Global Gauge Anomalies in Coset Models of Conformal Field Theory

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Abstract: We study the occurrence of global gauge anomalies in the coset models of two-dimensional conformal field theory that are based on gauged WZW models. A complete classification of the non-anomalous theories for a wide family of gauged rigid adjoint or twisted-adjoint symmetries of WZW models is achieved with the help of Dynkin's classification of Lie subalgebras of simple Lie algebras.

1. Introduction

Bosonic sigma models with the metric action functional possess rigid symmetries induced by isometries of their target space. Such rigid symmetries may be gauged by the minimal coupling to the gauge fields of the isometry group. The gauged action is then invariant under arbitrary local gauge transformations. The minimal coupling does not work, however, for the topological Wess–Zumino term in the action functional of the sigma model, if such is present. In particular, it was shown in [1,2] for the two-dimensional sigma model with the Wess–Zumino term corresponding to a closed 3-form H on the target space that the gauging of rigid symmetries requires satisfying certain conditions. Such conditions assure the absence of local gauge anomalies and guarantee the existence of a gauging procedure that results in an action functional invariant under infinitesimal local gauge transformations. The infinitesimal gauge invariance of the gauged action implies its invariance under all "small" local gauge transformations, i.e., the ones that are homotopic to unity. As was observed in [3], it is possible, however, that the gauged action exhibits global gauge anomalies that lead to its non-invariance under some "large" local gauge transformations non-homotopic to unity. The phenomenon was analyzed in detail for sigma models on closed worldsheets in [3] and on worldsheets with boundaries and defects in [4]. In the case of Wess–Zumino–Witten (WZW) models of conformal field theory with Lie group $G = \tilde{G}/Z$ as the target, where $\tilde{G}$ is the universal covering group of G and Z is a subgroup of the center $\tilde{Z}$ of $\tilde{G}$, with the Wess–Zumino term corresponding to the bi-invariant closed 3-form $H_k = \frac{k}{12\pi} \mathrm{tr}(g^{-1}dg)^3$, the local

gauge anomalies are absent for a restricted class of rigid symmetries. These include the symmetries induced by the adjoint action $g \mapsto hgh^{-1}$ on G for $h \in \tilde{G}/\tilde{Z}$, or by its twisted versions $g \mapsto hg\omega(h)^{-1}$, for $h \in \tilde{G}/Z^\omega$, where ω is an automorphism of $\tilde{G}$ and $Z^\omega = \{z \in \tilde{Z} \mid z\omega(z)^{-1} \in Z\}$ is the subgroup of elements in $\tilde{Z}$ that acts trivially. In these cases, the global gauge anomalies may occur for the target groups G that are not simply connected (corresponding to the so called non-diagonal WZW models). They are detected by a cohomology class $\varphi \in H^2(\tilde{G}/Z^\omega \times G, U(1))$ that can be easily computed. Class φ is invariant under the action of $\gamma \in \tilde{G}/Z^\omega$ on $\tilde{G}/\tilde{Z}^\omega \times G$ given by

$$(h, g) \mapsto (\gamma h \gamma^{-1}, \gamma g \omega(\gamma)^{-1}). \tag{1.1}$$

The simplest case when the anomaly class is nontrivial corresponds to $G = SU(3)/\mathbb{Z}_3$ at level $k = 1$ or to $G = SU(4)/\mathbb{Z}_4$ at level $k = 2$, both with $\omega = Id$. Some other cases with global gauge anomalies for $\omega = Id$ were cited in [3]. In Sect. 3 of the present paper, we obtain the full list of connected compact simple target groups G for which the WZW model with the gauged (twisted) adjoint action of $\tilde{G}/\tilde{Z}^\omega$ exhibits global gauge anomalies. In the twisted case, we consider only outer automorphisms ω since for inner automorphisms the twisted adjoint action may be reduced to the untwisted one by conjugating it with a right translation on G which is a rigid symmetry of the WZW theory. The classes of outer automorphisms of $\tilde{G}$ modulo inner automorphisms are generated by automorphisms of the Lie algebra $\mathfrak{g}$ that preserve the set of simple roots inducing a symmetry of the Dynkin diagram of $\mathfrak{g}$. Global gauge anomalies occur only for (non-simply connected) groups G with Lie algebras $\mathfrak{g} = A_r$, D_r, $\mathfrak{e}_6$ in the Cartan classification of simple Lie algebras.[1]

Gauged WZW models serve to construct coset G/H models [5,6] of the two-dimensional conformal field theory [7–10]. In such models, one restricts the gauging to the (possibly twisted) adjoint action on the target group of the subgroup $\Gamma = \tilde{H}/(Z^\omega \cap \tilde{H}) \subset \tilde{G}/Z^\omega$, where $\tilde{H}$ a closed connected subgroup of $\tilde{G}$ (simply-connected or not). Global gauge anomalies are now detected by the pullback cohomology class in $H^2(\Gamma \times G, U(1))$. Sects. 4 and 5 are devoted to finding out when the latter is nontrivial for groups G as before and for a wide class of subgroups $\tilde{H} \subset \tilde{G}$ (the nontriviality of the pullback class depends only on the subgroup $\tilde{H}$ modulo conjugation by elements of $\tilde{G}$ and it may occur only if the original anomaly class φ is nontrivial, hence for Lie algebras $\mathfrak{g}$ enumerated above). Closed connected subgroups $\tilde{H} \subset \tilde{G}$ are in one-to-one correspondence to Lie subalgebras $\mathfrak{h} \subset \mathfrak{g}$. We obtain the complete list of cases with global gauge anomalies for subgroups $\tilde{H}$ with the Lie algebra $\mathfrak{h}$ which is a semisimple regular subalgebra of $\mathfrak{g}$ (i.e. such that the roots of $\mathfrak{h}$ form a subset of roots of $\mathfrak{g}$). The complete classification (modulo conjugation) of regular subalgebras of simple Lie algebras was obtained in the classical work of Dynkin [11]. The complete classification of all semisimple subalgebras of simple Lie algebras is not known explicitly, except for low ranks and may be complicated. We give the complete list of non-regular semisimple subalgebras $\mathfrak{h}$ of $\mathfrak{g} = \mathfrak{e}_6$ corresponding to subgroups $\tilde{H} \subset \tilde{G}$ that lead to global gauge anomalies. For $\mathfrak{g} = A_r$ and $\mathfrak{g} = D_r$, we limit ourselves to a few examples of anomalous subgroups $\tilde{H} \subset \tilde{G}$ for which $\mathfrak{h}$ is a non-regular semisimple subalgebra of $\mathfrak{g}$.

As discussed in [3] for the untwisted case, the presence of global gauge anomalies of the type studied here renders the G/H coset models inconsistent on the quantum level

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[1] We consider the compact real forms $\mathfrak{g}$ of complex simple Lie algebras that are in one-to-one correspondence with their complexifications $\mathfrak{g}^\mathbb{C}$.

(barring accidental degeneracies of the affine characters). Hence the importance of the classification of the anomalous cases.

2. No-Anomaly Condition

The WZ contribution to the action of the WZW model corresponding to the closed 3-form H_k on a connected compact simple Lie group $G = \tilde{G}/Z$ with $Z \subset \tilde{Z}$ may be defined (modulo 2π) whenever the periods of H_k (i.e. its integrals over closed 3-cycles) belong to $2\pi\mathbb{Z}$. For the standard normalization of the invariant negative-definite quadratic form tr on the Lie algebra $\mathfrak{g}$ in which long roots (viewed as elements of $i\mathfrak{t}_{\mathfrak{g}}$, where $\mathfrak{t}_{\mathfrak{g}}$ is the Cartan subalgebra of $\mathfrak{g}$) have length squared 2, this happens for levels $k \in K_G \subset \mathbb{Z}$. If $G = \tilde{G}$ then $K_G = \mathbb{Z}$ whereas K_G may be a proper subset of $\mathbb{Z}$ if $G = \tilde{G}/Z$ with Z nontrivial (i.e. $\neq \{1\}$). Sets K_G of admissible levels are explicitly known [12,13]. Besides, for $G = SO(2r)/\mathbb{Z}_2$ with r even (where $K_G = \mathbb{Z}$ when $4|r$ and $K_G = 2\mathbb{Z}$ if $4 \nmid r$), there are two different consistent choices of the WZ term of the action. The details of the construction of the WZ contribution $\exp\left[i S_\Sigma^{WZ}(g)\right]$ to the Feynman amplitude of the sigma-model field $g : \Sigma \to G$ defined on a closed oriented worldsheet Σ, discussed e.g. in [14,15], will not interest us here beyond the fact that the result is invariant under the composition of fields g with the left or right action of (fixed) elements of group G. The action functional with the (twisted) adjoint symmetry of the WZW model gauged is a functional of field g and of gauge-field A, a $\mathfrak{g}$-valued 1-form on Σ. It has the form

$$S_\Sigma^{WZ}(g, A) = S_\Sigma^{WZ}(g) + \frac{k}{4\pi} \int \mathrm{tr}\left((g^{-1}dg)\omega(A) + (dg)g^{-1}A + g^{-1}Ag\omega(A)\right) \quad (2.1)$$

(for the untwisted case, $\omega = Id$). The local gauge transformations $h : \Sigma \to \tilde{G}/Z^\omega$ act on the sigma model and gauge fields by

$$^h g = hg\omega(h)^{-1}, \quad {}^h A = hAh^{-1} + hdh^{-1}. \quad (2.2)$$

Note that $Z^\omega = \tilde{Z}$ for $\omega = Id$. It is easy to show that the invariance of the gauged Feynman amplitudes under such transformations:

$$\exp\left[i S_\Sigma^{WZ}({}^h g, {}^h A)\right] = \exp\left[i S_\Sigma^{WZ}(g, A)\right] \quad (2.3)$$

is equivalent to the identity

$$\frac{\exp\left[i S_\Sigma^{WZ}({}^h g)\right]}{\exp\left[i S_\Sigma^{WZ}(g) + \frac{ik}{4\pi} \int_\Sigma \mathrm{tr}\left(g^{-1}dg\omega(h^{-1}dh) + (dg)g^{-1}h^{-1}dh + g^{-1}(h^{-1}dh)g\omega(h^{-1}dh)\right)\right]} = 1, \quad (2.4)$$

see Appendix A. The ratio on the left hand side belongs always to $U(1)$. It coincides with the evaluation of the anomaly class $\varphi \in H^2(\tilde{G}/Z^\omega \times G, U(1))$ on the 2-cocycle that is the image of the fundamental class of Σ under the map $(h, g) : \Sigma \to \tilde{G}/Z^\omega \times G$.

A simple analysis [3] of the structure of cohomology group $H^2(\tilde{G}/Z^\omega \times G, U(1))$ based on the Künneth Theorem shows that class φ is trivial if and only if identity (2.4) holds for $\Sigma = S^1 \times S^1$ and

$$h(e^{i\sigma_1}, e^{i\sigma_2}) = e^{i\sigma_1 \tilde{M}}, \quad g(e^{i\sigma_1}, e^{i\sigma_2}) = e^{i\sigma_2 M} \quad (2.5)$$

where $\tilde{M}$, $M \in \mathrm{it}_{\mathfrak{g}}$ and are such that, in terms of the exponential map with values in $\tilde{G}$,

$$\tilde{z} \equiv e^{2i\pi \tilde{M}} \in Z^{\omega} \quad \text{and} \quad z \equiv e^{2i\pi M} \in Z. \tag{2.6}$$

Both $\tilde{M}$ and M have to belong to the coweight lattice $P^{\vee}(\mathfrak{g}) \subset \mathrm{it}_{\mathfrak{g}}$ dual to the weight lattice of $\mathfrak{g}$ and composed of $M \in \mathrm{it}_{\mathfrak{g}}$ s.t. $\exp[2i\pi M] \in \tilde{Z}$. For (h, g) given by Eqs. (2.5), the left hand side of Eq. (2.4) is easily computable giving rise to the identity

$$c_{\tilde{z}\omega(\tilde{z})^{-1}, z} \exp\left[-2i\pi k \, \mathrm{tr}(M\omega(\tilde{M}))\right] = 1 \tag{2.7}$$

which holds for all $\tilde{M}$, $M \in P^{\vee}(\mathfrak{g})$ as above if and only if there are no global gauge anomalies for the WZW model with gauged (twisted) adjoint action of $\tilde{G}/\tilde{Z}^{\omega}$ on the target group G. In Eq. (2.7),

$$Z^2 \ni (z, z') \mapsto c_{z,z'} \in U(1) \tag{2.8}$$

is a k-dependent bihomomorphism in $Hom(Z \otimes Z, U(1))$ whose explicit form may be extracted from Appendix 2 of [12]. For cyclic $Z \equiv \mathbb{Z}_p$ generated by $z_0 = e^{2i\pi\theta}$ for $\theta \in P^{\vee}(\mathfrak{g})$,

$$c_{z_0^m, z_0^n} = \exp[-i\pi kmn \, \mathrm{tr}(\theta^2)]. \tag{2.9}$$

For the only case with non-cyclic Z, we shall explicit $c_{z,z'}$ in Sect. 3.4.4. In the untwisted case with $\omega = Id$, condition (2.7) reduces to the requirement that

$$\exp\left[-2i\pi k \, \mathrm{tr}(M\tilde{M})\right] = 1. \tag{2.10}$$

If we gauge only the adjoint action of $\tilde{H}/(\tilde{Z}^{\omega} \cap \tilde{H})$ then there are no global gauge anomalies if and only if identity (2.7) holds under the additional restriction that, as an element of $\tilde{G}$, $\exp[2i\pi \tilde{M}] \in \tilde{H}$.

It is enough to check the above conditions for $\tilde{M}$, M in different classes modulo the coroot lattice $Q^{\vee}(\mathfrak{g})$ (composed of $\tilde{M} \in \mathrm{it}_{\mathfrak{g}}$ s.t. $\exp[2i\pi \tilde{M}] = 1$ in $\tilde{G}$) since $\mathrm{tr}\,\tilde{M}M \in \mathbb{Z}$ if $\tilde{M} \in P^{\vee}(\mathfrak{g})$ and $M \in Q^{\vee}(\mathfrak{g})$ or *vice versa*. In particular, if $Z = \{1\}$, i.e. if G is simply connected, then conditions (2.7) and (2.10) are always satisfied so that there are no global gauge anomalies in that case. In the sequel, we shall describe for each Lie algebra $\mathfrak{g}$ the center $\tilde{Z}$ of the corresponding simply connected group $\tilde{G}$ in terms of coweights of $\mathfrak{g}$. Then choosing a Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$, we shall restrict elements $\tilde{M}$ by requiring that $e^{2i\pi \tilde{M}} \in \tilde{H}$. Note that $e^{2i\pi \tilde{M}} \in \tilde{H}$ if and only if $e^{2i\pi \tilde{M}} \in g\tilde{H}g^{-1}$ for $g \in \tilde{G}$ and $e^{2i\pi \tilde{M}} \in \tilde{Z}$. Hence the no-anomaly conditions coincide for conjugate subgroups $\tilde{H} \subset \tilde{G}$. Thus it is enough to consider one Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$ in each class of subalgebras related by inner automorphisms of $\mathfrak{g}$. We may also require that the Cartan subalgebra $\mathfrak{t}_{\mathfrak{h}}$ of $\mathfrak{h}$ be contained in the Cartan subalgebra $\mathfrak{t}_{\mathfrak{g}}$ of $\mathfrak{g}$. Then $e^{2i\pi \tilde{M}} \in \tilde{H}$ if and only if there is $q^{\vee} \in Q^{\vee}(\mathfrak{g})$ such that $\tilde{M} + q^{\vee} \in \mathrm{it}_{\mathfrak{h}}$. This is the condition that we shall impose on $\tilde{M}$.

The no-anomaly conditions for Lie subalgebras $\mathfrak{h} \subset \mathfrak{g}$ related by outer automorphisms ω' of $\mathfrak{g}$ are also related. Indeed, it is easy to see that the expression on the right hand side of Eq. (2.4) for gauge transformation h and fields g coincides with the similar expression for gauge transformation $\omega'(h)$ and field $\omega'(g)$ if in the latter case subgroup

$Z \subset \tilde{Z}$ is replaced by $\omega'(Z)$ and the twist ω by $\omega'\omega\omega'^{-1}$. The only exception is the case of $G = SO(2r)/\mathbb{Z}_2$ for even r and odd k where one may also have to interchange the two different consistent choices of the theory, see Sect. 3.4.4.

Summarizing: the necessary and sufficient condition for the absence of global gauge anomalies requires that Eq. (2.7) holds for all $\tilde{M}, M \in P^\vee(\mathfrak{g})$ such that

$$\tilde{z} \equiv e^{2i\pi\tilde{M}} \in Z^\omega \cap \tilde{H} \quad \text{and} \quad z \equiv e^{2i\pi M} \in Z. \tag{2.11}$$

In the untwisted case, this reduces to the condition[2]

$$k \operatorname{tr}(M\tilde{M}) \in \mathbb{Z} \quad \text{for all} \quad \tilde{M}, M \in P^\vee(\mathfrak{g}) \ s.t. \ \tilde{z} \in \tilde{H}, \ z \in Z. \tag{2.12}$$

The no-anomaly conditions for subgroups $\tilde{H} \subset \tilde{G}$ corresponding to Lie subalgebras $\mathfrak{h} \subset \mathfrak{g}$ related by inner (outer) automorphisms of $\mathfrak{g}$ coincide (are simply related).

3. Cases with $\mathfrak{h} = \mathfrak{g}$

As the first step, we shall consider the cases with $\mathfrak{h} = \mathfrak{g}$ for all simple algebras $\mathfrak{g}$ according to the Cartan classification, and for arbitrary nontrivial subgroups $Z \subset \tilde{Z}$. If there are no global gauge anomalies in that case, then the anomalies are absent also for other $\mathfrak{h} \subset \mathfrak{g}$. In other words, upon restricting $\mathfrak{h}$ to a smaller subalgebra, the anomalies may only disappear. In this way, a lot of trivial cases can be already treated without specifying the subalgebra $\mathfrak{h}$. We shall then consider in the next section the classification of subalgebras $\mathfrak{h} \subset \mathfrak{g}$ up to conjugation only for the remaining cases: those with possible anomalies.

3.1. Case $A_r = \mathfrak{su}(r+1)$, $r \geq 1$. Lie algebra $\mathfrak{g} = A_r$, corresponding to group $\tilde{G} = SU(r+1)$, is composed of traceless anti-hermitian matrices of size $r+1$. Its Cartan subalgebra $\mathfrak{t}_\mathfrak{g}$ may be taken as the subalgebra of diagonal traceless matrices with imaginary entries. We define $e_i \in i\mathfrak{t}_\mathfrak{g}$, $i = 1, \ldots, r+1$, as a diagonal matrix with the j's diagonal entry equal to δ_{ij}, so that $\operatorname{tr}(e_i e_j) = \delta_{ij}$. Roots (viewed as elements of $i\mathfrak{t}_\mathfrak{g}$) and coroots of $\mathfrak{su}(r+1)$ have then the form $e_i - e_j$ for $i \neq j$ and the standard choice of simple roots is $\alpha_i = e_i - e_{i+1}$, $i = 1 \ldots r$. The center $\tilde{Z} \cong \mathbb{Z}_{r+1}$ may be generated by $z = e^{2i\pi\theta}$ with $\theta = \lambda_r^\vee = (1/(r+1))\sum_{i=1}^{r+1} e_i - e_{r+1}$ where $\lambda_i^\vee$ denotes the i-th simple coweight satisfying $\operatorname{tr}(\lambda_i^\vee \alpha_j) = \delta_{ij}$. Subgroups Z of $\tilde{Z}$ are of the form $Z \cong \mathbb{Z}_p$ with $p|(r+1)$, and may be generated by $z^q = e^{2i\pi q\theta}$ for $r+1 = pq$. The admissible levels for the WZW model based on group $G = \tilde{G}/\mathbb{Z}_p$ are:

$$\begin{aligned} k \in 2\mathbb{Z} \quad &\text{if } p \text{ even and } q \text{ odd,} \\ k \in \mathbb{Z} \quad &\text{otherwise,} \end{aligned} \tag{3.1}$$

see [12,13]. If we now represent M and $\tilde{M}$ in the Euclidian space spanned by vectors e_i,

$$M = aq\theta = \left(\frac{a}{p}, \ldots, \frac{a}{p}, -\frac{ar}{p}\right), \quad a \in \mathbb{Z}, \tag{3.2}$$

$$\tilde{M} = \tilde{a}\theta = \left(\frac{\tilde{a}}{r+1}, \ldots, \frac{\tilde{a}}{r+1}, -\frac{\tilde{a}r}{r+1}\right), \quad \tilde{a} \in \mathbb{Z}, \tag{3.3}$$

the condition for M in (2.11) is satisfied and $e^{2i\pi\tilde{M}} \in \tilde{Z}$.

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[2] In the conformal filed theory terminology [16], condition (2.12) means that the monodromy charge $Q_J(\tilde{J})$ for the simple currents $\tilde{J}$ and J corresponding to the central elements $\tilde{z}$ and z has to vanish modulo 1.

3.1.1. Untwisted case. If $\omega = Id$, the global gauge invariance for $\mathfrak{h} = \mathfrak{g}$ is assured if

$$k \operatorname{tr}(M\tilde{M}) = k\frac{ra\tilde{a}}{p} \in \mathbb{Z}. \tag{3.4}$$

In particular, $k \in p\mathbb{Z}$ is a sufficient condition for the absence of global anomalies. Recall that p divides $r + 1$. This implies that p and r are relatively prime. Hence $k \in p\mathbb{Z}$ is also a necessary condition for the absence of the anomalies if there are no further restrictions on the values of $\tilde{a}$, i.e. if $\mathfrak{h} = \mathfrak{g}$. Taking into account restrictions (3.1), this leads to the first result:

Proposition 3.1. *The untwisted coset models corresponding to Lie algebra* $\mathfrak{g} = \mathfrak{su}(r+1)$, *subgroups* $Z \cong \mathbb{Z}_p$, $r + 1 = pq$, *and arbitrary subalgebras* $\mathfrak{h}$ *do not have global gauge anomalies if* $k \in p\mathbb{Z}$. *The models with* $\mathfrak{h} = \mathfrak{g}$ *and with* $k \notin p\mathbb{Z}$ *for* $p > 1$ *odd or* q *even, or with* $k \in 2\mathbb{Z}\backslash p\mathbb{Z}$ *for* $p > 2$ *even and* q *odd are anomalous.*

3.1.2. Twisted case. For $r > 1$, there is one nontrivial outer automorphism of $\mathfrak{su}(r + 1)$. It maps simple root α_i to α_{r+1-i} so that for $\tilde{M}$ given by Eq. (3.3),

$$\omega(\tilde{M}) = \omega(\tilde{a}\theta) = \left(\frac{\tilde{a}r}{r + 1}, \frac{-\tilde{a}}{r + 1}, \dots, \frac{-\tilde{a}}{r + 1}\right), \quad \tilde{a} \in \mathbb{Z}. \tag{3.5}$$

The condition

$$e^{2i\pi\tilde{M}} \omega(e^{-2i\pi\tilde{M}}) = e^{4i\pi\tilde{a}\theta} \in Z \tag{3.6}$$

reduces to the requirement

$$q|\tilde{a} \quad \text{for} \quad q \quad \text{odd} \quad \text{and} \quad \frac{q}{2}|\tilde{a} \quad \text{for} \quad q \quad \text{even.} \tag{3.7}$$

It follows that $Z^\omega \cong \mathbb{Z}_p$ for q odd and $Z^\omega \cong \mathbb{Z}_{2p}$ for q even. From Eq. (2.9), we obtain

$$c_{\tilde{z}\omega(\tilde{z})^{-1},z} = \exp\left[-2i\pi k\frac{\tilde{a}ar}{p}\right] \tag{3.8}$$

and from Eqs. (3.2) and (3.5),

$$\exp[-2i\pi k \operatorname{tr}(M\omega(\tilde{M}))] = \exp\left[-2i\pi k\frac{a\tilde{a}}{p}\right] \tag{3.9}$$

so that the no-anomaly condition (2.7) reduces to the identity

$$\exp[-2i\pi ka\tilde{a}q] = 1 \tag{3.10}$$

which always holds implying

Proposition 3.2. *The twisted coset models corresponding to Lie algebra* $\mathfrak{g} = \mathfrak{su}(r + 1)$, *subgroups* $Z \cong \mathbb{Z}_p$, $r + 1 = pq$, *and arbitrary subalgebras* $\mathfrak{h}$ *do not have global gauge anomalies.*

3.2. Case $B_r = \mathfrak{so}(2r + 1)$, $r \geq 2$. Lie algebra $\mathfrak{g} = B_r$, corresponding to group $\tilde{G} = Spin(2r + 1)$, is composed of real antisymmetric matrices of size $2r + 1$. The Cartan algebra $\mathfrak{t}_\mathfrak{g}$ may be taken as composed of r blocks

$$\begin{pmatrix} 0 & -t_i \\ t_i & 0 \end{pmatrix} \tag{3.11}$$

placed diagonally, with the last diagonal entry vanishing. Let $e_i \in i\mathfrak{t}_\mathfrak{g}$ denote the matrix corresponding to $t_j = i\delta_{ij}$. With the normalization such that $\text{tr}(e_i e_j) = \delta_{ij}$, roots of $\mathfrak{g}$ have the form $\pm e_i \pm e_j$ for $i \neq j$ and $\pm e_i$, and one may choose $\alpha_i = e_i - e_{i+1}$ for $i = 1 \ldots r - 1$ and $\alpha_r = e_r$ as the simple roots. The center $\tilde{Z} \cong \mathbb{Z}_2$ is generated by $z = e^{2i\pi\theta}$ with $\theta = \lambda_1^\vee = e_1$, and the only nontrivial subgroup of the center is $Z = \tilde{Z}$. If we describe M and $\tilde{M}$ in the Euclidean space spanned by vectors e_i, it is enough to take

$$M = a\theta = (a, 0, \ldots, 0), \quad \tilde{M} = \tilde{a}\theta = (\tilde{a}, 0, \ldots, 0), \quad a, \tilde{a} \in \mathbb{Z}. \tag{3.12}$$

Lie algebra $\mathfrak{so}_{r+1}$ does not have nontrivial outer automorphisms. For $\omega = Id$, the global gauge invariance is assured if

$$k \, \text{tr}(M\tilde{M}) = k a \tilde{a} \in \mathbb{Z} \tag{3.13}$$

which is always the case leading to

Proposition 3.3. *The coset models corresponding to Lie algebra $g = \mathfrak{so}(2r + 1)$ and any subalgebra $\mathfrak{h}$ do not have global gauge anomalies.*

3.3. Case $C_r = \mathfrak{sp}(2r)$, $r \geq 3$. Lie algebra $\mathfrak{g} = C_r$, corresponding to group $\tilde{G} = Sp(2r)$, is composed of antihermitian matrices X of size $2r$ such that ΩX is symmetric, with Ω built of r blocks

$$\omega = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{3.14}$$

placed diagonally. The Cartan algebra $\mathfrak{t}_\mathfrak{g}$ may be taken as composed of r blocks $t_i \omega$ placed diagonally. Let $e_i \in i\mathfrak{t}_\mathfrak{g}$ denote the matrix corresponding to $t_j = i\delta_{ij}$. With the normalization $\text{tr}(e_i e_j) = 2\delta_{ij}$, roots of $\mathfrak{g}$ have the form $(1/2)(\pm e_i \pm e_j)$ for $i \neq j$ and $\pm e_i$. The simple roots may be chosen as $\alpha_i = (1/2)(e_i - e_{i+1})$ for $i = 1, \ldots r - 1$ and $\alpha_r = e_r$. The center $\tilde{Z} \cong \mathbb{Z}_2$ is generated by $z = e^{2i\pi\theta}$ with $\theta = \lambda_r^\vee = (1/2)\sum_{i=1}^r e_i$, and its only nontrivial subgroup is $Z = \tilde{Z}$. We then take M and $\tilde{M}$ in the Euclidian space spanned by vectors e_i of the form

$$M = a\theta = \left(\frac{a}{2}, \ldots, \frac{a}{2}\right) \quad \tilde{M} = \tilde{a}\theta = \left(\frac{\tilde{a}}{2}, \ldots, \frac{\tilde{a}}{2}\right) \quad a, \tilde{a} \in \mathbb{Z}. \tag{3.15}$$

Lie algebra $\mathfrak{sp}(2r)$ does not have nontrivial outer automorphisms. For $\omega = Id$, taking into account the normalization of tr, we obtain:

$$k \, \text{tr}(M\tilde{M}) = k\frac{a\tilde{a}r}{2}, \tag{3.16}$$

ensuring the global gauge invariance if it is an integer. The admissible levels k are

$$k \in \mathbb{Z} \quad \text{if } r \text{ is even}, \tag{3.17}$$
$$k \in 2\mathbb{Z} \quad \text{if } r \text{ is odd}, \tag{3.18}$$

see [12,13], so that the above condition is always satisfied leading to

Proposition 3.4. *The coset models corresponding to Lie algebra $\mathfrak{g} = \mathfrak{sp}(2r)$ and any subalgebra $\mathfrak{h}$ do not have global gauge anomalies.*

3.4. Case $D_r = \mathfrak{so}(2r)$, $r \geq 4$. Lie algebra $\mathfrak{g} = D_r$, corresponding to group $\tilde{G} = Spin(2r)$, is composed of real antisymmetric matrices of size $2r$. The Cartan algebra $\mathfrak{t_g}$ may be taken as composed of r blocks

$$\omega = \begin{pmatrix} 0 & -t_i \\ t_i & 0 \end{pmatrix} \tag{3.19}$$

placed diagonally. Let us denote by $e_i \in i\mathfrak{t_g}$ the matrix corresponding to $t_j = i\delta_{ij}$. With the normalization $\mathrm{tr}(e_i e_j) = \delta_{ij}$, roots of $\mathfrak{g}$ have the form $\pm e_i \pm e_j$ for $i \neq j$, and the simple roots may be chosen as $\alpha_i = e_i - e_{i+1}$ for $i = 1 \ldots r - 1$ and $\alpha_r = e_{r-1} + e_r$.

Case of r odd. If r is odd, the center $\tilde{Z} \cong \mathbb{Z}_4$ is generated by $z = e^{2i\pi\theta}$ with $\theta = \lambda_r^\vee = (1/2) \sum_{i=1}^r e_i$. The possible nontrivial subgroups are $Z = \tilde{Z}$ and $Z \cong \mathbb{Z}_2$, generated by z^2. In particular, $Spin(2r)/\mathbb{Z}_2 = SO(2r)$. Taking the general form of M and $\tilde{M}$ in the Euclidian space spanned by vectors e_i,

$$\begin{aligned} M = a\theta = \left(\frac{a}{2}, \ldots, \frac{a}{2} \right), \quad & a \in \mathbb{Z} \text{ if } Z \cong \mathbb{Z}_4, \\ & a \in 2\mathbb{Z} \text{ if } Z \cong \mathbb{Z}_2, \\ \tilde{M} = \tilde{a}\theta = \left(\frac{\tilde{a}}{2}, \ldots, \frac{\tilde{a}}{2} \right), \quad & \tilde{a} \in \mathbb{Z}. \end{aligned} \tag{3.20}$$

The admissibility condition for the levels in the corresponding WZW models are [13]:

$$k \in 2\mathbb{Z} \quad \text{if } Z \cong \mathbb{Z}_4, \tag{3.21}$$
$$k \in \mathbb{Z} \quad \text{if } Z \cong \mathbb{Z}_2. \tag{3.22}$$

3.4.1. Untwisted case. If $\omega = Id$ then the global gauge invariance is assured if the quantity

$$k \, \mathrm{tr}(M\tilde{M}) = k \frac{a\tilde{a}r}{4}, \tag{3.23}$$

is an integer. The latter holds for

$$k \in 4\mathbb{Z} \quad \text{if } Z \cong \mathbb{Z}_4, \tag{3.24}$$
$$k \in 2\mathbb{Z} \quad \text{if } Z \cong \mathbb{Z}_2. \tag{3.25}$$

Comparing to to the admissibility conditions (3.21), we deduce the following

Proposition 3.5. *The untwisted coset models corresponding to Lie algebra $\mathfrak{g} = \mathfrak{so}(2r)$, r odd, and any subalgebra $\mathfrak{h}$ do not have global gauge anomalies for*

$$k \in 4\mathbb{Z} \quad \text{if } Z \cong \mathbb{Z}_4 \tag{3.26}$$
$$k \in 2\mathbb{Z} \quad \text{if } Z \cong \mathbb{Z}_2. \tag{3.27}$$

The models with $\mathfrak{h} = \mathfrak{g}$ and $k \in 2\mathbb{Z}$ with odd $k/2$ for $Z \cong \mathbb{Z}_4$ or with k odd for $Z \cong \mathbb{Z}_2$ are anomalous.

3.4.2. Twisted case. There is only one nontrivial outer automorphism ω of $\mathfrak{so}(2r)$ with odd r, It exchanges the simple roots α_{r-1} and α_r and does not change the other ones. Thus, taking M and $\tilde{M}$ given by (3.20), we get

$$\omega(\tilde{M}) = \tilde{a} \, \omega(\lambda_r^\vee) = \tilde{a}\lambda_{r-1}^\vee = -\tilde{a}\lambda_r^\vee + \tilde{a}q^\vee = -\tilde{M} + \tilde{a}q^\vee \tag{3.28}$$

Table 1. Subgroups of $\tilde{Z}(Spin(2r)) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, r even, and their generators

Subgroup Z	Type	Generator(s) z_i
$\tilde{Z}$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	z_1, z_2
$Z_1 := \mathbb{Z}_2 \times \{1\}$	$\mathbb{Z}_2$	z_1
$Z_2 := \{1\} \times \mathbb{Z}_2$	$\mathbb{Z}_2$	z_2
Z_{diag}	$\mathbb{Z}_2$	$z_1 z_2$

where $q^\vee \in Q^\vee(D_r)$. The condition

$$e^{2i\pi \tilde{M}} \omega(e^{-2i\pi \tilde{M}}) = e^{4i\pi \tilde{a}\theta} \ \in \ Z \tag{3.29}$$

is always satisfied whatever the subgroup $Z \cong \mathbb{Z}_4$ or $\mathbb{Z}_2$ considered. From Eq. (2.9), we obtain

$$c_{\tilde{z}\omega(\tilde{z})^{-1}, z} = \exp\left[-i\pi k \frac{\tilde{a}ar}{2}\right] \tag{3.30}$$

and from Eqs. (3.20) and (3.28),

$$\exp\left[-2i\pi k \operatorname{tr}(M\omega(\tilde{M}))\right] = \exp\left[+i\pi k \frac{a\tilde{a}r}{2}\right] \tag{3.31}$$

so that the no-anomaly condition (2.7) always holds implying

Proposition 3.6. *The twisted coset models corresponding to Lie algebra $\mathfrak{g} = \mathfrak{so}(2r)$, r odd, subgroups $Z \cong \mathbb{Z}_4$ or $\mathbb{Z}_2$, and arbitrary subalgebras $\mathfrak{h}$ do not have global gauge anomalies.*

Case of r even. If r is even, the center $\tilde{Z} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ is generated by $z_1 = e^{2i\pi\theta_1}$ with $\theta_1 = \lambda_r^\vee = (1/2)\sum_{i=1}^r e_i$ and $z_2 = e^{2i\pi\theta_2}$ with $\theta_2 = \lambda_1^\vee = e_1$. The possible nontrivial subgroups are given in Table 1.

Here, $SO(2r) = Spin(2r)/Z_2$. The general form of M and $\tilde{M}$ in the Euclidian space spanned by vectors e_i is

$$
\begin{aligned}
M &= a_1\theta_1 + a_2\theta_2 = \left(\frac{a_1}{2} + a_2, \frac{a_1}{2}, \ldots, \frac{a_1}{2}\right), && a_1, a_2 \in \mathbb{Z} \text{ if } Z = \tilde{Z}, \\
& && a_1 \in \mathbb{Z}, a_2 = 0 \text{ if } Z = Z_1, \\
& && a_1 = 0, a_2 \in \mathbb{Z} \text{ if } Z = Z_2, \\
& && a_1 = a_2 \in \mathbb{Z} \text{ if } Z = Z_{\text{diag}}, \\
\tilde{M} &= \tilde{a}_1\theta_1 + \tilde{a}_2\theta_2 = \left(\frac{\tilde{a}_1}{2} + \tilde{a}_2, \frac{\tilde{a}_1}{2}, \ldots, \frac{\tilde{a}_1}{2}\right), && \tilde{a}_1, \tilde{a}_2 \in \mathbb{Z}.
\end{aligned}
\tag{3.32}
$$

In this case, the conditions for admissible levels of the WZW model are [13]:

$$
\begin{aligned}
k \in \mathbb{Z} \quad & \text{if } r/2 \text{ is even for any } Z, \\
& \quad r/2 \text{ is odd for } Z = Z_2, \\
k \in 2\mathbb{Z} \quad & \text{if } r/2 \text{ is odd and } Z = \tilde{Z}, Z_1 \text{ or } Z_{\text{diag}}.
\end{aligned}
\tag{3.33}
$$

3.4.3. Untwisted case. If $\omega = Id$ then the global gauge invariance is assured if

$$k \, \text{tr}(M\tilde{M}) = k \left(\frac{a_1\tilde{a}_1 r}{4} + \frac{a_1\tilde{a}_2}{2} + \frac{a_2\tilde{a}_1}{2} + a_2\tilde{a}_2 \right), \qquad (3.34)$$

is an integer. This holds for $k \in 2\mathbb{Z}$, whatever the subgroup considered. Comparing to the admissibility conditions (3.33), we deduce the following

Proposition 3.7. *The untwisted coset models corresponding to Lie algebra $\mathfrak{g} = \mathfrak{so}(2r)$, r even, and any subalgebra $\mathfrak{h}$ do not have global gauge anomalies if $k \in 2\mathbb{Z}$. The models with $\mathfrak{h} = \mathfrak{g}$ and with k odd for $r/2$ even and any nontrivial Z, or with k odd for $r/2$ odd and $Z = Z_2$, are anomalous.*

3.4.4. Twisted case. For $r > 4$, there is only one nontrivial outer automorphism ω of $\mathfrak{so}(2r)$, which is the same as the one described in the case of r odd: it interchanges the simple roots α_{r-1} and α_r. Thus, taking M and $\tilde{M}$ given by (3.32), we get

$$\begin{aligned}
\omega(\tilde{M}) &= \tilde{a}_1\omega(\theta_1) + \tilde{a}_2\omega(\theta_2) = \tilde{a}_1\lambda^{\vee}_{r-1} + \tilde{a}_2\lambda^{\vee}_1 \\
&= \tilde{a}_1\lambda^{\vee}_r + (\tilde{a}_1 + \tilde{a}_2)\lambda^{\vee}_1 + \tilde{a}_1 q^{\vee} = \tilde{M} + \tilde{a}_1\theta_2 + \tilde{a}_1 q^{\vee}
\end{aligned} \qquad (3.35)$$

where $q^{\vee} \in Q^{\vee}(D_r)$. The condition

$$e^{2i\pi\tilde{M}} \omega(e^{-2i\pi\tilde{M}}) = e^{-2i\pi\tilde{a}_1\theta_2} \in Z \qquad (3.36)$$

is satisfied for arbitrary $\tilde{a}_1$ if $Z = \tilde{Z}$ or Z_2, and for $\tilde{a}_1 = 0 \bmod 2$ if $Z = Z_1$ or Z_{diag}. For $Z = \tilde{Z}$, the expression for bihomomorphism (2.8) extracted from [12] reads:

$$c_{z_1^{m_1} z_2^{m_2}, z_1^{n_1} z_2^{n_2}} = \left(\pm \exp\left[\frac{i\pi k}{2} \right] \right)^{m_1 n_2 - m_2 n_1} \exp\left[-\frac{i\pi k}{2}(m_1 n_1 \frac{r}{2} + m_1 n_2 + m_2 n_1 + 2m_2 n_2) \right] \qquad (3.37)$$

for $m_i, n_i \in \mathbb{Z}$, with the sign $\pm$ corresponding to the two choices of WZ action functional. For the cyclic subgroups of $\tilde{Z}$, the above expression reduces to the one given by Eq. (2.9). We have:

$$c_{\tilde{z}\omega(\tilde{z})^{-1}, z} = (\pm 1)^{a_1\tilde{a}_1} \exp[i\pi k(a_1\tilde{a}_1 + a_2\tilde{a}_1)] \qquad (3.38)$$

and, from Eqs. (3.32) and (3.35),

$$\exp[-2i\pi k \, \text{tr}(M\omega(\tilde{M}))] = \exp\left[-i\pi k \left((\frac{r}{2} + 1)a_1\tilde{a}_1 + a_1\tilde{a}_2 + a_2\tilde{a}_1 \right) \right]. \qquad (3.39)$$

Hence the no-anomaly condition (2.7) requires that

$$(\pm 1)^{a_1\tilde{a}_1} \exp\left[-i\pi k \left(\frac{r}{2}a_1\tilde{a}_1 + a_1\tilde{a}_2 \right) \right] = 1 \qquad (3.40)$$

Considering each subgroup Z and the corresponding values of a_1, a_2, $\tilde{a}_1$, and $\tilde{a}_2$, and recalling the conditions (3.33) for the admissible levels of the corresponding WZW model, we deduce the

Proposition 3.8. *The twisted coset model corresponding to Lie algebra* $\mathfrak{g} = \mathfrak{so}(2r)$, *$r > 4$ even and arbitrary subalgebra do not have anomalies for $Z = \tilde{Z}$ (+ theory), Z_1 and Z_{diag} if k is even, and for $Z = Z_2$ if $k \in \mathbb{Z}$. The twisted models with $\mathfrak{h} = \mathfrak{g}$ for $Z = \tilde{Z}$ (- theory) and k even, and for $Z = \tilde{Z}$, Z_1 or Z_{diag} and k odd, $r/2 > 2$ even, are anomalous.*

For $r = 4$, there are more nontrivial outer automorphisms, because the symmetries of the diagram of D_4 form the permutation group S_3 (the well known "triality"). They belong to two conjugacy classes, the one composed of cyclic permutations of order 2,

$$\omega_1 : \alpha_3 \to \alpha_4 \to \alpha_3, \qquad \omega_2 : \alpha_1 \to \alpha_3 \to \alpha_1, \qquad \omega_3 : \alpha_1 \to \alpha_4 \to \alpha_1, \quad (3.41)$$

and the one containing cyclic permutations of order 3,

$$\omega_4 : \alpha_1 \to \alpha_4 \to \alpha_3 \to \alpha_1, \qquad \omega_4^{-1} : \alpha_1 \to \alpha_3 \to \alpha_4 \to \alpha_1. \qquad (3.42)$$

The no-anomaly conditions for twists ω and $\omega'\omega\omega'^{-1}$ in the same conjugacy class are related, as was discussed at the end of Sect. 2: they coincide if in the latter case subgroup $Z \subset \tilde{Z}$ is replaced by $\omega'(Z)$. The only exception is the case $Z = \tilde{Z}$ for odd k where one has also to interchange the $\pm$ theories if ω' is cyclic of order 2. It is straightforward to see that

$$\omega_4\omega_1\omega_4^{-1} = \omega_2, \quad \omega_4^{-1}\omega_1\omega_4 = \omega_3 \qquad (3.43)$$

and

$$\omega_4(Z_1) = Z_{diag}, \quad \omega_4(Z_2) = Z_1, \quad \omega_4(Z_{diag}) = Z_2. \qquad (3.44)$$

The results of Proposition 3.8 still hold for $r = 4$ and twist ω_1 and the ones for $r = 4$ and twists ω_2 and ω_3 follow from the latter by using the above remark (or by a direct calculation) giving:

Proposition 3.9. *The twisted coset models corresponding to Lie algebra $\mathfrak{g} = \mathfrak{so}(8)$ with twist ω_1 and arbitrary subalgebra do not have anomalies for $Z = \tilde{Z}$ (+ theory), Z_1 and Z_{diag} if k is even, and for $Z = Z_2$ if $k \in \mathbb{Z}$. The models with $\mathfrak{h} = \mathfrak{g}$ for $Z = \tilde{Z}$ (- theory) and k even, and for $Z = \tilde{Z}$, Z_1 or Z_{diag} and k odd are anomalous. The results for twist ω_2 (ω_3) are as the ones for twist ω_1 except for the permutation (3.44) of the subgroups $Z \to \omega_4(Z)$ ($Z \to \omega_4^{-1}(Z)$).*

For the cyclic outer automorphism ω_4 of order 3, taking M and $\tilde{M}$ given by Eqs. (3.32), we obtain:

$$\omega_4(\tilde{M}) = (\tilde{a}_1 + \tilde{a}_2)\theta_1 + \tilde{a}_1\theta_2 + \tilde{a}_1 q^\vee \qquad (3.45)$$

where $q^\vee \in Q^\vee(D_4)$. The condition

$$e^{2i\pi\tilde{M}} \omega_4(e^{-2i\pi\tilde{M}}) = \exp[2i\pi(-\tilde{a}_2\theta_1 + (\tilde{a}_2 - \tilde{a}_1)\theta_2)] \in Z \qquad (3.46)$$

is satisfied for arbitrary $\tilde{a}_1, \tilde{a}_2$ if $Z = \tilde{Z}$, and for $\tilde{a}_1 = \tilde{a}_2$, $\tilde{a}_2 = 0$, $\tilde{a}_1 = 0$, all *mod* 2, if $Z = Z_1$, Z_2 or Z_{diag} respectively. Expression (3.37) for the bihomomorphism gives here:

$$c_{\tilde{z}\omega_4(\tilde{z})^{-1}, z} = (\pm 1)^{-a_2\tilde{a}_2 + a_1\tilde{a}_1 - a_1\tilde{a}_2} \exp[i\pi k(a_1\tilde{a}_1 + a_2\tilde{a}_1 - a_2\tilde{a}_2)] \qquad (3.47)$$

From Eqs. (3.32) and (3.35),

$$\exp[-2i\pi k\,\mathrm{tr}(M\omega_4(\tilde{M}))] = \exp\left[-i\pi k\,(a_1\tilde{a}_1 + a_2\tilde{a}_1 + a_2\tilde{a}_2)\right] \tag{3.48}$$

so that the no-anomaly condition (2.7) becomes

$$(\pm 1)^{-a_2\tilde{a}_2 - a_1\tilde{a}_2 + a_1\tilde{a}_1} = 1. \tag{3.49}$$

Considering each subgroup Z and the corresponding values of a_1, a_2, $\tilde{a}_1$, and $\tilde{a}_2$, and recalling the admissible values (3.33) of the level, we deduce

Proposition 3.10. *The twisted coset models corresponding to Lie algebra* $\mathfrak{g} = \mathfrak{so}(8)$, *outer automorphism* ω_4 *and arbitrary subalgebra do not have anomalies for* $Z = \tilde{Z}$ *(+ theory) and* $Z = Z_1$, Z_2 *or* Z_{diag}. *The models with* $\mathfrak{h} = \mathfrak{g}$ *and* $Z = \tilde{Z}$ *(- theory) is anomalous.*

The results for the twist ω_4^{-1} may be deduced from the above proposition if we observe that ω_4^{-1} may be obtained from ω_4 by the conjugation by any cyclic outer automorphism ω' of order 2. Hence the conditions for the absence or the presence of anomalies for the theory twisted by ω_4^{-1} are as for the ones for the twist ω_4 except for the exchange of the $\pm$ theories for $Z = \tilde{Z}$ and k odd leading to

Proposition 3.11. *The twisted coset models corresponding to Lie algebra* $\mathfrak{g} = \mathfrak{so}(8)$, *outer automorphism* ω_4^{-1} *and arbitrary subalgebra do not have anomalies for* $Z = \tilde{Z}$ *($(-)^k$ theory) and* $Z = Z_1$, Z_2 *or* Z_{diag}. *The models with* $\mathfrak{h} = \mathfrak{g}$, *and* $Z = \tilde{Z}$ *($(-)^{k+1}$ theory) is anomalous.*

This may be confirmed by a direct calculation.

3.5. Case $\mathfrak{e}_6$.

The imaginary part $i\mathfrak{t}_\mathfrak{g}$ of the complexification of the Cartan subalgebra $\mathfrak{t}_\mathfrak{g}$ of $\mathfrak{g} = \mathfrak{e}_6$ may be identified with the subspace of $\mathbb{R}^7$ orthogonal to the vector $(1, \ldots, 1, 0)$, with the scalar product inherited from $\mathbb{R}^7$. The simple roots may be taken as $\alpha_i = e_i - e_{i+1}$ for $i = 1 \ldots 5$ and $\alpha_6 = (1/2)(-e_1 - e_2 - e_3 + e_4 + e_5 + e_6) + (1/\sqrt{2})e_7$, where e_i are the vectors of the canonical basis of $\mathbb{R}^7$. The center $\tilde{Z} \cong \mathbb{Z}_3$ is generated by $z = e^{2i\pi\theta}$ with $\theta = \lambda_5^\vee = (1/6)(e_1 + e_2 + e_3 + e_4 + e_5 - 5e_6) + (1/\sqrt{2})e_7$. The only nontrivial subgroup is $Z = \tilde{Z}$. The general form of M and $\tilde{M}$ in the Euclidian space spanned by vectors e_i is

$$M = a\theta = \left(\frac{a}{6}, \ldots, \frac{a}{6}, \frac{-5a}{6}, \frac{a}{\sqrt{2}}\right) \quad a \in \mathbb{Z},$$
$$\tilde{M} = \tilde{a}\theta = \left(\frac{\tilde{a}}{6}, \ldots, \frac{\tilde{a}}{6}, \frac{-5\tilde{a}}{6}, \frac{\tilde{a}}{\sqrt{2}}\right) \quad \tilde{a} \in \mathbb{Z}. \tag{3.50}$$

3.5.1. Untwisted case.

If $\omega = Id$ then the global gauge invariance is assured if

$$k\,\mathrm{tr}(M\tilde{M}) = k\frac{4a\tilde{a}}{3}, \tag{3.51}$$

is an integer. This holds for $k \in 3\mathbb{Z}$. Since all integer levels $k \in \mathbb{Z}$ are admissible [12,13], we deduce

Proposition 3.12. *The untwisted coset models corresponding to Lie algebra* $\mathfrak{g} = \mathfrak{e}_6$ *and arbitrary subalgebra* $\mathfrak{h}$ *do not have global gauge anomalies if* $k \in 3\mathbb{Z}$. *The models* $Z = \mathbb{Z}_3$, $\mathfrak{h} = \mathfrak{g}$ *and* $k \in \mathbb{Z}\backslash 3\mathbb{Z}$ *are anomalous.*

3.5.2. Twisted case. There is only one nontrivial outer automorphism ω of $\mathfrak{e}_6$, which exchanges the simple roots α_1 and α_2 with α_5 and α_4 and does not change the other ones. Thus, taking M and $\tilde{M}$ given by (3.50), we get

$$\omega(\tilde{M}) = \tilde{a}\omega(\lambda_5^{\vee}) = \tilde{a}\lambda_1^{\vee} = -\tilde{a}\lambda_5^{\vee} + \tilde{a}q^{\vee} = -\tilde{M} + \tilde{a}q^{\vee} \tag{3.52}$$

where $q^{\vee} \in Q^{\vee}(\mathfrak{e}_6)$. The condition

$$e^{2i\pi \tilde{M}} \omega(e^{-2i\pi \tilde{M}}) = e^{4i\pi \tilde{a}\theta} \in Z \tag{3.53}$$

is always satisfied for $Z = \tilde{Z}$. From Eq. (2.9), we obtain

$$c_{\tilde{z}\omega(\tilde{z})^{-1},z} = \exp[-2i\pi k \frac{4\tilde{a}a}{3}] \tag{3.54}$$

and from Eqs. (3.50) and (3.52),

$$\exp[-2i\pi k \, \mathrm{tr}(M\omega(\tilde{M}))] = \exp\left[+2i\pi k \frac{4a\tilde{a}}{3}\right] \tag{3.55}$$

so that the no-anomaly condition (2.7) always holds implying

Proposition 3.13. *The twisted coset models corresponding to Lie algebra $\mathfrak{g} = \mathfrak{e}_6$, subgroup $Z \cong \mathbb{Z}_3$ and arbitrary subalgebras $\mathfrak{h}$ do not have global gauge anomalies.*

3.6. Case $\mathfrak{e}_7$. The imaginary part $i\mathfrak{t}_{\mathfrak{g}}$ of the complexification of the Cartan subalgebra $\mathfrak{t}_{\mathfrak{g}}$ of $\mathfrak{g} = \mathfrak{e}_7$ may be identified with the subspace of $\mathbb{R}^8$ orthogonal to the vector $(1, \ldots, 1)$ with the simple roots $\alpha_i = e_i - e_{i+1}$ for $i = 1 \ldots 6$ and $\alpha_7 = (1/2)(-e_1 - e_2 - e_3 - e_4 + e_5 + e_6 + e_7 + e_8)$, where e_i are the vectors of the canonical basis of $\mathbb{R}^8$. The center $\tilde{Z} \cong \mathbb{Z}_2$ is generated by $z = e^{2i\pi\theta}$ with $\theta = \lambda_1^{\vee} = (1/4)(3, -1, \ldots, -1, 3)$. The only nontrivial subgroup is $Z = \tilde{Z}$. The general form of M and $\tilde{M}$ in the Euclidian space generated by e_i is

$$\begin{aligned} M = a\theta &= \left(\frac{3a}{4}, \frac{-a}{4}, \ldots, \frac{-a}{4}, \frac{3a}{4}\right) \quad a \in \mathbb{Z}, \\ \tilde{M} = \tilde{a}\theta &= \left(\frac{3\tilde{a}}{4}, \frac{-\tilde{a}}{4}, \ldots, \frac{-\tilde{a}}{4}, \frac{3\tilde{a}}{4}\right) \quad \tilde{a} \in \mathbb{Z}. \end{aligned} \tag{3.56}$$

Lie algebra $\mathfrak{e}_7$ does not have nontrivial outer automorphisms so that we may take $\omega = Id$. The global gauge invariance is then assured if the quantity

$$k \, \mathrm{tr}(M\tilde{M}) = k\frac{3a\tilde{a}}{2}, \tag{3.57}$$

is an integer. This holds for $k \in 2\mathbb{Z}$. The condition for admissible levels also requires in this case that $k \in 2\mathbb{Z}$ [12,13] so that we deduce:

Proposition 3.14. *The coset models corresponding to Lie algebra $\mathfrak{g} = \mathfrak{e}_7$ and any subalgebra $\mathfrak{h}$ do not have global gauge anomalies.*

3.7. Case $\mathfrak{g}_2$, $\mathfrak{f}_4$ and $\mathfrak{e}_8$. The center of the simply connected groups corresponding to Lie algebras $\mathfrak{g} = \mathfrak{g}_2$, $\mathfrak{f}_4$ or $\mathfrak{e}_8$ is trivial : $\tilde{Z} \cong \{1\}$ so that there are no nontrivial subgroups Z in that case and we infer:

Proposition 3.15. *The coset models corresponding to Lie algebras* $\mathfrak{g} = \mathfrak{g}_2$, $\mathfrak{f}_4$ *or* $\mathfrak{e}_8$ *and any subalgebra* $\mathfrak{h}$ *do not have global gauge anomalies.*

4. Regular Subalgebras

Looking back at the previous section, the global gauge anomalies of the coset models may appear only for $\mathfrak{g} = A_r$, D_r and $\mathfrak{e}_6$ in the untwisted case, and only for $\mathfrak{g} = D_r$ with even r in the twisted case (note that these are all simply laced Lie algebras). Now we have to specify the Lie subalgebra $\mathfrak{h}$ of a simple algebra $\mathfrak{g}$ to see in which cases the anomalies survive the restriction of the symmetry group. The first class of semisimple subalgebras that we shall consider are the regular ones, introduced by Dynkin in [11]. A Lie subalgebra $\mathfrak{h}$ of an algebra $\mathfrak{g}$ is called regular if, for a choice of the Cartan subalgebra $t_\mathfrak{g} \subset \mathfrak{g}$ (defined up to conjugation), its complexification is of the form

$$\mathfrak{h}^\mathbb{C} = t_\mathfrak{h}^\mathbb{C} \oplus \left(\underset{\alpha \in \Delta_\mathfrak{h} \subset \Delta_\mathfrak{g}}{\oplus} \mathbb{C} e_\alpha \right) \tag{4.1}$$

where $t_\mathfrak{h} \subset t_\mathfrak{g}$ is a Cartan subalgebra of $\mathfrak{h}$. Subalgebra $\mathfrak{h}$ is semisimple if $\alpha \in \Delta_\mathfrak{h}$ implies that $-\alpha \in \Delta_\mathfrak{h}$ and if $\alpha \in \Delta_\mathfrak{h}$ span $t_\mathfrak{h}^\mathbb{C}$. $\Delta_\mathfrak{h}$ is then the set of roots of $\mathfrak{h}$.

Construction of regular subalgebras. There is a nice diagrammatic method to obtain all the regular semisimple subalgebras of a given semisimple algebra (up to conjugation), proposed by Dynkin in [11] and summarized in [17]. We briefly describe it here:

1. Take the Dynkin diagram of the ambient algebra $\mathfrak{g}$, and adjoin to it a node corresponding to the lowest root $\delta = -\phi$ (negative of the highest root ϕ) of $\mathfrak{g}$, obtaining the extended Dynkin diagram of $\mathfrak{g}$.
2. Remove arbitrarily one root from this diagram, in order to obtain at most $r+1$ different diagrams, which may split into orthogonal subdiagrams.
3. Reapply the firsts two steps to each connected subdiagram obtained above, until no new diagram appears. This way one gets all the regular subalgebras $\mathfrak{h} \subset \mathfrak{g}$ of maximal rank.
4. Remove again an arbitrarily root from each diagram, and apply the full procedure to each connected subdiagram obtained this way (including the last step).

The algorithm stops when no root can be removed, hence one will obtain all the regular subalgebras of $\mathfrak{g}$.

4.1. Regular semisimple subalgebras of A_r. The semisimple regular subalgebras of A_r are given in [11] (Chapter II, Table 9) and have the form:

$$\mathfrak{h} = A_{r_1} \oplus \cdots \oplus A_{r_m}, \quad r_1 + 1 + \cdots + r_m + 1 \le r + 1 \tag{4.2}$$

The embedding of $\mathfrak{h}$ in $\mathfrak{g}$ realizing the ideals A_{r_i} as diagonal blocks in the matrices of A_r is unique up to an inner automorphism of A_r. Taking M and $\tilde{M}$ as given in Eqs. (3.2) and (3.3) we must require that $\tilde{M} + q^\vee \in it_\mathfrak{h}$, for some $q^\vee \in Q^\vee(A_r)$. Looking block by block, we obtain the conditions

$$\frac{\tilde{a}(r_i + 1)}{r + 1} \in \mathbb{Z} \quad \forall i = 1, \ldots, m \tag{4.3}$$

and that

$$\frac{\tilde{a}}{r+1} \in \mathbb{Z} \tag{4.4}$$

if the inequality in (4.2) is strict. The latter condition implies that (3.4) holds eliminating possible global gauge anomalies. We may then limit ourselves to the case when the inequality in (4.2) is saturated. This implies that For $i = 1, \ldots, m$, we may then rewrite conditions (4.3) as

$$\tilde{a}(r_i + 1) = q_i(r + 1) \quad q_i \in \mathbb{Z}. \tag{4.5}$$

In what follows, we shall denote by, respectively, $u_1 \wedge \cdots \wedge u_n$ and $u_1 \vee \cdots \vee u_n$ the greatest common divisor and the least common multiple of $u_1, \ldots, u_n$. Dividing both sides of Eq. (4.5) by $(r + 1) \wedge (r_i + 1)$, we get

$$\tilde{a} \frac{r_i + 1}{(r+1) \wedge (r_i + 1)} = q_i \frac{r+1}{(r+1) \wedge (r_i + 1)} \tag{4.6}$$

so that $\frac{r+1}{(r+1)\wedge(r_i+1)} | \frac{\tilde{a}(r_i+1)}{(r+1)\wedge(r_i+1)}$. Using the fact that $\frac{r+1}{(r+1)\wedge(r_i+1)}$ and $\frac{r_i+1}{(r+1)\wedge(r_i+1)}$ are relatively prime, we infer that $\frac{r+1}{(r+1)\wedge(r_i+1)} | \tilde{a}$, i.e. that

$$\tilde{a} \in \frac{r+1}{(r+1) \wedge (r_i+1)} \mathbb{Z} \quad \forall i = 1, \ldots, m \tag{4.7}$$

which leads, according to Proposition B.1 of Appendix B, to the condition

$$\tilde{a} \in \left(\frac{r+1}{(r+1) \wedge (r_1+1)} \vee \cdots \vee \frac{r+1}{(r+1) \wedge (r_m+1)} \right) \mathbb{Z} \tag{4.8}$$

This property can be reformulated, using Proposition B.2 of Appendix B, as

$$\tilde{a} \in \left(\frac{r+1}{(r+1) \wedge (r_1+1) \wedge \cdots \wedge (r_m+1)} \right) \mathbb{Z} \tag{4.9}$$

Since we assumed that $r_1 + 1 + \cdots + r_m + 1 = r + 1$, condition (4.9) may be simplified to

$$\tilde{a} \in \left(\frac{r+1}{(r_1+1) \wedge \cdots \wedge (r_m+1)} \right) \mathbb{Z} \tag{4.10}$$

In order to guarantee that the quantity (3.4) is an integer for every a and $\tilde{a}$, ensuring the global gauge invariance, it is enough to compute it for $a = 1$ and

$$\tilde{a} = \frac{r+1}{(r_1+1) \wedge \cdots \wedge (r_m+1)}. \tag{4.11}$$

Denoting $(r_1 + 1) \wedge \cdots \wedge (r_m + 1) = l$, and $r + 1 = pq$, the quantity (3.4) becomes

$$k \operatorname{tr}(M\tilde{M}) = k \frac{rq}{l} = kr \frac{q/(q \wedge l)}{l/(q \wedge l)}. \tag{4.12}$$

Finally, recalling that $l|(r + 1)$ and, consequently, $\frac{l}{q \wedge l}$ and r are relatively prime, we infer that the right hand side of Eq. (4.12) is be an integer if and only if

$$k \in \frac{l}{q \wedge l} \mathbb{Z}. \tag{4.13}$$

Taking into account condition (3.1) for admissible levels, we are now able to state

Proposition 4.1. *The untwisted coset models built with Lie algebra* $\mathfrak{g} = A_r$, *subgroup* $Z \cong \mathbb{Z}_p$ *for* $(r + 1) = pq$ *and any regular subalgebra* $\mathfrak{h} = A_{r_1} \oplus \cdots \oplus A_{r_m}$ *do not have global gauge anomalies for*

- $r_1 + 1 + \cdots r_m + 1 < r + 1 \quad k \in \begin{cases} 2\mathbb{Z} \text{ if } p \text{ even and } q \text{ odd} \\ \mathbb{Z} \text{ otherwise} \end{cases}$

- $r_1 + 1 + \cdots r_m + 1 = r + 1 \quad k \in \begin{cases} \dfrac{l}{q \wedge l}\mathbb{Z} \cap 2\mathbb{Z} \text{ if } p \text{ even and } q \text{ odd} \\ \dfrac{l}{q \wedge l}\mathbb{Z} \text{ otherwise} \end{cases}$

where $l = (r_1 + 1) \wedge \cdots \wedge (r_m + 1)$. *The other untwisted models with admissible levels are anomalous.*

Example 1: $\mathfrak{g} = A_4 = \mathfrak{su}(5)$. The center $\tilde{Z} \cong \mathbb{Z}_5$ of the corresponding group has only one nontrivial subgroup, $Z = \tilde{Z} \cong \mathbb{Z}_5$, so with $p = 5$ odd and $q = 1$ odd with the previous notations. The admissible levels are $k \in \mathbb{Z}$, according to (3.1). Following Proposition 3.1, the regular subalgebra $\mathfrak{h} = \mathfrak{g}$ leads to the condition $k \in 5\mathbb{Z}$ for non-anomalous models. Then, applying the last proposition above, the cases $\mathfrak{h} = A_1$, $A_1 \oplus A_1 \equiv 2A_1$, A_2 and A_3 leads to non-anomalous models for every $k \in \mathbb{Z}$, because here we have $r_1 + 1 + \cdots r_m + 1 < r + 1 = 5$. For $\mathfrak{h} = A_2 \oplus A_1$, we have an equality. However, $l = (r_1 + 1) \wedge (r_2 + 1) = 3 \wedge 2 = 1$, so $l/(l \wedge q) = 1$ and the model has no anomalies for every $k \in \mathbb{Z}$. Consequently, the only anomalous models corresponding to $\mathfrak{g} = A_4$ and $\mathfrak{h}$ regular are those with $\mathfrak{h} = \mathfrak{g}$, $Z = \tilde{Z}$ and $k \in \mathbb{Z} \backslash 5\mathbb{Z}$.

Example 2: $\mathfrak{g} = A_5 = \mathfrak{su}(6)$. Here the center $\tilde{Z} \cong \mathbb{Z}_6$ has three nontrivial subgroups : $Z \cong \mathbb{Z}_6$, $\mathbb{Z}_3$ and $\mathbb{Z}_2$ with the respective admissible levels $k \in 2\mathbb{Z}$, $\mathbb{Z}$ and $2\mathbb{Z}$. The models corresponding to the case $\mathfrak{h} = \mathfrak{g}$ will be non-anomalous for

$$k \in \begin{cases} 6\mathbb{Z} & \text{if } Z \cong \mathbb{Z}_6 \\ 3\mathbb{Z} & \text{if } Z \cong \mathbb{Z}_3 \\ 2\mathbb{Z} & \text{if } Z \cong \mathbb{Z}_2. \end{cases} \tag{4.14}$$

Regular subalgebras $\mathfrak{h} = A_1$, $2A_1$, A_2, $A_2 \oplus A_1$, A_3 and A_4 correspond to the strict inequality for ranks in the proposition above, so there will be no anomalies for these models with

$$k \in \begin{cases} 2\mathbb{Z} & \text{if } Z \cong \mathbb{Z}_6 \text{ or } \mathbb{Z}_2 \\ \mathbb{Z} & \text{if } Z \cong \mathbb{Z}_3. \end{cases} \tag{4.15}$$

Computation shows that $\mathfrak{h} = 2A_2$ leads to non-anomalous models for the same k as for $\mathfrak{h} = \mathfrak{g}$, and that the models corresponding to $\mathfrak{h} = A_3 \oplus A_1$ and to $3A_1$ have no anomalies for $k \in 2\mathbb{Z}$ if $Z \cong \mathbb{Z}_6$ or $\mathbb{Z}_2$ and for $k \in \mathbb{Z}$ if $Z \cong \mathbb{Z}_3$. Thus, the anomalous models corresponding to $\mathfrak{g} = A_5$ have either $\mathfrak{h} = \mathfrak{g}$ or $\mathfrak{h} = 2A_2$, where $k \in 2\mathbb{Z} \backslash 6\mathbb{Z}$ for $Z \cong \mathbb{Z}_6$ and $k \in \mathbb{Z} \backslash 3\mathbb{Z}$ for $Z \cong \mathbb{Z}_3$.

4.2. Regular semisimple subalgebras of D_r. The semisimple regular subalgebras of D_r are given in [11] (Chapter II, Table 9) and have the form:

$$\mathfrak{h} = A_{r_1} \oplus \cdots \oplus A_{r_m} \oplus D_{s_1} \oplus \cdots \oplus D_{s_n} \tag{4.16}$$

where $r_1 + 1 + \cdots + r_m + 1 + s_1 + \cdots s_n \leq r$.[3] The embedding of D_{s_i} subalgebras realizes them as diagonal blocks in D_r. Instead of giving an explicit embedding of subalgebras A_{r_i}, it is enough to see that A_l is trivially embedded in D_{l+1}, by sending the l simple roots $\alpha_i^{A_l}$ of A_l to the l first simple roots $\alpha_i^{D_{l+1}}$ of D_{l+1}. Then, the Serre construction allows us to reconstruct the full structure of A_l, embedded in D_{l+1}, which is then easily embedded in D_r as a diagonal block. The embedding of $\mathfrak{h}$ into $\mathfrak{g}$ described above is unique, up to inner automorphisms of $\mathfrak{g}$, except for even r if there are no D_{s_i} and $r_1 + 1 + \cdots + r_m + 1 = r$ with all r_i odd. In the latter case there is a second independent embedding of $A_{r_1} \oplus \cdots \oplus A_{r_m}$ into D_r that sends the simple roots of A_{r_m} to the last $r_m + 1$ simple roots of D_r omitting α_{r-1}. That embedding is related to the previous one by the outer automorphism ω of D_r that permutes roots α_{r_1} and α_r, but not by an inner automorphism. Recall that the coroot lattice $Q^\vee(D_r)$ is composed of vectors

$$q^\vee = \sum_{i=1}^{r} q_i^\vee e_i \quad \text{with} \quad q_i^\vee \in \mathbb{Z} \quad \text{and} \quad \sum_{i=1}^{r} q_i^\vee \in 2\mathbb{Z}. \tag{4.17}$$

Case of r odd. Taking M and $\tilde{M}$ as given in (3.20), we shall impose the condition $e^{2i\pi\tilde{M}} \in \tilde{H}$. On the Lie-algebra level, we have to show that for some $q^\vee \in Q^\vee(\mathfrak{g})$, $\tilde{M} + q^\vee$ belongs to $i\mathfrak{t}_\mathfrak{h}$. Looking block by block, we infer that

$$\frac{\tilde{a}(r_i + 1)}{2} \in \mathbb{Z}, \quad i = 1, \ldots, m, \tag{4.18}$$

and that

$$\frac{\tilde{a}}{2} \in \mathbb{Z} \tag{4.19}$$

if $r_1 + 1 + \cdots + r_m + 1 + s_1 + \cdots s_n < r$. The condition that the sum of components of vectors in $Q^\vee(\mathfrak{so}(2r))$ is even imposes the additional requirement that

$$\frac{\tilde{a}r}{2} \in 2\mathbb{Z}, \tag{4.20}$$

i.e. $\tilde{a} \in 4\mathbb{Z}$, in the absence of D_{s_i} components in $\mathfrak{h}$, [in that case conditions (4.18) and (4.19) imply already that $\tilde{a} \in 2\mathbb{Z}$]. Re-examining the quantity (3.23) which has to be an integer with the above restrictions in mind and taking into account the conditions for admissible levels, we deduce

Proposition 4.2. *The untwisted coset models built with Lie algebra* $\mathfrak{g} = \mathfrak{so}(2r)$, *r odd, and a regular subalgebra* $\mathfrak{h} = A_{r_1} \oplus \cdots \oplus A_{r_m} \oplus D_{s_1} \oplus \cdots \oplus D_{s_n}$ *do not have global gauge anomalies for the following cases*

- $r_1 + 1 + \cdots + r_m + 1 + s_1 + \cdots s_n = r$ *with all* r_i *odd and* $k \in \begin{cases} 4\mathbb{Z} & \text{if } Z \cong \mathbb{Z}_4 \\ 2\mathbb{Z} & \text{if } Z \cong \mathbb{Z}_2 \end{cases}$

- $\begin{aligned} &r_1 + 1 + \cdots + r_m + 1 + s_1 + \cdots s_n < r \text{ or} \\ &r_1 + 1 + \cdots + r_m + 1 + s_1 + \cdots s_n = r \text{ and some } r_i \text{ even} \end{aligned} \Big\} k \in \begin{cases} 2\mathbb{Z} & \text{if } Z \cong \mathbb{Z}_4 \\ \mathbb{Z} & \text{if } Z \cong \mathbb{Z}_2 \end{cases}$

The other untwisted models with admissible levels are not globally gauge invariant.

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[3] To take into account all the possible cases with this formula, we may need to consider D_2 instead of $2A_1$ and D_3 instead of A_3 to respect the inequality. See examples below.

Remark. In particular, the global gauge anomalies present if $\mathfrak{h} = \mathfrak{g}$ for $Z = \mathbb{Z}_4$ and $k \in 2\mathbb{Z}, k/2$ odd, or for $Z = \mathbb{Z}_2$ and k odd, disappear for $\mathfrak{h} = A_{r_1} \oplus \cdots \oplus A_{r_m} \oplus D_{s_1} \oplus \cdots \oplus D_{s_n}$ if $r_1 + 1 + \cdots + r_m + 1 + s_1 + \cdots s_n < r$ or if $r_1 + 1 + \cdots + r_m + 1 + s_1 + \cdots s_n = r$ with some r_i even. Note that if there no D_{s_i} and $r_1 + 1 + \cdots + r_m + 1 = r$ then all r_i cannot be odd.

Example: $\mathfrak{g} = D_5 = \mathfrak{so}(10)$. The admissible levels are $k \in 2\mathbb{Z}$ for $Z = \tilde{Z} \cong \mathbb{Z}_4$ and $k \in \mathbb{Z}$ for $Z \cong \mathbb{Z}_2$. According to Proposition 3.5, there are no gauge anomalies in the case $\mathfrak{h} = \mathfrak{g}$ for

$$k \in \begin{cases} 4\mathbb{Z} & \text{if } Z \cong \mathbb{Z}_4 \\ 2\mathbb{Z} & \text{if } Z \cong \mathbb{Z}_2. \end{cases} \tag{4.21}$$

For regular subalgebra $\mathfrak{h} = A_1, 2A_1 \cong D_2, A_2, A_3 \cong D_3, D_4$, the inequality on the ranks is strict so there are no anomalies for

$$k \in \begin{cases} 2\mathbb{Z} & \text{if } Z = \mathbb{Z}_4 \\ \mathbb{Z} & \text{if } Z = \mathbb{Z}_2. \end{cases} \tag{4.22}$$

In the case $\mathfrak{h} = A_4$ and $A_2 \oplus A_1$, the rank inequality is saturated and there is one r_i even, so (4.22) still gives the no-anomaly condition for k. D_5 admits also $D_3 \oplus D_2 \cong A_3 \oplus 2A_1$, $A_1 \oplus D_3 \cong A_3 \oplus A_1, A_2 \oplus D_2 \cong A_2 \oplus 2A_1, 2D_2 \cong 4A_1$ and $A_1 \oplus D_2 \cong 3A_1$, see [17] or the method described above, where only the left hand sides respect the inequality for ranks and should be used to extract the no-anomaly conditions. For $A_2 \oplus D_2, 2D_2$ and $A_1 \oplus D_2$ either the inequality for ranks is saturated and there is an even r_i or the inequality for ranks is strict, hence there are no anomalies for levels satisfying (4.22). Finally, for $D_3 \oplus D_2$ and $A_1 \oplus D_3$ the rank inequality is saturated by there is no even r_i and the gauge anomalies persist for $Z \cong \mathbb{Z}_4$ if $k \in 2\mathbb{Z}\backslash 4\mathbb{Z}$ and for $Z \cong \mathbb{Z}_2$ if k odd.

Case of r even. Taking M and $\tilde{M}$ as given in (3.32) and following the same reasoning as for the case of r odd, we get the same conditions:

$$\frac{\tilde{a}_1(r_i + 1)}{2} \in \mathbb{Z}, \quad i = 1, \ldots, m, \tag{4.23}$$

and, if $r_1 + 1 + \cdots + r_m + 1 + s_1 + \cdots s_n < r$,

$$\frac{\tilde{a}_1}{2} \in \mathbb{Z} \tag{4.24}$$

Additionally, if there are no D_{s_i} components in $\mathfrak{h}$, then

$$\tilde{a}_1 \frac{r}{2} + \tilde{a}_2 \in 2\mathbb{Z} \quad \text{for the 1}^{\text{st}} \text{ embedding}$$
$$\tilde{a}_1 \left(\frac{r}{2} - 1\right) + \tilde{a}_2 \in 2\mathbb{Z} \quad \text{for the 2}^{\text{nd}} \text{ embedding} \tag{4.25}$$

(the last two conditions differ only if all r_i are odd and the rank inequality is saturated because in the other cases $\tilde{a}_1$ has to be even). Examining the quantity (3.34) which has to be an integer with this information in mind and taking into account the admissibility conditions for the levels, we deduce

Proposition 4.3. *The untwisted coset models built with Lie algebra* $\mathfrak{g} = \mathfrak{so}(2r)$*, r even, and a regular subalgebra* $\mathfrak{h} = A_{r_1} \oplus \cdots \oplus A_{r_m} \oplus D_{s_1} \oplus \cdots \oplus D_{s_n}$ *do not have global gauge anomalies for the following cases*

- $r_1 + 1 + \cdots + r_m + 1 + s_1 + \cdots s_n = r$ *with all r_i odd*

$$k \in \begin{cases} 2\mathbb{Z} \text{ for any } Z \\ \mathbb{Z} \text{ if } r/2 \text{ even, no } D_{s_i} \text{ and } Z = Z_1 \quad \text{for the } 1^{\text{st}} \text{ embedding} \\ \mathbb{Z} \text{ if } r/2 \text{ even, no } D_{s_i} \text{ and } Z = Z_{diag} \text{ for the } 2^{\text{nd}} \text{ embedding} \end{cases}$$

- $\left. \begin{array}{l} r_1 + 1 + \cdots + r_m + 1 + s_1 + \cdots s_n < r \text{ or} \\ r_1 + 1 + \cdots + r_m + 1 + s_1 + \cdots s_n = r \text{ and some } r_i \text{ even} \end{array} \right\}$

$$k \in \begin{cases} 2\mathbb{Z} \text{ if } Z = \tilde{Z}, \ Z_1 \text{ or } Z_{diag} \\ \mathbb{Z} \text{ if } Z = Z_2 \\ \mathbb{Z} \text{ if } r/2 \text{ even, no } D_{s_i} \text{ and any } Z \end{cases}$$

The other untwisted models with admissible levels are not globally gauge invariant.

Remark. In particular, the global gauge anomalies present if $\mathfrak{h} = \mathfrak{g}$ for $Z = Z_2$ and k odd disappear for $\mathfrak{h} = A_{r_1} \oplus \cdots \oplus A_{r_m} \oplus D_{s_1} \oplus \cdots \oplus D_{s_n}$ if $r_1 + 1 + \cdots + r_m + 1 + s_1 + \cdots s_n < r$ or if $r_1 + 1 + \cdots + r_m + 1 + s_1 + \cdots s_n = r$ with some r_i even.

Example: $\mathfrak{g} = D_4 = \mathfrak{so}(8)$. Here r and $r/2$ are both even, so all levels $k \in \mathbb{Z}$ are admissible for all Z and there are no anomalies in the case $\mathfrak{h} = \mathfrak{g}$ for k even according to Proposition 3.7, whereas the cases with k odd are anomalous. The possible (proper, nontrivial) subalgebras $\mathfrak{h}$ are: A_1, A_2, $2A_1$, A_3 (the latter two with 2 inequivalent embeddings), D_2, D_3, $2D_2$ and $A_1 \oplus D_2$. Note that the two embeddings of $2A_1$ and that of D_2 are related by the outer automorphisms of D_4 and similarly for the two embeddings of A_3 and the one of D_3. For regular subalgebra $\mathfrak{h} = A_1$ or A_2, the inequality on ranks is strict and there are no D_{s_i} so there are no anomalies for $k \in \mathbb{Z}$ for all Z. For D_2 or D_3, the rank inequality is still strict and there are no anomalies for k even and all Z and for k odd and $Z = Z_2$. For $A_1 \oplus D_2$ or $2D_2$, the rank inequality is saturated and there are no anomalies for even k and any Z. Finally, for $2A_1$ or A_3 the rank inequality is saturated and there are no D_{s_i} so there are no anomalies for k even and any Z and for k odd and $Z = Z_1$ for the 1st embedding and $Z = Z_{diag}$ for the 2nd one.

Recall from Sect. 3.4.4 that the twisted coset models for $\mathfrak{g} = \mathfrak{so}(2r) = \mathfrak{h}$ with $r > 4$ even have gauge anomalies for $Z = \tilde{Z}$ (- theory) if k is even and for $Z = \tilde{Z}$, Z_1 or Z_{diag} if k is odd for $r/2$ even. These are the cases where the no-anomaly condition (3.40) may be violated. The restriction $e^{2i\pi \tilde{M}} \in \tilde{H}$ for $\mathfrak{h} = A_{r_1} \oplus \cdots \oplus A_{r_m} \oplus D_{s_1} \oplus \cdots \oplus D_{s_n}$ if $r_1 + 1 + \cdots + r_m + 1 + s_1 + \cdots s_n < r$ or if $r_1 + 1 + \cdots + r_m + 1 + s_1 + \cdots s_n = r$ with some r_i even imposes the condition $\tilde{a}_1 \in 2\mathbb{Z}$ removing the anomalies in the case $Z = \tilde{Z}$ (- theory) for k even and, if, additionally, there are no D_{s_i} components in $\mathfrak{h}$, also for $Z \neq Z_2$ and k odd. If there are no D_{s_i} and $r_1 + 1 + \cdots + r_m + 1 = r$ with all r_i odd then for k odd ($r/2$ even) the anomalies for $Z = \tilde{Z}$ are removed for the + theory in the case of the 1st embedding and for the − theory in the case of the 2nd embedding, and for $Z = Z_1$, Z_{diag} in the case of both embeddings. We obtain this way

Proposition 4.4. *The twisted coset models built with Lie algebra* $\mathfrak{g} = \mathfrak{so}(2r)$*, $r > 4$ even, and a regular subalgebra* $\mathfrak{h} = A_{r_1} \oplus \cdots \oplus A_{r_m} \oplus D_{s_1} \oplus \cdots \oplus D_{s_n}$ *do not have global gauge anomalies for the following cases*

- $r_1 + 1 + \cdots + r_m + 1 + s_1 + \cdots s_n = r$ with all r_i odd

$$k \in \begin{cases} 2\mathbb{Z} \text{ if } Z = \tilde{Z} \text{ (+ theory) or } Z = Z_1, \ Z_{diag} \\ \mathbb{Z} \quad \text{if } Z = Z_2 \\ \mathbb{Z} \quad \text{if } r/2 \text{ even, no } D_{s_i} \text{ and } Z = \tilde{Z} \text{ (+ theory) for the } 1^{st} \text{ embedding} \\ \mathbb{Z} \quad \text{if } r/2 \text{ even, no } D_{s_i} \text{ and } Z = \tilde{Z} \text{ (- theory) for the } 2^{nd} \text{ embedding} \\ \mathbb{Z} \quad \text{if } r/2 \text{ even, no } D_{s_i} \text{ and } Z = Z_1, \ Z_{diag} \end{cases}$$

- $\left. \begin{array}{l} r_1 + 1 + \cdots + r_m + 1 + s_1 + \cdots s_n < r \text{ or} \\ r_1 + 1 + \cdots + r_m + 1 + s_1 + \cdots s_n = r \text{ and some } r_i \text{ even} \end{array} \right\}$

$$k \in \begin{cases} 2\mathbb{Z} \text{ if } Z = \tilde{Z}, \ Z_1 \text{ or } Z_{diag} \\ \mathbb{Z} \quad \text{if } Z = Z_2 \\ \mathbb{Z} \quad \text{if } r/2 \text{ even, no } D_{s_i} \text{ and } Z = \tilde{Z}, \ Z_1, \ Z_{diag} \end{cases}$$

The other twisted models with admissible levels are not globally gauge invariant.

The above results also hold for the coset model with $\mathfrak{g} = \mathfrak{so}(8)$ with twist ω_1, see (3.41). Hence, for $\mathfrak{h} = A_1$ or A_2 there are no gauge anomalies. For D_2 or D_3 there are no anomalies if k is even for any Z and if k is odd for $Z = Z_2$. For $A_1 \oplus D_2$ or $2D_2$ there are no anomalies if k is even for $Z = \tilde{Z}$ (+ theory) or $Z = Z_1, \ Z_2, \ Z_{diag}$ or if k is odd and $Z = Z_2$. Finally, for $2A_1$ or A_3 there are no anomalies for $Z = \tilde{Z}$ (+ theory for the 1st embedding, − theory for the 2nd one) and for $Z = Z_1, \ Z_2, \ Z_{diag}$. In accordance with the discussion of Sect. 3.4.4, we may obtain the result for twist ω_2 from the one for ω_1 by applying the permutation $Z \to \omega_4(Z)$ induced by the outer automorphism ω_4 on the cyclic subgroups of $\tilde{Z}$, see Eqs. (3.44), and on the one $\mathfrak{h} \to \omega_4(\mathfrak{h})$ on subalgebras (modulo inner automorphisms) induced by the action (3.42) of ω_4 on simple roots:

$$\omega_4(A_1) = A_1, \quad \omega_4(A_2) = A_2, \quad \omega_4((2A_1)^{(1)}) = (2A_1)^{(2)}, \quad \omega_4((2A_1)^{(2)}) = D_2,$$
$$\omega_4(A_3^{(1)}) = A_3^{(2)}, \quad \omega_4(A_3^{(2)}) = D_3, \quad \omega_4(D_2) = (2A_1)^{(1)}, \quad \omega_4(D_3) = A_3^{(1)},$$
$$\omega_4(2D_2) = 2D_2, \quad \omega_4(A_1 \oplus D_2) = A_1 \oplus D_2. \tag{4.26}$$

where the superscript (i), $i = 1, 2$, labels the independent embeddings. Similarly, the result for twist ω_3 from the one for ω_1 by applying the inverse permutations $Z \to \omega_4^{-1}(Z)$ and $\mathfrak{h} \to \omega_4^{-1}(\mathfrak{h})$. For twists ω_4, ω_4^{-1}, the the remaining gauge anomalies are lifted if $\mathfrak{h} = A_1$ or A_2 imposing the restrictions $\tilde{a}_1, \tilde{a}_2 \in 2\mathbb{Z}$ resulting in

Proposition 4.5. *The twisted coset models built with Lie algebra $\mathfrak{g} = \mathfrak{so}(8)$ with twist ω_4 have global gauge anomalies for regular subalgebras $\mathfrak{h} = 2A_1, A_3, D_2, D_3, 2D_2, A_1 \oplus D_2$ and $Z = \tilde{Z}$ (- theory). The other cases of coset models with Lie algebra $\mathfrak{so}(8)$ and twist ω_4 are without anomalies.*

Similarly

Proposition 4.6. *The twisted coset models built with Lie algebra $\mathfrak{g} = \mathfrak{so}(8)$ with twist ω_4^{-1} have global gauge anomalies for regular subalgebras $\mathfrak{h} = 2A_1, A_3, D_2, D_3, 2D_2$ and $A_1 \oplus D_2$ and $Z = \tilde{Z}$ $((-)^k$ theory). The other cases of coset models with Lie algebra $\mathfrak{so}(8)$ and twist ω_4^{-1} are without anomalies.*

4.3. Regular semisimple subalgebras of $\mathfrak{e}_6$. In this case with fixed rank $r = 6$, one can establish a complete list of regular semisimple subalgebras, up to conjugation, with an embedding, however, that is not explicit [11,17]. We shall only need the embedding of simple roots in the ambient algebra which is enough to reconstruct the full embedding using the Serre construction. The element M and $\tilde{M}$ will be described employing the explicit realization of the coweight and coroot lattices of $\mathfrak{e}_6$,

$$P^\vee(\mathfrak{e}_6) = \left\{ \left(\frac{a}{6} + q_1, \ldots, \frac{a}{6} + q_6, \frac{b}{\sqrt{2}} \right) \left| \begin{array}{l} a, b, q_1, \ldots, q_6 \in \mathbb{Z} \\ a + q_1 + \cdots + q_6 = 0 \\ a + b \in 2\mathbb{Z} \end{array} \right. \right\} \quad (4.27)$$

and the coroot lattice $Q^\vee(\mathfrak{e}_6)$ is defined the same way but adding the condition $a \in 3\mathbb{Z}$. We shall consider only the untwisted coset models because the twisted ones are non-anomalous, see Proposition 3.13. Taking M and $\tilde{M}$ in $P^\vee(\mathfrak{e}_6)$ with the corresponding coefficients, the quantity (3.51) becomes

$$k \, \text{tr}(M\tilde{M}) = k \frac{a\tilde{a}}{3} + m, \quad \text{with } m \in \mathbb{Z} \quad (4.28)$$

Now, specifying a subalgebra $\mathfrak{h} \subset \mathfrak{e}_6$ and requiring that $e^{2i\pi \tilde{M}} \in \tilde{Z} \cap \tilde{H}$, two possibilities arise: if one can show that $\tilde{a} \in 3\mathbb{Z}$ then the previous quantity is an integer for every $k \in \mathbb{Z}$ and all the corresponding coset models are globally gauge invariant. Otherwise, if there exist an element $\tilde{M}$ such that $\tilde{a} \notin 3\mathbb{Z}$, then we have to require $k \in 3\mathbb{Z}$ to have a globally gauge invariant coset model, and the other coset models are anomalous. Before examining the anomaly problem for every regular subalgebra of $\mathfrak{e}_6$, one can make four remarks:

- if there are no anomalies for a given subalgebra $\mathfrak{h}$ of $\mathfrak{e}_6$ ($\tilde{a} \in 3\mathbb{Z}$), then the regular subalgebras that are smaller (and will be obtained from the Dynkin diagram of $\mathfrak{h}$ by the procedure described above) lead also to the condition $\tilde{a} \in 3\mathbb{Z}$, inheriting it from $\mathfrak{h}$. In other words, the regular subalgebra with no anomalies protects the cases of its regular subalgebras. Consequently, we will look only at the cases where the anomalies are present and treat the problem by decreasing rank.
- Among the regular subalgebras generated by the algorithm described at the beginning of Sect. 4, many can still be mapped into each other by the conjugations that normalize $\mathfrak{t}_{\mathfrak{e}_6}$ (and induce on it Weyl group transformations) and, as a result, they lead to the same condition for the absence of anomalies. We may then consider only one regular subalgebra in each class of subalgebras related by Weyl group transformations. In particular, there are Weyl group transformations that permute the simple roots α_i and $\delta = -\phi$ according to the symmetries of the extended Dynkin diagrams (see, e.g., Appendix B of [18]) and they permit to restrict the count of regular subalgebras.
- The subalgebras related by the outer automorphism of $\mathfrak{e}_6$ lead to the same no-anomaly condition, see the remark at the end of Sect. 2.
- Since $e^{2i\pi \tilde{M}} \in \tilde{Z} \cap \tilde{H}$ if and only if $\tilde{M} \in P^\vee(\mathfrak{g})$ and $\tilde{M} + q^\vee \in i\mathfrak{t}_{\mathfrak{h}} \subset i\mathfrak{t}_{\mathfrak{g}}$ for some $q^\vee \in Q^\vee(\mathfrak{e}_6)$, it is enough to check the no-anomaly condition (2.12) only for $\tilde{M} \in P^\vee(\mathfrak{g})$ perpendicular to the orthogonal complement $i\mathfrak{t}_{\mathfrak{h}}^\perp$ of $i\mathfrak{t}_{\mathfrak{h}}$ in $i\mathfrak{t}_{\mathfrak{g}}$.

We now consider the regular semisimple subalgebras, beginning by those of rank 6 and then decreasing the rank. Subspace $i\mathfrak{t}_{\mathfrak{h}}^\perp$ (which is small for high ranks) is computed for each subalgebra and we look at the consequences of the condition $\tilde{M} \perp i\mathfrak{t}_{\mathfrak{h}}^\perp$ on $\tilde{M}$.

Table 2. $\mathrm{i}\mathfrak{t}_{\mathfrak{h}}^{\perp}$ for the regular subalgebras of $\mathfrak{e}_6$ of rank 5 and 4 and consequences for $\tilde{a}$; the simple roots α_i of $\mathfrak{e}_6$ and its lowest root δ are used to generate the regular subalgebras [17]

$\mathfrak{h}$	Simple roots of $\mathfrak{h}$	Basis of $\mathrm{i}\mathfrak{t}_{\mathfrak{h}}^{\perp}$	$\tilde{M}$
D_5	$\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_6$	$\left(1, 1, 1, 1, 1, -5, 3\sqrt{2}\right)$	$\tilde{a} \in 3\mathbb{Z}$
$A_3 \oplus 2A_1$	$\alpha_1, \alpha_2, \alpha_3 \oplus \delta \oplus \alpha_5$	$(1, 1, 1, 1, -2, -2, 0)$	$\tilde{a} \in 3\mathbb{Z}$
$A_4 \oplus A_1$	$\alpha_1, \alpha_2, \alpha_3, \alpha_4 \oplus \delta$	$(1, 1, 1, 1, 1, -5, 0)$	$\tilde{a} \in 3\mathbb{Z}$
A_5	$\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$	$(0, 0, 0, 0, 0, 0, 1)$	$\tilde{a} \in 2\mathbb{Z}$
$2A_2 \oplus A_1$	$\alpha_1, \alpha_2 \oplus \alpha_4, \alpha_5 \oplus \alpha_6$	$\left(1, 1, 1, -1, -1, -1, 3\sqrt{2}\right)$	$\tilde{a} \in \mathbb{Z}$
$2A_2$	$\alpha_1, \alpha_2 \oplus \alpha_4, \alpha_5$	$(1, 1, 1, -1, -1, -1, 0)$ $(0, 0, 0, 0, 0, 0, 1)$	$\tilde{a} \in 2\mathbb{Z}$

Upon using the protection property and the Weyl transformations described above, as well as the outer automorphism of $\mathfrak{e}_6$, only a few cases have to be treated. The explicit computation is given in Table 2. The subalgebras of rank 6 are not represented because we have $\mathfrak{t}_{\mathfrak{h}}^{\perp} = \emptyset$, so there is no supplementary condition for $\tilde{M}$ and there are always anomalies if $k \notin 3\mathbb{Z}$. Only subalgebras of rank 5 and 4 have potential anomalies, the ones of lower ranks being protected by a possible inclusion into non-anomalous subalgebras.

We are thus able to state

Proposition 4.7. *The untwisted coset models built with Lie algebra $\mathfrak{g} = \mathfrak{e}_6$ and any regular subalgebra $\mathfrak{h}$ do not have global gauge anomalies for every $k \in \mathbb{Z}$, except for the cases $\mathfrak{h} = \mathfrak{e}_6, A_5 \oplus A_1, 3A_2$, of rank 6, $A_5, 2A_2 \oplus A_1$, of rank 5, and $2A_2$ of rank 4, where the only globally gauge invariant models are those with $k \in 3\mathbb{Z}$.*

5. R-Subalgebras and S-Subalgebras

The regular subalgebras are not the only possible Lie subalgebras for a given ambient Lie algebra. We can use them, however, to classify all the remaining ones. Let $\mathfrak{h}$ be a semisimple subalgebra of $\mathfrak{g}$. Let $\mathcal{R}(\mathfrak{h})$ be a minimal regular subalgebra of $\mathfrak{g}$ containing $\mathfrak{h}$ (up to conjugation). If $\mathcal{R}(\mathfrak{h}) = \mathfrak{g}$, then $\mathfrak{h}$ is called an S-subalgebra. Otherwise, it is called an R-subalgebra. For the exceptional simple algebras, the classification of R- and S-subalgebras has been achieved by Dynkin in [11]. The case of other simple algebras was discussed in [19] with less explicit results. In this section, we first treat completely the case of non-regular subalgebras of the exceptional Lie algebra $\mathfrak{g} = \mathfrak{e}_6$ which may have anomalies and then we consider some examples of non-regular subalgebras of classical Lie algebras.

Dynkin index. Consider a simple Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$ of a semisimple Lie algebra $\mathfrak{g}$ and the corresponding embedding ι. The relation

$$\mathrm{tr}_{\mathfrak{g}}(\iota(X))^2 = j \, \mathrm{tr}_{\mathfrak{h}} X^2 \quad \text{for} \quad X \in \mathfrak{h} \tag{5.1}$$

where the invariant quadratic forms $\mathrm{tr}_{\mathfrak{g}}$ and $\mathrm{tr}_{\mathfrak{h}}$ have the normalizations described in the beginning of Sect. 2, defines the scalar factor j (independent of X), called Dynkin index, which is always an integer [11]. Moreover, j is invariant under composition of ι with inner (and outer) automorphisms of $\mathfrak{g}$, so that it depends on the class of equivalent embeddings.

5.1. Simple nonregular subalgebras of $\mathfrak{e}_6$.

Subalgebras of rank 1. According to Dynkin, the subalgebra $\mathfrak{h} = A_1$ can be embedded in several different ways in $\mathfrak{e}_6$, as regular, R- and S-subalgebra and the embedding ι is fully characterized by the embedding of the simple coroot $\alpha^\vee$ of A_1. Recall the compatibility condition for $\tilde{M}$ in the anomaly problem

$$e^{2i\pi \tilde{M}} \in \tilde{H} \cap \tilde{Z} \subseteq \mathcal{Z}(\tilde{H}), \tag{5.2}$$

where $\mathcal{Z}(\tilde{H}) = \{1, e^{2i\pi\iota(\lambda^\vee)}\}$ with $\lambda^\vee = \frac{1}{2}\alpha^\vee$ is the center of $\tilde{H}$ which is either trivial (if $1 = e^{2i\pi\iota(\lambda^\vee)}$ and $\tilde{H} \cong SO(3)$) or is isomorphic to $\mathbb{Z}_2$ (if $1 \neq e^{2i\pi\iota(\lambda^\vee)}$ and $\tilde{H} \cong SU(2)$). Looking at the embedding of $\lambda^\vee$ in $\mathfrak{e}_6$, three possibilities can occur

1. If $\iota(\lambda^\vee) \notin P^\vee(\mathfrak{e}_6)$ then $\tilde{Z} \cap \tilde{H} = \{1\}$ and $\tilde{M}$ is a coroot of $\mathfrak{e}_6$, so the quantity (3.51) is always an integer and there are no anomalies for this model.
2. If $\iota(\lambda^\vee) \in Q^\vee(\mathfrak{e}_6)$ then $\tilde{M}$ is still only a coroot of $\mathfrak{e}_6$, and there are no anomalies too.
3. If $\iota(\lambda^\vee) \in P^\vee(\mathfrak{e}_6)\backslash Q^\vee(\mathfrak{e}_6)$ then anomalies are possible and we have to check that the quantity (3.51) is an integer for $\tilde{M} = \iota(\lambda^\vee)$ looking at the corresponding value for $\tilde{a}$, see Eq. (4.28).

The explicit embeddings are given in [11] (Chapter III, Table 18), and the computation of the intersection with the roots of $\mathfrak{e}_6$ is done in Table 3 for each subalgebra of rank 1: the possibility 3 never occurs, so there are no anomalies for the corresponding coset models for any $k \in \mathbb{Z}$.

Simple S-subalgebras of rank > 1. Following [11] (Chapter IV, Table 24), there exist four S-subalgebras of $\mathfrak{e}_6$ of rank > 1: $\mathfrak{h} = A_2, \mathfrak{g}_2, C_4$ and $\mathfrak{f}_4$. For the cases $\mathfrak{g}_2$ and $\mathfrak{f}_4$, the center of the corresponding group is $\mathcal{Z}(\tilde{H}) \cong \{1\}$. Then $\tilde{M}$ can be only a coroot of $\mathfrak{e}_6$ and the quantity (3.51) is always an integer. For the two remaining cases, the explicit embedding is still given in [11], and the strategy is the same as for rank one: we look how the generating element $\iota(\lambda^\vee)$ of $\mathcal{Z}(\tilde{H})$ intersects with the lattices of $\mathfrak{e}_6$ and check which possibility occurs among those listed in the case of rank one (except that we would also have to check that for the low multiples of $\lambda^\vee$ if $\iota(\lambda^\vee)$ were not in $Q^\vee(\mathfrak{g})$). The results are described in Table 4 from which we infer that there are no gauge anomalies for all simple S-subalgebras of $\mathfrak{e}_6$.

Simple R-subalgebras of rank > 1. We only need to look at the R-subalgebras $\mathfrak{h}$ with potential anomalies. Indeed, the subalgebra $\mathcal{R}(\mathfrak{h})$ is regular, so has been already treated. If $\mathcal{R}(\mathfrak{h})$ corresponds to a model without anomalies, then it protects also the R-subalgebra $\mathfrak{h}$ included in it and there will be no anomalies for the model built with $\mathfrak{h}$. The list of the R-subalgebras of $\mathfrak{e}_6$ is given in [11] (Chapter IV, Table 25), but without explicit embedding. There remain five cases with potential anomalies: $\mathfrak{h} = A_2$, with $\mathcal{R}(\mathfrak{h}) = A_5, 2A_2, 3A_2$, and $\mathfrak{h} = A_3$ or C_3 with $\mathcal{R}(\mathfrak{h}) = A_5$. If $\mathcal{R}(\mathfrak{h})$ is simple, then the embedding of $\mathfrak{h}$ in $\mathcal{R}(\mathfrak{h})$ is given in [17] (Table XIII), considering $\mathfrak{h}$ as an S-subalgebra of $\mathcal{R}(\mathfrak{h})$.

If $\mathcal{R}(\mathfrak{h})$ is only semisimple, the problem of the embedding is treated in [20], where several inequivalent embeddings of $\mathfrak{h}$ in $\mathfrak{e}_6$ appear. For the $\mathfrak{h} = A_2$ and $\mathcal{R}(\mathfrak{h}) = 3A_2$, the two inequivalent embeddings are the following, denoting by $\tilde{\alpha}_1^\vee$ and $\tilde{\alpha}_2^\vee$ the simple coroots of A_2.

$$\iota_1(\tilde{\alpha}_1^\vee) = \alpha_1^\vee + \alpha_5^\vee + \delta^\vee \qquad \iota_2(\tilde{\alpha}_1^\vee) = \alpha_1^\vee + \alpha_4^\vee + \delta^\vee \tag{5.3}$$

$$\iota_1(\tilde{\alpha}_2^\vee) = \alpha_2^\vee + \alpha_4^\vee + \alpha_6^\vee \qquad \iota_2(\tilde{\alpha}_2^\vee) = \alpha_2^\vee + \alpha_5^\vee + \alpha_6^\vee \tag{5.4}$$

Table 3. The embedding of element $\lambda^\vee$ for rank 1 subalgebras and its intersection with the lattices of $\mathfrak{e}_6$

$\mathcal{R}(\mathfrak{h})$	Index	$\iota(\lambda^\vee)$	Compatibility
A_1	1	$\left(0,0,0,0,0,0,\frac{1}{\sqrt{2}}\right)$	$\notin P^\vee(\mathfrak{e}_6)$
$2A_1$	2	$\left(\frac{1}{2},0,0,0,0,\frac{-1}{2},\frac{1}{\sqrt{2}}\right)$	$\notin P^\vee(\mathfrak{e}_6)$
$3A_1$	3	$\left(\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{-1}{4},\frac{-1}{4},\frac{-1}{4},\frac{3}{2\sqrt{2}}\right)$	$\notin P^\vee(\mathfrak{e}_6)$
A_2	4	$\left(0,0,0,0,0,0,\sqrt{2}\right)$	$\in Q^\vee(\mathfrak{e}_6)$
$A_2 \oplus A_1$	5	$\left(\frac{1}{2},0,0,0,0,\frac{-1}{2},\sqrt{2}\right)$	$\notin P^\vee(\mathfrak{e}_6)$
$A_2 \oplus 2A_1$	6	$\left(\frac{1}{2},\frac{1}{2},0,0,\frac{-1}{2},\frac{-1}{2},\sqrt{2}\right)$	$\notin P^\vee(\mathfrak{e}_6)$
$2A_2$	8	$\left(1,0,0,0,0,-1,\sqrt{2}\right)$	$\in Q^\vee(\mathfrak{e}_6)$
$2A_2 \oplus A_1$	9	$\left(\frac{3}{4},\frac{1}{4},\frac{1}{4},\frac{-1}{4},\frac{-1}{4},\frac{-3}{4},\frac{5}{2\sqrt{2}}\right)$	$\notin P^\vee(\mathfrak{e}_6)$
A_3	10	$\left(\frac{1}{2},0,0,0,0,\frac{-1}{2},\frac{3}{\sqrt{2}}\right)$	$\notin P^\vee(\mathfrak{e}_6)$
$A_3 \oplus A_1$	11	$\left(\frac{1}{2},\frac{1}{2},0,0,\frac{-1}{2},\frac{-1}{2},\frac{3}{\sqrt{2}}\right)$	$\notin P^\vee(\mathfrak{e}_6)$
$A_3 \oplus 2A_1$	12	$\left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{-1}{2},\frac{-1}{2},\frac{-1}{2},\frac{3}{\sqrt{2}}\right)$	$\in Q^\vee(\mathfrak{e}_6)$
A_4	20	$\left(1,0,0,0,0,-1,2\sqrt{2}\right)$	$\in Q^\vee(\mathfrak{e}_6)$
$A_4 \oplus A_1$	21	$\left(1,\frac{1}{2},0,0,\frac{-1}{2},-1,2\sqrt{2}\right)$	$\notin P^\vee(\mathfrak{e}_6)$
D_4	28	$\left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{-1}{2},\frac{-1}{2},\frac{-1}{2},\frac{5}{\sqrt{2}}\right)$	$\in Q^\vee(\mathfrak{e}_6)$
$D_5(a_1)$	30	$\left(1,\frac{1}{2},0,0,\frac{-1}{2},-1,\frac{5}{\sqrt{2}}\right)$	$\notin P^\vee(\mathfrak{e}_6)$
A_5	35	$\left(\frac{3}{2},\frac{1}{2},0,0,\frac{-1}{2},\frac{-3}{2},\frac{5}{\sqrt{2}}\right)$	$\notin P^\vee(\mathfrak{e}_6)$
$A_5 \oplus A_1$	36	$\left(\frac{3}{2},\frac{1}{2},\frac{1}{2},\frac{-1}{2},\frac{-1}{2},\frac{-3}{2},\frac{5}{\sqrt{2}}\right)$	$\in Q^\vee(\mathfrak{e}_6)$
D_5	60	$\left(\frac{3}{2},\frac{1}{2},\frac{1}{2},\frac{-1}{2},\frac{-1}{2},\frac{-3}{2},\frac{7}{\sqrt{2}}\right)$	$\in Q^\vee(\mathfrak{e}_6)$
$\mathfrak{e}_6(a_1)$	84	$\left(2,1,0,0,-1,-2,4\sqrt{2}\right)$	$\in Q^\vee(\mathfrak{e}_6)$
$\mathfrak{e}_6$	156	$\left(\frac{5}{2},\frac{3}{2},\frac{1}{2},\frac{-1}{2},\frac{-3}{2},\frac{-5}{2},\frac{11}{\sqrt{2}}\right)$	$\in Q^\vee(\mathfrak{e}_6)$

Table 4. The embedding of element $\lambda^\vee$ for simple S-subalgebras of $\mathfrak{e}_6$ and its intersection with the lattices

$\mathfrak{h}$	$\mathcal{R}(\mathfrak{h})$	Index	$\iota(\lambda^\vee)$	Compatibility
A_2	$\mathfrak{e}_6$	9	$\left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{-1}{2},\frac{-1}{2},\frac{-1}{2},\frac{3}{\sqrt{2}}\right)$	$\in Q^\vee(\mathfrak{e}_6)$
C_4	$\mathfrak{e}_6$	1	$\left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{-1}{2},\frac{-1}{2},\frac{-1}{2},\frac{1}{\sqrt{2}}\right)$	$\in Q^\vee(\mathfrak{e}_6)$

where we have exchanged $\alpha_4^\vee$ and $\alpha_5^\vee$. The other possible exchanges are equivalent to ι_1 or ι_2 [20]. For $\mathcal{R}(\mathfrak{h}) = 2A_2$, the two embeddings are given by similar formulas but with omission of $\alpha_6^\vee$ and $\delta^\vee$. Again, in order to find $Z(\tilde{H}) \cap \tilde{Z}$, we have to check how the generating element $\iota(\lambda^\vee)$ of $\mathcal{Z}(\tilde{H})$ intersects with the lattices of $\mathfrak{e}_6$. An explicit calculation is done in Table 5, and this time potential anomalies occur. Then, looking at the value of $\tilde{a}$ for $\tilde{M} = \iota(\lambda^\vee)$, we deduce an, eventually more restrictive, condition on level k required to avoid the anomalies (to exclude the anomalies in the case of $A_3 \subset A_5$, we also have to observe that $\iota(2\lambda^\vee) \in Q^\vee(\mathfrak{e}_6)$).

Table 5. The embedding of element $\lambda^\vee$ for simple R-subalgebras of $\mathfrak{e}_6$ and its intersection with the lattices

$\mathfrak{h}$	$\mathcal{R}(\mathfrak{h})$	Index	$\iota(\lambda^\vee)$	Compatibility	$\tilde{a}$
A_2	$2A_2(\iota_1)$	2	$\left(\frac{1}{3}, \frac{1}{3}, \frac{-2}{3}, \frac{2}{3}, \frac{-1}{3}, \frac{-1}{3}, 0\right)$	$\notin P^\vee(\mathfrak{e}_6)$	
A_2	$2A_2(\iota_2)$	2	$\left(\frac{1}{3}, \frac{1}{3}, \frac{-2}{3}, \frac{1}{3}, \frac{1}{3}, \frac{-2}{3}, 0\right)$	$\in P^\vee(\mathfrak{e}_6)\backslash Q^\vee(\mathfrak{e}_6)$	2
A_2	$3A_2(\iota_1)$	3	$(0, 0, -1, 1, 0, 0, 0)$	$\in Q^\vee(\mathfrak{e}_6)$	
A_2	$3A_2(\iota_2)$	3	$\left(0, 0, -1, \frac{2}{3}, \frac{2}{3}, \frac{-1}{3}, 0\right)$	$\notin P^\vee(\mathfrak{e}_6)$	
A_2	A_5	5	$\left(\frac{2}{3}, \frac{2}{3}, \frac{-1}{3}, \frac{2}{3}, \frac{-1}{3}, \frac{-4}{3}, 0\right)$	$\in P^\vee(\mathfrak{e}_6)\backslash Q^\vee(\mathfrak{e}_6)$	4
A_3	A_5	2	$\left(\frac{1}{2}, \frac{1}{2}, \frac{-1}{2}, \frac{1}{2}, \frac{-1}{2}, \frac{-1}{2}, 0\right)$	$\notin P^\vee(\mathfrak{e}_6)$	
C_3	A_5	1	$\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{-1}{2}, \frac{-1}{2}, \frac{-1}{2}, 0\right)$	$\notin P^\vee(\mathfrak{e}_6)$	

In case of potential anomalies, the explicit value of $\tilde{a}$ that enters quantity (4.28) is given

This way, we obtain the general result for simple nonregular subalgebras of $\mathfrak{e}_6$

Proposition 5.1. *The untwisted coset models with* $\mathfrak{g} = \mathfrak{e}_6$ *and any simple, nonregular subalgebra* $\mathfrak{h}$ *do not have global gauge anomalies for* $k \in \mathbb{Z}$ *except for the R-subalgebras* $\mathfrak{h} = A_2$ *with* $\mathcal{R}(\mathfrak{h}) = A_5$ *and* $\mathfrak{h} = A_2$ *with* $\mathcal{R}(\mathfrak{h}) = 2A_2$ *embedded via* ι_2. *For those subalgebras, the global gauge invariance requires that* $k \in 3\mathbb{Z}$.

5.2. Semisimple nonregular subalgebras of $\mathfrak{e}_6$. Let $\mathfrak{h}$ be a semisimple subalgebra of $\mathfrak{e}_6$:

$$\mathfrak{h} = \bigoplus_{i=1}^{n} \mathfrak{h}_i \tag{5.5}$$

where the $\mathfrak{h}_i$ are simple, and the corresponding subgroups are denoted by $\tilde{H}_i$. The case $n = 1$ has been already treated above, so we now deal with $n \geq 2$. First, suppose that one of the $\mathfrak{h}_i$ considered as a simple subalgebra leads to anomalies: there exists $\tilde{M}_i$ such that $e^{2i\pi \tilde{M}_i} \in \tilde{H}_i \cap \tilde{Z}$ which imposes $k \in 3\mathbb{Z}$ to ensure that the quantity (3.51) is integral. Then, taking $\tilde{M} = \tilde{M}_i$ but now embedded in $\mathfrak{h}$, we shall still have to impose $k \in 3\mathbb{Z}$ to have a globally gauge invariant model with semisimple Lie algebra $\mathfrak{h}$. In other words, semisimple algebras composed of simple ideals with at least one leading to anomalies are also anomalous. However, the inverse is not true: one can have a semisimple subalgebra corresponding to an anomalous model with all its simple ideals without any anomaly. For example, the model with regular subalgebra $2A_2$ of $\mathfrak{e}_6$ is anomalous for $k \in \mathbb{Z}\backslash 3\mathbb{Z}$ whereas the one with A_2 (still regular) is globally gauge invariant for every $k \in \mathbb{Z}$. Thus we need to check all the cases where all the simple ideals correspond to models without anomaly. To do that, we need to consider the elements $\sum_{i=1}^{n} \alpha_i \iota(\lambda_i^\vee)$ where $\alpha_i \in \mathbb{Z}$ and λ_i are the generating elements of the center of the $\tilde{H}_i$, which have all been described above in the simple case (Tables 3, 4, 5), and $\iota : \mathfrak{h} \to \mathfrak{e}_6$ is the embedding. Comparing how these elements are compatible with the coroot and coweight lattices of $\mathfrak{e}_6$, the anomaly problem is reduced to the three possibilities described in the simple case 5.1.

S-subalgebras. In [11] (Chapter V, Table 39) one can find all the S-subalgebra of $\mathfrak{e}_6$ and their including relations. It turns out that subalgebra $\mathfrak{h} = \mathfrak{g}_2 \oplus A_2$ (with the explicit embedding given in [11], Chapter V, Table 35) leads to an anomaly if $k \in \mathbb{Z}\backslash 3\mathbb{Z}$, and that the other semisimple nonsimple S-subalgebras of $\mathfrak{e}_6$ are protected.

Table 6. Semisimple nonsimple R-subalgebras of $\mathfrak{e}_6$ with possible anomalies and the conditions on k required for their absence

$\mathfrak{h}$	$\mathcal{R}(\mathfrak{h})$	$\{\mathcal{R}(\mathfrak{h}_i)\}$	Indices	No anomaly for
$A_2 \oplus A_1$	$A_5 \oplus A_1$	A_5, A_1	2,1	$k \in 3\mathbb{Z}$
$A_3 \oplus A_1$	$A_5 \oplus A_1$	A_5, A_1	2,1	$k \in \mathbb{Z}$
$C_3 \oplus A_1$	$A_5 \oplus A_1$	A_5, A_1	1,1	$k \in \mathbb{Z}$
$A_1 \oplus A_1$	$A_5 \oplus A_1$	A_5, A_1	35,1	$k \in \mathbb{Z}$
$(A_2(\iota_1) \oplus A_1) \oplus A_1$	$A_5 \oplus A_1$	A_5, A_1	2,3,1	$k \in \mathbb{Z}$
$(A_2(\iota_2) \oplus A_1) \oplus A_1$	$A_5 \oplus A_1$	A_5, A_1	2,3,1	$k \in 3\mathbb{Z}$
$(2A_1) \oplus A_1$	$A_5 \oplus A_1$	A_5, A_1	8,3,1	$k \in \mathbb{Z}$
$A_1 \oplus (2A_2)$	$A_2 \oplus (2A_2)$	$A_2, 2A_2$	4,1,1	$k \in 3\mathbb{Z}$
$A_2 \oplus A_2(\iota_1)$	$A_2 \oplus (2A_2)$	$A_2, 2A_2$	1,2	$k \in \mathbb{Z}$
$A_2 \oplus A_2(\iota_2)$	$A_2 \oplus (2A_2)$	$A_2, 2A_2$	1,2	$k \in 3\mathbb{Z}$
$A_1 \oplus A_2(\iota_2)$	$A_2 \oplus (2A_2)$	$A_2, 2A_2$	4,2	$k \in 3\mathbb{Z}$
$A_1 \oplus A_1 \oplus A_2$	$A_2 \oplus A_2 \oplus A_2$	A_2, A_2, A_2	4,4,1	$k \in \mathbb{Z}$
$A_2(\iota_2) \oplus A_1$	A_5	A_5	2,3	$k \in 3\mathbb{Z}$
$A_1 \oplus A_2(\iota_2)$	$A_1 \oplus (2A_2)$	$A_1, 2A_2$	1,2	$k \in 3\mathbb{Z}$
$A_1 \oplus A_1 \oplus A_2$	$A_1 \oplus A_2 \oplus A_2$	A_1, A_2, A_2	1,4,1	$k \in \mathbb{Z}$

R-subalgebras. The end of [17] proposes a method to construct all the semisimple R-subalgebras: the idea is to take the semisimple S-subalgebras of the semisimple regular subalgebras of $\mathfrak{e}_6$, treating each semisimple ideal independently. The semisimple S-subalgebras are described for the classical algebras up to rank 6 in [17], which is enough to construct all the semisimple R-subalgebras of $\mathfrak{e}_6$. However, we only need to treat the R-subalgebras $\mathfrak{h}$ where the regular subalgebras $\mathcal{R}(\mathfrak{h})$ lead to an anomaly problem, because the other cases are protected against anomalies. The computation is given in Table 6, using the fact that one ideal leads to an anomaly or computing the elements of the center as described before. Note that for the nonsimple S-subalgebra $A_2 \oplus A_1 \subset A_5$, A_2 is actually embedded in $A_2 \oplus A_2$ [17], so the question of the two inequivalent embeddings ι_1 and ι_2 arises also here, as in 5.1. Working by decreasing rank, we have excluded some algebras from this Table since they are protected by the ones of higher rank that do not have anomalies.

Putting all that together, we obtain the following result:

Proposition 5.2. *The untwisted coset models with $\mathfrak{g} = \mathfrak{e}_6$ and any nonregular nonsimple semisimple subalgebra $\mathfrak{h}$ do not have global gauge anomaly for $k \in \mathbb{Z}$, except for the S-subalgebra $\mathfrak{h} = \mathfrak{g}_2 \oplus A_2$ and the R-subalgebras appearing in Table 6 with the condition $k \in 3\mathbb{Z}$ which exhibit global gauge anomaly for $k \in \mathbb{Z}\backslash 3\mathbb{Z}$.*

5.3. Examples of nonregular subalgebras of classical Lie algebras. The semisimple nonregular subalgebra of classical algebra has been classified explicitly in [17] only up to rank 6. The general classification proposed by Dynkin in [19] is less explicit and does not allow us to treat the anomaly problem in a general form as for regular subalgebras. Here we only give some example of classical algebras, but the method is always the same once the explicit embedding of a subalgebra is known : as for $\mathfrak{e}_6$, we need to look how the embedding of the generating element of the center of the considered subalgebra is compatible with the coroot lattice of the ambient algebra.

Nonregular semisimple subalgebras of A_4. The coroot lattice of A_4 is given by

$$P^\vee(A_4) = \left\{ \left(\frac{a}{5} + q_1, \ldots, \frac{a}{5} + q_4, -\frac{4a}{5} - q_1 - \cdots - q_4 \right) \middle| a, q_1, \ldots q_4 \in \mathbb{Z} \right\} \quad (5.6)$$

and the coweight lattice $Q^\vee(A_4)$ is given by the same formula but with $a = 0$. According to [17], A_4 admits two S-subalgebras which are simple: A_1 and B_2. For $\mathfrak{h} = A_1$, the embedding of the generating element $\lambda^\vee$ of the center of the corresponding group is given by

$$\iota(\lambda^\vee) = (2, 1, 0, -1, -2) \in Q^\vee(A_4) \tag{5.7}$$

so the quantity $k \, \mathrm{tr}(M\tilde{M})$ will be integral for every $k \in \mathbb{Z}$ and there will be no anomaly for this model. For $\mathfrak{h} = B_2$, one have

$$\iota(\lambda^\vee) = (1, 0, 0, 0, -1) \in Q^\vee(A_4) \tag{5.8}$$

which leads to the same conclusion. As we have seen in the regular case, all regular subalgebras of A_4 (except A_4) leads to non-anomalous models. We immediately conclude that all the R-subalgebra of A_4 are protected by their regular $\mathcal{R}(\mathfrak{h})$, so there is also no anomaly for these models. Finally, the only anomalous models corresponding to $\mathfrak{g} = A_4$ and an arbitrary semisimple subalgebra are those with $\mathfrak{h} = \mathfrak{g}$, $Z = \tilde{Z} \cong Z_5$ and $k \in \mathbb{Z}\backslash 5\mathbb{Z}$.

S-subalgebras of A_5. The coroot lattice of A_5 is given by

$$P^\vee(A_5) = \left\{ \left(\frac{a}{6} + q_1, \ldots, \frac{a}{6} + q_5, -\frac{5a}{6} - q_1 - \cdots - q_5 \right) \middle| a, q_1, \ldots q_5 \in \mathbb{Z} \right\} \tag{5.9}$$

and the coweight lattice $Q^\vee(A_5)$ is given by the same formula but with $a = 0$. According to [17], A_5 admits six S-subalgebras : $A_1, A_2, A_3, C_3, A_1 \oplus A_1$ and $A_2 \oplus A_1$. For $\mathfrak{h} = A_1$, one has

$$\iota(\lambda^\vee) = \left(\frac{5}{2}, \frac{3}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}, -\frac{5}{2} \right), \tag{5.10}$$

see Table VI of [17], whereas for $\mathfrak{h} = A_2$, A_3 and C_3, one has

$$\iota(\lambda^\vee) = \left(\frac{2}{3}, \frac{2}{3}, -\frac{1}{3}, \frac{2}{3}, -\frac{1}{3}, -\frac{1}{3} \right), \left(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2} \right),$$
$$\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2} \right), \tag{5.11}$$

respectively, see the last 3 entries of Table 5 above. In all 4 cases, $\iota(\lambda^\vee) \in P^\vee(A_5)\backslash Q^\vee(A_5)$. Taking $\iota(\lambda^\vee) = \tilde{M}$ with $\tilde{a} = 3, 4, 3, 3$, respectively, and appropriate $\tilde{q}_i$, and $M \in P^\vee(A_5)$ such that $e^{2i\pi M} \in Z \cong \mathbb{Z}_p$, we obtain

$$\mathrm{tr}(M\tilde{M}) = \frac{5a\tilde{a}}{p} + n, \tag{5.12}$$

where $n \in \mathbb{Z}$. There will be no anomaly for k such that $k \, \mathrm{tr}(M\tilde{M}) \in \mathbb{Z}$. For $\tilde{a} = 3$, this imposes on k the same restrictions that the admissibility conditions (3.1), so that the untwisted coset theories corresponding to the S-subalgebras $\mathfrak{h} = A_1, A_3, C_3 \subset A_5$ do not have anomalies. For the S-subalgebra $\mathfrak{h} = A_2$, we obtain the non-anomalous models with admissible levels for

$$k \in \begin{cases} \mathbb{Z} \cap 2\mathbb{Z} = 2\mathbb{Z} & \text{if } Z \cong \mathbb{Z}_2 \\ 3\mathbb{Z} & \text{if } Z \cong \mathbb{Z}_3 \\ 3\mathbb{Z} \cap 2\mathbb{Z} = 6\mathbb{Z} & \text{if } Z \cong \mathbb{Z}_6 \end{cases} \tag{5.13}$$

The other untwisted models corresponding to the S-subalgebra $\mathfrak{h} = A_2 \subset A_5$ and non-trivial subgroups Z are anomalous.

There are no conceptual or technical difficulties to obtain the no-anomaly conditions on k for other subalgebras of A_5, and also for other classical algebra $\mathfrak{g}$, once the embeddings are known, but there is no general result so each case has to be treated separately. The previous examples show that different anomaly conditions could appear according to the subalgebra considered.

6. Conclusions

We have studied above the conditions for the absence of global gauge anomaly in the coset models of conformal field theory derived from WZW models with connected simple compact groups $G = \tilde{G}/Z$ as the targets by gauging a subgroup of the rigid adjoint or twisted-adjoint symmetries $G \ni g \mapsto h g \omega(h)^{-1} \in G$, where ω is a, possible trivial, automorphism of G. The full group of such symmetries is equal to $\tilde{G}/Z^\omega$, where Z^ω is the maximal subgroup of the center $\tilde{Z}$ of the universal covering group $\tilde{G}$ of G for which the (twisted) adjoint action is well defined. We considered both the coset models where the full group $\tilde{G}/Z^\omega$ was gauged and the ones where the gauging concerned only a closed connected subgroup of $\tilde{G}/Z^\omega$. Global gauge anomalies obstructing the invariance of the Feynman amplitudes of the theory under "large" gauge transformations non-homotopic to unity may appear only for non-simply connected groups G corresponding to Lie algebras $\mathfrak{g}$ of types A_r, D_r and $\mathfrak{e}_6$ (that are all simply-laced). Using the results [11,17,20] on the classification of semisimple Lie subalgebras of simple Lie algebras, we obtained a complete list of non-anomalous coset models (without boundaries) for groups G with the Lie algebra A_r, D_r or $\mathfrak{e}_6$ if the gauged symmetry subgroup $\subset \tilde{G}/Z^\omega$ corresponds to a regular Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$ or, for $\mathfrak{g} = \mathfrak{e}_6$, to any semisimple Lie subalgebra. The global gauge anomalies that appear in the other coset model should render them inconsistent on the quantum level, as was argued in [3].

Appendices

A. Gauge-Invariance Condition

Here we prove the equivalence between relations (2.3) and (2.4). From Eq. (2.1), we have to show that

$$
\frac{k}{4\pi} \int \mathrm{tr} \left(({}^h g^{-1} d^h g) \omega({}^h A) + (d^h g)^h g^{-1h} A + {}^h g^{-1h} A^h g \omega({}^h A) \right)
$$
$$
- \frac{k}{4\pi} \int \mathrm{tr} \left((g^{-1} dg) \omega(A) + (dg) g^{-1} A + g^{-1} A g \omega(A) \right)
$$
$$
= -\frac{k}{4\pi} \int_\Sigma \mathrm{tr} \left(g^{-1} dg \omega(h^{-1} dh) + (dg) g^{-1} h^{-1} dh + g^{-1}(h^{-1} dh) g \omega(h^{-1} dh) \right)
$$

(A.1)

But

$$
\frac{k}{4\pi} \int \mathrm{tr} \left(({}^h g^{-1} d^h g) \omega({}^h A) + (d^h g)^h g^{-1h} A + {}^h g^{-1 Ah} g \omega({}^h A) \right)
$$
$$
- \frac{k}{4\pi} \int \mathrm{tr} \left((g^{-1} dg) \omega(A) + (dg) g^{-1} A + g^{-1} A g \omega(A) \right)
$$

$$= \frac{k}{4\pi} \int \mathrm{tr} \left(\omega(h)g^{-1}h^{-1} \, d(hg\omega(h)^{-1})\omega(hAh^{-1} + hdh^{-1}) \right.$$
$$+ d(hg\omega(h)^{-1})\omega(h)g^{-1}h^{-1}(hAh^{-1} + hdh^{-1})$$
$$\left. + \omega(h)g^{-1}h^{-1}(hAh^{-1} + hdh^{-1})hg\omega(h)^{-1}\omega(hAh^{-1} + hdh^{-1}) \right)$$
$$- \frac{k}{4\pi} \int \mathrm{tr} \left((g^{-1}dg)\omega(A) + (dg)g^{-1}A + g^{-1}Ag\omega(A) \right)$$
$$= \frac{k}{4\pi} \int \mathrm{tr} \left(g^{-1}((h^{-1}dh)g + (dg) - g\omega(h^{-1}dh))\omega(A + (dh^{-1})h) \right.$$
$$+ ((dh)g + h(dg) - hg\omega(h^{-1}dh))g^{-1}h^{-1}(hAh^{-1} + hdh^{-1})$$
$$\left. + \omega(h)g^{-1}h^{-1}(hAh^{-1} + hdh^{-1})hg\omega(h)^{-1}\omega(hAh^{-1} + hdh^{-1}) \right)$$
$$- \frac{k}{4\pi} \int \mathrm{tr} \left((g^{-1}dg)\omega(A) + (dg)g^{-1}A + g^{-1}Ag\omega(A) \right)$$
$$= \frac{k}{4\pi} \int \mathrm{tr} \left(g^{-1}(h^{-1}dh)g\omega(A - h^{-1}dh) - (g^{-1}dg)\omega(h^{-1}dh) - \omega(h^{-1}dh)\omega(A) \right.$$
$$+ (h^{-1}dh)(A - h^{-1}dh) - (dg)g^{-1}(h^{-1}dh) - g\omega(h^{-1}dh)g^{-1}(A - h^{-1}dh)$$
$$- g^{-1}(h^{-1}dh)g\omega(A - h^{-1}dh)$$
$$\left. - g^{-1}(A - h^{-1}dh)g\omega(h^{-1}dh) - g^{-1}(h^{-1}dh)g\omega(h^{-1}dh) \right)$$
$$= \frac{k}{4\pi} \int \mathrm{tr} \left(-(g^{-1}dg)\omega(h^{-1}dh) \right.$$
$$\left. - (dg)g^{-1}(h^{-1}dh) - g^{-1}(h^{-1}dh)g\omega(h^{-1}dh) \right) \tag{A.2}$$

which establishes identity (A.1).

B. Arithmetical Properties

For $a, b \in \mathbb{Z}$ we denote $a \wedge b$ the greatest common divisor and $a \vee b$ the least common multiple of a and b.

Proposition B.1. *Let $k_1, \ldots k_s \in \mathbb{Z}$ and $k \in \mathbb{Z}$ such that $\forall i = 1 \ldots s, k \in k_i\mathbb{Z}$, then*

$$k \in (k_1 \vee \cdots \vee k_s)\,\mathbb{Z} \tag{B.1}$$

The demonstration is done by induction on s.

Proposition B.2. *Let $k_1, \ldots k_s \in \mathbb{Z}$ such that $\forall i = 1 \ldots s, k_i = \dfrac{a}{a \wedge b_i}$ with $a, b_1, \ldots,$ $b_s \in \mathbb{Z}$, then*

$$k_1 \vee \cdots \vee k_s = \frac{a}{a \wedge b_1 \wedge \cdots \wedge b_s} \tag{B.2}$$

The demonstration is done by induction on s:

- $s = 2$

$$\frac{a}{a \wedge b_1} \vee \frac{a}{a \wedge b_2} = \frac{\dfrac{a^2}{(a \wedge b_1)(a \wedge b_2)}}{\dfrac{a}{a \wedge b_1} \wedge \dfrac{a}{a \wedge b_2}} \tag{B.3}$$

using $ab = (a \wedge b)(a \vee b)$. Then we can rewrite the denominator:

$$\frac{a}{a \wedge b_1} \wedge \frac{a}{a \wedge b_2} = a \frac{(a \wedge b_2) \wedge (a \wedge b_1)}{(a \wedge b_2)(a \wedge b_1)}, \qquad \text{(B.4)}$$

thus

$$\frac{a}{a \wedge b_1} \vee \frac{a}{a \wedge b_2} = \frac{a}{a \wedge b_1 \wedge a \wedge b_2} = \frac{a}{a \wedge b_1 \wedge b_2}. \qquad \text{(B.5)}$$

- Suppose the result true for $s \geq 2$, the result for $s+1$ is trivially true, using the induction hypothesis at rank s, then 2.

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