$\begin{aligned} P_*(k) &= \frac{i}{2\pi} \oint_{\mathcal{C}_*(k)} (H(k) - z1)^{-1} dz \\ &= \sum_{n \in \mathcal{I}_*} |u_n(k, \cdot)\rangle \langle u_n(k, \cdot)|. \end{aligned}$$

where $\mathcal{C}_*(k)$ intersects the real line in $E_-(k)$ and $E_+(k)$ (**GAP CONDITION**).

A model for Quantum Hall insulators

Quantum Hall insulators: For $d \in \{2, 3\}$, consider $A_b(x) = \frac{b}{2c} x \wedge \hat{e}$ with $\hat{e}$ a unit vector

$$H_{\Gamma,b} = \frac{1}{2} (-i\nabla_x - A_b(x))^2 + V_{\Gamma}(x) \quad \text{acting in } L^2(\mathbb{R}^d)$$

where V_{Γ} is periodic with respect to $\Gamma = \text{Span}_{\mathbb{Z}} \{a_1, \dots, a_d\}$.

Assume moreover the **commensurability condition** for the magnetic field $B = b\hat{e}$:

$$\frac{1}{c} B \cdot (\gamma \wedge \gamma') \in 2\pi\mathbb{Q} \quad \text{for all } \gamma, \gamma' \in \Gamma$$

Ordinary translations are replaced by the **magnetic translations** [Zak64]

$$(T_{\gamma}^b \psi)(x) := e^{i\gamma \cdot A_b(x)} \psi(x - \gamma) \quad \gamma \in \Gamma.$$

These commute with $H_{\Gamma,b}$, but satisfy the **pseudo-Weyl relations**

$$T_{\gamma}^b T_{\gamma'}^b = e^{\frac{i}{c} B \cdot (\gamma \wedge \gamma')} T_{\gamma'}^b T_{\gamma}^b \quad \gamma, \gamma' \in \Gamma.$$

The **$\mathbb{Z}^d$ -symmetry** is recovered at the price of considering a smaller lattice $\Gamma_b \subset \Gamma$.

Assumptions

Consider $A = A_\Gamma + A_b$ (**periodic + linear**) with A_b satisfying the commensurability condition, and set

$$H(\kappa) = \frac{1}{2}(-i\nabla_y - A(y) + \kappa)^2 + V_\Gamma(y) \quad \text{acting in } \mathcal{H}_f^b, \quad \kappa \in \mathbb{C}^d,$$

where $\mathcal{H}_f^b := \{\psi \in L_{\text{loc}}^2(\mathbb{R}^d) : T_\gamma^b \psi = \psi, \text{ for all } \gamma \in \Gamma_b\}$.

Assumption 1: The magnetic potential $A = A_\Gamma + A_b$ and the scalar potential V_Γ are such that the family of operators $H(\kappa)$ is an **entire analytic family in the sense of Kato with compact resolvent**.

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If $A = A_\Gamma$ is Γ -periodic, with fundamental unit cell Y , then it is sufficient to assume either:

- $A \in L^\infty(Y; \mathbb{R}^2)$ when $d = 2$ or $A \in L^4(Y; \mathbb{R}^3)$ when $d = 3$, and $\text{div } A, V_\Gamma \in L^2_{\text{loc}}(\mathbb{R}^d)$ when $2 \leq d \leq 3$;
- $A \in L^r(Y; \mathbb{R}^2)$ with $r > 2$ and $V_\Gamma \in L^p(Y)$ with $p > 1$ when $d = 2$, or $A \in L^3(Y; \mathbb{R}^3)$ and $V_\Gamma \in L^{3/2}(Y)$ when $d = 3$ (compare [BS]).

[BS] BIRMAN, SUSLINA: Algebra i Analiz **11** (1999). St. Petersburg Math. J. **11**, (2000).

Lemma: Let $P_*(k)$ be the spectral projector of $H(k)$ corresponding to a set $\sigma_*(k) \subset \mathbb{R}$ such that the **gap condition** is satisfied. Then the family $\{P_*(k)\}_{k \in \mathbb{R}^d}$ has the following properties:

(P₁) the map $k \mapsto P_*(k)$ is **analytic** from $\mathbb{R}^d$ to $\mathcal{B}(\mathcal{H}_f^b)$;

(P₂) the map $k \mapsto P_*(k)$ is **τ -covariant**, i. e.

$$P_*(k + \lambda) = \tau(\lambda) P_*(k) \tau(\lambda)^{-1} \quad \forall k \in \mathbb{R}^d, \quad \forall \lambda \in \Gamma_b^*.$$

where $\tau : \Gamma_b^* \simeq \mathbb{Z}^d \longrightarrow \mathcal{U}(\mathcal{H}_f^b)$ is a unitary representation.

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For the sake of simpler slides, here we consider the case $\tau \equiv \mathbb{1}$.

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Concretely, for the operator H_Γ

$$\tau(\lambda)f(y) = e^{i\lambda \cdot y} f(y)$$

for $f \in L^2_{\text{per}}(\mathbb{R}^d, dy) = \mathcal{H}_f^0$.

For the sake of simpler slides, here we consider the case $\tau \equiv \mathbb{1}$.

Lemma: Let $P_*(k)$ be the spectral projector of $H(k)$ corresponding to a set $\sigma_*(k) \subset \mathbb{R}$ such that the **gap condition** is satisfied. Then the family $\{P_*(k)\}_{k \in \mathbb{R}^d}$ has the following properties:

(P₁) the map $k \mapsto P_*(k)$ is **analytic** from $\mathbb{R}^d$ to $\mathcal{B}(\mathcal{H}_f^b)$;

(P₂) the map $k \mapsto P_*(k)$ is **τ -covariant**, i. e.

$$P_*(k + \lambda) = \tau(\lambda) P_*(k) \tau(\lambda)^{-1} \quad \forall k \in \mathbb{R}^d, \quad \forall \lambda \in \Gamma_b^*.$$

where $\tau : \Gamma_b^* \simeq \mathbb{Z}^d \longrightarrow \mathcal{U}(\mathcal{H}_f^b)$ is a unitary representation.

In view of (P₁) and (P₂) the ranges of $P_*(k)$ define a (smooth) Hermitian vector bundle over $\mathbb{T}_*^d := \mathbb{R}^d / \Gamma_b$. For $d = 2$, it is characterized by the **Chern number**

$$c_1(P) := \frac{1}{2\pi i} \int_{\mathbb{T}_*^2} \text{Tr}_{\mathcal{H}_f^b} \left(P(k) [\partial_{k_1} P(k), \partial_{k_2} P(k)] \right) dk_1 \wedge dk_2$$

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Wannier functions and their localization

IDEA: Wannier functions provide a reasonable compromise between **localization in energy** and **localization in position space**, as far as compatible with the uncertainty principle.

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They are associated at the energy window corresponding to $P_*(\cdot)$ via

Definition: A **Bloch frame** is a collection $\{\phi_1, \dots, \phi_m\} \subset L^2(\mathbb{T}_*^d, \mathcal{H}_f^b)$ such that:

$(\phi_1(k), \dots, \phi_m(k))$ is an orthonormal basis of $\text{Ran } P_*(k)$ for a.e. $k \in \mathbb{R}^d$

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In general, a Bloch frame mixes different Bloch bands

$$\phi_a(k) = \sum_{n \in \mathcal{I}_*} \underbrace{u_n(k)}_{\text{Bloch funct.}} U_{na}(k) \quad \text{for some unitary matrix } U(k).$$

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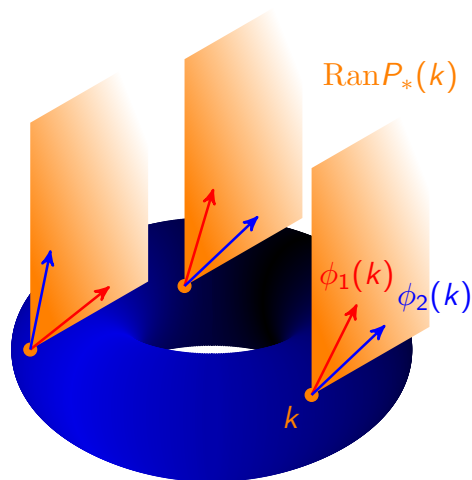
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The ambiguity in the choice is dubbed **Bloch gauge ambiguity**.

unitary matrix $U(k)$.

The Bloch bundle



Competition between **regularity** and **periodicity** is encoded by the **Bloch bundle**

$$\mathcal{E} = (E \rightarrow \mathbb{T}_*^d).$$

Existence of continuous Bloch frames is **topologically obstructed** if $c_1(P_*) \neq 0$.

Definition (CWFs): The composite Wannier functions $\{w_1, \dots, w_m\} \subset L^2(\mathbb{R}^d)$ associated to a Bloch frame $\{\phi_1, \dots, \phi_m\}$ are defined as

$$w_a(x) := (\mathcal{U}^{-1} \phi_a)(x) = \frac{1}{|\mathbb{B}_b|} \int_{\mathbb{B}_b} dk e^{ik \cdot x} \phi_a(k, x).$$

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$$\int_{\mathbb{R}^d} \langle x \rangle^{2s} |w_{a,\gamma}(x)|^2 dx \quad \longleftrightarrow \quad \|\phi_a\|_{H^s(\mathbb{T}_*^d, \mathcal{H}_f^b)}^2$$

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Hence, by Sobolev theorem, for $s > d/2$ the existence of s -localized CWFs is topologically obstructed.

But, for $2 \leq d \leq 3$, still there might exist 1-localized CWFs, i. e. $\langle X^2 \rangle_w < +\infty$.

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$$\int_{\mathbb{R}^d} \langle x \rangle^{2s} |w_{a,\gamma}(x)|^2 dx \iff \|\phi_a\|_{H^s(\mathbb{T}_*^d, \mathcal{H}_f^b)}^2$$

Hence, by Sobolev theorem

NO! In the topologically non-trivial case, does NOT exist any system of 1-localized composite Wannier functions!

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The constructive theorem

Theorem 1 [Monaco, GP, Pisante, Teufel '17]

Assume $d \leq 3$. Consider a magnetic periodic Schrödinger operator in $L^2(\mathbb{R}^d)$ satisfying the previous assumptions (**Kato smallness** + **commensurability** + **gap**).

Then we construct a Bloch frame in $H^s(\mathbb{T}^d; \mathcal{H}^m)$ for every $s < 1$ and, correspondingly, a system of CWFs $\{w_{a,\gamma}\}$ such that

$$\sum_{a=1}^m \int_{\mathbb{R}^d} \langle x \rangle^{2s} |w_{a,\gamma}(x)|^2 dx < +\infty \quad \forall \gamma \in \Gamma, \forall s < 1.$$

The Localization-Topology Correspondence

Theorem 2 [Monaco, GP, Pisante, Teufel '17]

Assume $d \leq 3$. Consider a magnetic periodic Schrödinger operator in $L^2(\mathbb{R}^d)$ satisfying the previous assumptions (Kato smallness + commensurability + gap).

The following statements are equivalent:

- **Finite second moment**: there exist CWFs $\{w_{a,\gamma}\}$ such that

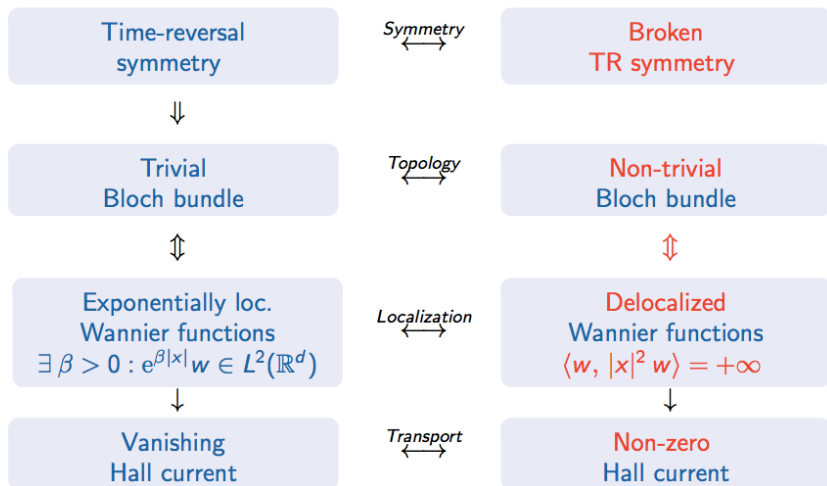
$$\sum_{a=1}^m \int_{\mathbb{R}^d} \langle x \rangle^2 |w_{a,\gamma}(x)|^2 dx < +\infty \quad \forall \gamma \in \Gamma;$$

- **Exponential localization**: there exist CWFs $\{w_{a,\gamma}\}$ and $\alpha > 0$ such that

$$\sum_{a=1}^m \int_{\mathbb{R}^d} e^{2\beta|x|} |w_{a,\gamma}(x)|^2 dx < +\infty \quad \forall \gamma \in \Gamma, \beta \in [0, \alpha);$$

- **Trivial topology**: the family $\{P_*(k)\}_k$ corresponds to a **trivial Bloch bundle**.

Synopsis: symmetry, localization, transport, topology



References

- First column: existence of exponentially localized CWFs

[Ko]	KOHN, W., <i>Phys. Rev.</i> 115 (1959)	($m = 1, d = 1$, even)
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- Second column: **power-law decay** of CWFs in the **topological phase**

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Our result applies to all $d \leq 3$ and is largely **model-independent**, since it covers both continuous and tight-binding models.

Grassmanian reinterpretation

$$G_m(\mathcal{H}) := \{P \in \mathcal{B}(\mathcal{H}) : P^2 = P = P^*, \text{Tr } P = m\} \quad (\text{Grassmann manifold})$$

$$W_m(\mathcal{H}) := \{J : \mathbb{C}^m \rightarrow \mathcal{H} \text{ linear isometry}\} \quad (\text{Stiefel manifold}),$$

Notice that $J = \sum_{a=1}^m |\psi_a\rangle \langle e_a|$, where $\{\psi_a\} \subset \mathcal{H}$ is an m -frame. There is a natural map $\pi : W_m(\mathcal{H}) \rightarrow G_m(\mathcal{H})$ sending each m -frame $\Psi = \{\psi_1, \dots, \psi_m\}$ into the orthogonal projection on its linear span, namely $\pi : J \mapsto JJ^* = \sum_{a=1}^m |\psi_a\rangle \langle \psi_a|$.

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At least formally, $W_m(\mathcal{H})$ is a principal bundle over $G_m(\mathcal{H})$ with projection π and fiber $\mathcal{U}(\mathbb{C}^m)$. The data P and Φ correspond to a commutative diagram:

$$\begin{array}{ccc} & W_m(\mathcal{H}) & \\ \Phi \nearrow & \downarrow \pi & \\ \mathbb{T}^d & \xrightarrow{P} & G_m(\mathcal{H}) \end{array}$$

where $P \in C^\infty(\mathbb{T}^d; \mathcal{B}(\mathcal{H}))$ and $\Phi \in H^s(\mathbb{T}^d; \mathcal{H}^m)$.

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Since the set of m -frames is **not a linear space**, the approximation of a given Sobolev map by smooth maps might be **topologically obstructed**.

where $P \in C^\infty(\mathbb{T}^d; \mathcal{B}(\mathcal{H}))$ and $\Phi \in H^s(\mathbb{T}^d; \mathcal{H}^m)$.

Approximation of a given Sobolev map

Theorem [Hang & Lin]

Let $2 \leq d \leq 3$. Consider a compact, boundaryless, smooth submanifold $M \subset \mathbb{R}^{\nu}$.

If $d = 3$, assume moreover that the homotopy group $\pi_2(M)$ is trivial.

Then, every Sobolev map $\Psi \in H^1(\mathbb{T}^d, M)$ can be approximated by a sequence $\{\Psi^{(\ell)}\}_{\ell \in \mathbb{N}} \subset C^\infty(\mathbb{T}^d, M)$ such that $\Psi^{(\ell)} \xrightarrow{H^1} \Psi$ as $\ell \rightarrow \infty$.

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- One has to reduce **covariance** to **periodicity**, while staying in a **k -independent fiber Hilber space [CHN]**

[HL] HANG, F.; LIN, F.H. : Topology of Sobolev mappings. II. *Acta Math.* 2003.

[CHN] CORNEAN, H.; HERBST, I., NENCIU, G. : *Ann. H. Poincaré*. 2016.

Approximation of a given Sobolev map

Theorem [Hang & Lin]

Let $2 \leq d \leq 3$. Consider a compact, boundaryless, smooth submanifold $M \subset \mathbb{R}^\nu$. If $d = 3$, assume moreover that **the homotopy group $\pi_2(M)$ is trivial**.

Then, every Sobolev map $\Psi \in H^1(\mathbb{T}^d, M)$ can be approximated by a sequence $\{\Psi^{(\ell)}\}_{\ell \in \mathbb{N}} \subset C^\infty(\mathbb{T}^d, M)$ such that $\Psi^{(\ell)} \xrightarrow{H^1} \Psi$ as $\ell \rightarrow \infty$.

- The result can be applied to the **(finite-dimensional)** Stiefel manifold $W_m(\mathbb{C}^n)$, using that $\pi_2(W_m(\mathbb{C}^n)) = 0$ whenever $n \geq m + 2$ (here $\mathbb{C}^n$ is regarded as a Galerkin truncation of $\mathcal{H}$).
- One has to reduce to a finite-dimensional space $V_n \simeq \mathbb{C}^n$
- One has to reduce **covariance** to **periodicity**, while staying in a **k -independent fiber Hilber space [CHN]**
- In our specific case, there is no obstruction to **construct** a Bloch frame with regularity $H^s(\mathbb{T}^d; \mathcal{H}^m)$ for every $s < 1$, essentially via parallel transport [construction in the paper].

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Part II

The Localization-Topology Correspondence: the non-periodic case

Work in progress with G. Marcelli and M. Moscolari

A modified paradigm

- ▶ Recall the **periodic** TKNN paradigm:

$$\underbrace{\sigma_{xy}^{(\text{Kubo})}}_{\text{Transport}} = -\frac{i}{(2\pi)^2} \int_{\mathbb{T}^2} \text{Tr} \left(P(k) \left[\partial_{k_1} P(k), \partial_{k_2} P(k) \right] \right) dk = \frac{1}{2\pi} \underbrace{C_1(\mathcal{E}_P)}_{\text{Topology}}$$

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- ▶ Inspired by [AS₂], we can write

$$\sigma_{xy}^{(\text{Kubo})} = \mathcal{T} \left(i P \left[[X_1, P], [X_2, P] \right] \right)$$

where the **trace per unit volume** is $\mathcal{T}(A) = \lim_{\Lambda_n \nearrow \mathbb{R}^d} |\Lambda_n|^{-1} \text{Tr}(\chi_{\Lambda_n} A \chi_{\Lambda_n})$.

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- Analogy with the **NCG approach to QHE** [Co, Be, BES, Ke, ...]

$$C_1(p) \simeq \tau \left(ip \left[\partial_1(p), \partial_2(p) \right] \right)$$

for p a projector in the **rotation C^* -algebra** $\simeq$ **NC torus**. Here $C_1(p) \in \mathbb{Z}$.

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Questions for experts in NCG:

Is $\mathcal{T}(\cdot)$ a **tracial state** over a C^* -subalgebra of $\mathcal{B}(L^2(\mathbb{R}^d))$?
Which one? NCG?

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Question we addressed:

Is there any relation between $C_1(P) = 0$ and the existence of a **well-localized** **GWB**??

Definition (Generalized Wannier basis) (compare with [NN93])

An orthogonal projector P acting in $L^2(\mathbb{R}^2)$ admits a **G-localized generalized Wannier basis (GWB)** if there exist:

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- (i) a **Delone set** $\Gamma \subseteq \mathbb{R}^2$, i. e. a discrete set such that $\exists 0 < r < R < \infty$ s.t.
 - (a) $\forall x \in \mathbb{R}^2$ there is at most one element of Γ in the ball of radius r centred in x (the set has **no accumulation points**);
 - (b) $\forall x \in \mathbb{R}^2$ there is at least one element of Γ in the ball of radius R centred in x (the set is **not sparse**);

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 - (b) $\forall x \in \mathbb{R}^2$ there is at least one element of Γ in the ball of radius R centred in x (the set is **not sparse**);
- (ii) a localization function G (typically $G(x) = (1 + |x|^2)^{s/2}$ for some $s \geq 1$), a constant $M > 0$ independent of $\gamma \in \Gamma$ and an **orthonormal basis of $\text{Ran } P$** , $\{\psi_{\gamma,a}\}_{\gamma \in \Gamma, 1 \leq a \leq m(\gamma) < \infty}$ with $m(\gamma) \leq m_* \ \forall \gamma \in \Gamma$, satisfying

$$\int_{\mathbb{R}^2} G(|x - \gamma|)^2 |\psi_{\gamma,a}(x)|^2 dx \leq M.$$

We call each $\psi_{\gamma,a}$ a **generalized Wannier function (GWF)**.

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Localization implies Chern triviality

Theorem - work in progress [Marcelli, Moscolari, GP]

Let P_μ be the Fermi projector of a reasonable Schrödinger operator in $L^2(\mathbb{R}^2)$.

Suppose that P_μ admits a generalized Wannier basis, $\{\psi_{\gamma,a}\}_{\gamma \in \Gamma, 1 \leq a \leq m(\gamma) < m_*}$, which is s_* -localized in the sense

$$\int_{\mathbb{R}^2} (1 + |x - \gamma|^2)^{s_*} |\psi_{\gamma,a}(x)|^2 dx \leq M.$$

Then, if $s_* \geq 7$ [provisional hypothesis], one has that

$$\mathcal{T} \left(i P_\mu \left[[X_1, P_\mu], [X_2, P_\mu] \right] \right) = 0.$$

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Clearly, the optimal statement would be for $s_* = 1$, as in the periodic case.

Technical difficulties.

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$d = 1$ in [NN93] it is proved that an exponentially localized GWB exists under general hypothesis (Γ discrete): for $d = 1$, $P_\mu X P_\mu$ has discrete spectrum [!!!], and a GWB is provided by its eigenfunctions

$$\{\psi_{\gamma,a}\} \quad \gamma \in \sigma_{\text{disc}}(P_\mu X P_\mu) =: \Gamma, a \in \{1, \dots, m(\gamma)\}.$$

A simple but relevant observation

- ▶ Let $\tilde{X}_j := P_\mu X_j P_\mu$ be the **reduced position operator**. Then, by simple algebra

$$P_\mu \left[[X_1, P_\mu], [X_2, P_\mu] \right] = \left[\tilde{X}_1, \tilde{X}_2 \right]. \quad (1)$$

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Cyclicity is recovered if it happens that

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- ▶ Boring details. Our estimates are far from being optimal.

Part III

The Localization-Topology Correspondence: $\mathbb{Z}_2$ topological insulators, periodic case

Based on joint work with D. Fiorenza and D. Monaco
and on work by H. Cornean, D. Monaco and S. Teufel

Time-reversal-symmetric systems

- ▶ Time-reversal symmetry (TRS) is represented by a anti-unitary operator Θ , such that

$$\Theta^2 = \pm \mathbb{1} \quad \text{and} \quad H = \Theta H \Theta^{-1}.$$

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Question:

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- ▶ TRS insulators, either bosonic or fermionic, are always **Chern trivial**. Hence, **exponentially localized CWF** exist.

Question:

Does exist a system of
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The answer crucially depends on the fact that the symmetry is **bosonic**/**fermionic**.

Definition: Time-Reversal-Symmetric Wannier functions

$$\Theta^2 = \mathbb{1}$$

Spin-0 particle. $\mathcal{H} = L^2(\mathbb{R}^d)$. Time-reversal operator $(\Theta\psi)(x) = \overline{\psi(x)}$.
A system of CWF $\{w_{\gamma,a}\}$ is TRS if each $w_{\gamma,a}$ is $\mathbb{R}$ -valued, namely

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$$w_b(x) = \sum_{a=1}^m \left(\Theta w_a(x) \right) \varepsilon_{ab}$$

where the matrix $\varepsilon = (\varepsilon_{ab})$ is unitary and skew-symmetric (**reshuffling matrix**).

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Example

For a single pair of $\mathbb{C}^2$ -valued Wannier functions w_1, w_2

where the n matrix).

$$\begin{pmatrix} w_1^\uparrow \\ w_1^\downarrow \end{pmatrix} = \begin{pmatrix} \overline{w_2^\downarrow} \\ \overline{w_2^\uparrow} \end{pmatrix}$$

symmetric (**reshuffling**)

Bosonic case

Theorem [D. Fiorenza, D. Monaco, GP]

Consider the bosonic case $\Theta^2 = +1$. Then, for any $d \leq 3$ one can always construct a basis of exponentially localized time-reversal-symmetric (i. e. $\mathbb{R}$ -valued) composite Wannier functions for P_μ .

In other words, there is no additional obstruction to impose the TRS constraint.

[FMP₁] D. FIORENZA, D. MONACO, G. PANATI *Ann. H. Poincaré* 17 (2016)

Fermionic case

Theorem [H. Cornean, D. Monaco, S. Teufel]

Consider the Fermionic case $\Theta^2 = -\mathbb{1}$.

[CMT]

H. CORNEAN, D. MONACO, S. TEUFEL *Rev. Math. Phys.* **29**(2017)

[GP]

G.M. GRAF, M. PORTA: *Commun. Math. Phys.* **324** (2013)

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D. FIORENZA, D. MONACO, G. PANATI *Commun. Math. Phys.* 343 (2016)

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L. FU, C.L. KANE *Phys. Rev. B* **74** (2006).

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Theorem [H. Cornean, D. Monaco, S. Teufel]

Consider the Fermionic case $\Theta^2 = -\mathbb{1}$.

- For $d = 1$, one can always construct a system of exponentially localized TRS composite Wannier functions.

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Fermionic case

Theorem [H. Cornean, D. Monaco, S. Teufel]

Consider the Fermionic case $\Theta^2 = -\mathbb{1}$.

- For $d = 1$, one can always construct a system of exponentially localized TRS composite Wannier functions.
- For $d = 2$, one can construct a system of TRS composite Wannier functions if and only if the Graf-Porta index $\mathcal{I}(P_\mu)$ vanishes [GP] .

[CMT]

H. CORNEAN, D. MONACO, S. TEUFEL *Rev. Math. Phys.* **29**(2017)

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L. FU, C.L. KANE *Phys. Rev. B* **74** (2006).

Fermionic case

Theorem [H. Cornean, D. Monaco, S. Teufel]

Consider the **Fermionic case** $\Theta^2 = -\mathbb{1}$.

- For $d = 1$, one can always construct a system of exponentially localized TRS composite Wannier functions.
- For $d = 2$, one can construct a system of TRS composite Wannier functions **if and only if the Graf-Porta index $\mathcal{I}(P_\mu)$ vanishes [GP]**.

Actually, one can prove that

$$\mathcal{I} = \delta = \Delta$$

(see **D. Monaco's** talk)

[CMT]

H. CORNEAN, D. MONACO, S. TEUFEL *Rev. Math. Phys.* **29**(2017)

[GP]

G.M. GRAF, M. PORTA: *Commun. Math. Phys.* **324** (2013)

[FMP₂]

D. FIORENZA, D. MONACO, G. PANATI *Commun. Math. Phys.* 343 (2016)

[FK]

L. FU, C.L. KANE *Phys. Rev. B* **74** (2006).

Synoptic table for $\mathbb{Z}_2$ top insulators $\Theta^2 = -1, d = 2$

Vanishing
FKM index

Topology
 $\longleftrightarrow$

Non-vanishing
FKM index



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$\updownarrow$

Exponentially loc.
TRS Wannier functions

Localization
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Kubo formula for spin transport

See talk by G. Marcelli

Thank you for your attention!!