Ann. Henri Poincaré 19 (2018), 709–741

1424-0637/18/030709-33
published online February 19, 2018
https://doi.org/10.1007/s00023-018-0657-7

Annales Henri Poincaré

Bulk–Edge Correspondence for Two-Dimensional Floquet Topological Insulators

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Abstract. Floquet topological insulators describe independent electrons on a lattice driven out of equilibrium by a time-periodic Hamiltonian, beyond the usual adiabatic approximation. In dimension two, such systems are characterized by integer-valued topological indices associated with the unitary propagator, alternatively in the bulk or at the edge of a sample. In this paper, we give new definitions of the two indices, relying neither on translation invariance nor on averaging, and show that they are equal. In particular, weak disorder and defects are intrinsically taken into account. Finally, indices can be defined when two driven samples are placed next to one another either in space or in time and then shown to be equal. The edge index is interpreted as a quantized pumping occurring at the interface with an effective vacuum.

1. Introduction

Bulk–edge correspondence is a crucial concept in the context of quantum Hall effect and topological insulators. From the topological point of view, the bulk properties of an infinite sample can be deduced by looking at the gapless modes, propagating at the edge of a sample with boundary, and *vice versa* [5,8,9]. This duality is commonly observed in physical systems where both bulk and edge index are well understood. Sometimes it is even assumed to fill the lack of interpretation of a bulk invariant, the physics at the edge being usually more intuitive. In any case, a proof of this correspondence is as much a mathematical challenge as a helpful identity for physics.

In analogy with topological insulators, it was recently realized that topological phases could arise in periodically driven systems. The initial proposal was to induce topology on a two-dimensional sample through a time-periodic perturbation of a trivial material, e.g., by irradiation of graphene [10,15] or semiconductor quantum wells by microwaves [13], but it was then realized that

a large class of time-periodic Hamiltonians of independent electrons may support topological properties, as long as the unitary propagator after one period is gapped [11,19].

For samples that are also space-translation invariant, Rudner et al. [19] defined a topological bulk index that is integer valued and equal to the number of edge modes that appear in the spectrum for associated dynamics on a strip geometry. Moreover, an explicit definition of the edge index and a proof of the bulk–edge correspondence was proposed in [19], but with the extra assumption that the unitary propagator is also periodic in time. Recently, the requirement of spatial invariance has been dropped and similar result was obtained for disordered systems [7], or [21] where averaging over fluxes threading the sample has been used. An interacting model was proposed in [12]. Finally, the bulk invariant has been generalized to the cases with time-reversal or chiral symmetry [3,4,6], and bulk–edge correspondence for one-dimensional chiral systems was studied in [1].

In this paper, we give new definitions for both the bulk and edge index that do not require space-translation invariance of the Hamiltonian, nor averaging, and show a general proof of the bulk–edge correspondence. We only assume that the Hamiltonian is local (short range), periodic and regular enough in time. The construction works as soon as the bulk one-period propagator has a spectral gap. If space-translation invariance is present though, the definition generalizes the existing one. If not, it applies to weakly disordered systems, see Remark 3.13. Moreover, in this approach the edge index is interpreted as a quantized pumping of charges after one cycle. Exploiting a duality between space and time (see Sect. 3.3), we show that this pumping actually occurs at the interface with an effective vacuum, computed from the original Hamiltonian and depending on the spectral gap under consideration.

The concept of topological pump and the study of periodically driven system in this context are not new, but until recently the adiabatic hypothesis has been always implied. From Thouless' original work [22] to more recent and abstract considerations [16], the driving was always assumed to be slow enough in order to use the adiabatic theorem. In particular, the time-dependent spectrum of the Hamiltonian is the relevant object of interest, and usually a persistent gap all along the driving is assumed. We stress that Floquet topological insulators and in particular the present work are not placed in this frame. Here the driving can be arbitrary and we do not make any assumption on the spectrum of the Hamiltonian, but only on the corresponding propagator. Finally note that this notion of non-adiabatic quantized pumping has already been observed in [21].

The paper is organized as follows. First Sect. 2 describes the context of Floquet topological insulators for which the construction applies. The main results are then stated in Sect. 3. The definition of bulk and edge indices, as well as the bulk–edge correspondence, is done in two steps. Inspired by [19], we first assume that the bulk propagator is periodic in time. The edge invariant is interpreted as charge pumping and can be identified with an index of pair of projections [2]. The bulk index is a mixture of commutative (in time) and

non-commutative (in space) expression of the odd Chern number [17]. For the general case, we define the bulk and edge index through a relative time evolution that allows to reduce matters to the previous case, by considering an effective Hamiltonian for each spectral gap of the bulk propagator. The index of an interface is also defined to provide a simple interpretation of this effective Hamiltonian.

Section 4 then studies the locality and continuity properties of bulk and edge propagators, required for the indices to be well defined, and compares these propagators. All this is established through the notion of confinement [5] and switch functions [2]. The proofs are finally detailed in Sect. 5, mostly following the statements of Sect. 3 but postponing some computations to "Appendix A." Although the mathematical expressions of the indices look similar to those for topological insulators, the operators involved are quite different and indeed describe another physics.

Finally, note that shortly after this work was completed an independent result on similar matters was proposed in [20]. Based on K-theory, it extends this bulk–edge correspondence to every dimension, but the physical interpretation is less immediate than in the functional analysis approach. Moreover, our work does not rely on any covariance property.

2. Floquet Topological Insulators

2.1. Bulk and Edge Hamiltonians

We consider a tight-binding model of independent electrons on the two-dimensional lattice $\mathbb{Z}^2$. The bulk Hilbert space is $\mathcal{H}_{\mathrm{B}} = \ell^2(\mathbb{Z}^2) \otimes \mathbb{C}^N$, where $\mathbb{C}^N$ accounts for internal degrees of freedom (sub-lattice, spin, orbital, etc.). For $\mathbf{m} \in \mathbb{Z}^2$, we denote by the usual ket notation $|\mathbf{m}\rangle \in \ell^2(\mathbb{Z}^2)$ the state localized at site $\mathbf{m}$ and $\langle \mathbf{m}|$ its corresponding bra. For any operator K on $\mathcal{H}_{\mathrm{B}}$ and $\mathbf{m}, \mathbf{n} \in \mathbb{Z}^2$, the kernel $K_{\mathbf{m},\mathbf{n}} \equiv \langle \mathbf{m}|K|\mathbf{n}\rangle$ is a matrix of size N. According to the context, $|\mathbf{m}| = |m_1| + |m_2|$ and $|K_{\mathbf{m},\mathbf{n}}|$ denotes the operator norm of finite matrices. The operator norm on the full Hilbert space $\mathcal{H}_{\mathrm{B}}$ is denoted by $\|K\|$.

The electrons are ruled by a family of one-particle Hamiltonians $H_B(t)$, namely a self-adjoint operator on $\mathcal{H}_{\mathrm{B}}$ for each $t \in \mathbb{R}$. In the context of Floquet topological insulators, we assume that it satisfies some further assumptions.

Definition 2.1 (*Bulk Hamiltonian*). Let $H_{\mathrm{B}}(t) : \mathcal{H}_{\mathrm{B}} \to \mathcal{H}_{\mathrm{B}}$ be a family of self-adjoint operators for $t \in \mathbb{R}$. We say that H_{B} is a bulk Hamiltonian if it is

1. *time-periodic:* $\exists T \in \mathbb{R}$ so that $H_{\mathrm{B}}(t+T) = H_{\mathrm{B}}(t)$ for all $t \in \mathbb{R}$,
2. *local:* $\exists\, \mu$, $C > 0$ independent of t so that for any $t \in [0, T]$ and $\mathbf{m}, \mathbf{n} \in \mathbb{Z}^2$

$$|H_{\mathrm{B}}(t)_{\mathbf{m},\mathbf{n}}| \leq C \mathrm{e}^{-\mu|\mathbf{m}-\mathbf{n}|}\,; \tag{2.1}$$

   μ is called the *locality exponent*,
3. *piecewise strongly continuous:* the map $t \mapsto H_{\mathrm{B}}(t)$ is strongly continuous except possibly for jump discontinuities.

Note that because of Condition 1, the parameter t is reduced to a compact interval so that the uniform bound in Condition 2 is equivalent to a family of time-dependent bounds for $t \in [0, T]$.

Remark 2.2 (Physical models covered). Any time-periodic Hamiltonian that for each t is a finite range or exponentially decaying hopping term is a bulk Hamiltonian in the sense of Definition 2.1. Moreover, piecewise constant Hamiltonians (e.g., as in [19]) are also allowed thanks to Condition 3. However we do not require space-translation invariance for a bulk Hamiltonian so that any disordered configuration can be implemented through H_{B} *a priori*, see Remark 3.13 below. Finally we do not require a spectral gap uniform in times, in contrast to adiabatic theory.

Remark 2.3 (Underlying topology). We define a norm on local operators which is suited to bulk Hamiltonians. For fixed μ let

$$\|A\|_\mu = \inf\{C \,|\, \forall t \in [0,T] \quad \forall \mathbf{m}, \mathbf{n} \in \mathbb{Z}^2 \quad |A(t)_{\mathbf{m},\mathbf{n}}| \leq C\mathrm{e}^{-\mu|\mathbf{m}-\mathbf{n}|}\}, \tag{2.2}$$

which satisfies $\|\cdot\|_\lambda \leq \|\cdot\|_\mu$ for $\lambda \leq \mu$.

This local norm will be used for homotopy considerations.

The edge system is described by considering only a half-plane, which we take to be $\mathbb{N} \times \mathbb{Z} \subset \mathbb{Z}^2$, so that the edge Hilbert space is $\mathcal{H}_{\mathrm{E}} = \ell^2(\mathbb{N} \times \mathbb{Z}) \otimes \mathbb{C}^N$. Bulk and edge spaces are related through the partial isometry

$$\iota : \mathcal{H}_{\mathrm{E}} \longrightarrow \mathcal{H}_{\mathrm{B}}, \qquad \iota^* : \mathcal{H}_{\mathrm{B}} \longrightarrow \mathcal{H}_{\mathrm{E}}, \tag{2.3}$$

where ι is the canonical injection of $\mathcal{H}_{\mathrm{E}}$ in $\mathcal{H}_{\mathrm{B}}$ and ι^* is the canonical truncation of $\mathcal{H}_{\mathrm{B}}$ to $\mathcal{H}_{\mathrm{E}}$. In particular, they satisfy

$$\iota^*\iota = \mathrm{Id}_{\mathcal{H}_{\mathrm{E}}}, \qquad \iota\iota^* = P_1, \tag{2.4}$$

where $P_1 : \mathcal{H}_{\mathrm{B}} \to \mathcal{H}_{\mathrm{B}}$ is the projection on states supported in the right half-plane $n_1 \geq 0$.

Definition 2.4 (*Edge Hamiltonian*). For a given bulk Hamiltonian $H_{\mathrm{B}}(t)$, the edge Hamiltonian $H_{\mathrm{E}}(t) : \mathcal{H}_{\mathrm{E}} \to \mathcal{H}_{\mathrm{E}}$ is the family of self-adjoint operators defined by

$$H_{\mathrm{E}}(t) = \iota^* H_{\mathrm{B}}(t) \iota. \tag{2.5}$$

Properties 1–3 of $\mathcal{H}_{\mathrm{B}}$, H_{B} are inherited to $\mathcal{H}_{\mathrm{E}}$, H_{E}. In particular, $\|H_{\mathrm{E}}\|_\mu \leq \|H_{\mathrm{B}}\|_\mu$.

As a sharp cut of the bulk space, this edge Hamiltonian corresponds to Dirichlet boundary condition, but an extra term confined near the boundary can actually be added to the previous definition without changing the topological aspects, see Proposition 5.1 below, allowing the implementation of other local boundary conditions or defects at the edge.

2.2. Propagator

The spectrum of a time-dependent Hamiltonian $H(t)$ at any given time will not be of importance. Instead we shall consider the time evolution operator generated by $H(t)$, see, e.g., [18, Theorem. X.69].

Definition 2.5 (*Propagator*) Let $H(t)$ be a family of bounded Hamiltonians on a Hilbert space $\mathcal{H}$, with $t \mapsto H(t)$ strongly continuous. The unitary propagator $U(t,s) \in \mathcal{U}(\mathcal{H})$ is a two-parameter family of unitary operators strongly continuous in t and s satisfying

$$U(t,t) = \mathrm{Id}_{\mathcal{H}}, \qquad U(t,r)U(r,s) = U(t,s), \tag{2.6}$$

and so that for any $\psi \in \mathcal{H}$, $\varphi_s(t) = U(t,s)\psi$ is the unique solution of

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\varphi_s(t) = H(t)\varphi_s(t), \qquad \varphi_s(s) = \psi. \tag{2.7}$$

where we have set $\hbar = 1$. $H(t)$ is called the generator of $U(t,s)$.

If the Hamiltonian has jump discontinuities, the propagator is defined piecewise but remains strongly continuous even at the discontinuity points thanks to (2.6). Note that in the case of a time-independent Hamiltonian H, the propagator is given by

$$U(t,s) = \mathrm{e}^{-\mathrm{i}(t-s)H} \tag{2.8}$$

and satisfies $U(t+\tau, s+\tau) = U(t,s)$ for any $\tau \in \mathbb{R}$. If $H(t+T) = H(t)$ is periodic in time, then that property survives for $\tau = T$, which implies $U(t+T,s) = U(t,0)U(T,s)$ by (2.6). As a result, the whole family $U(t,s)$ is determined by its restriction $U(t) \equiv U(t,0)$ to the compact interval $0 \leq t \leq T$; and its long-time behavior by just $U(T)$. The spectrum of $U(T)$ thus carries essential information about the solutions of (2.7). This is the so-called Floquet theory. Because $U(t)$ is unitary, its spectrum $\sigma[U(t)]$ belongs to $\mathcal{S}^1$, and at $t = T$ we denote

$$\mathrm{e}^{-\mathrm{i}\varepsilon T} \in \sigma[U(T)] \subset \mathcal{S}^1 \tag{2.9}$$

so that ε has the dimension of an energy. Because it is defined modulo $2\pi/T$, it is rather called *quasi-energy*, in analogy with quasi-momentum in Bloch theorem. Indeed the eigenstates of $U(T)$ provide solutions to (2.7) that are time-periodic up to the phase $\mathrm{e}^{-\mathrm{i}\varepsilon T}$.

2.3. Stroboscopic Gap Assumption

The topological aspects can be characterized through the propagator of a bulk Hamiltonian.

Definition 2.6 (*Floquet Topological Insulator*). We say that $H_{\mathrm{B}}(t)$, a bulk Hamiltonian in the sense of Definition 2.1, is a Floquet topological insulator if the corresponding unitary propagator at $t = T$, $U_{\mathrm{B}}(T)$ has a spectral gap.

As illustrated in Fig. 1, the spectrum of $U_{\mathrm{B}}(T)$ is typically constituted of one or several bands (of arbitrary nature) separated by gaps. By extension we also speak about quasi-energy ε when $\mathrm{e}^{-\mathrm{i}T\varepsilon}$ is in a spectral gap of $U_{\mathrm{B}}(T)$. Moreover, note that assumption of a "stroboscopic" spectral gap, i.e., only for

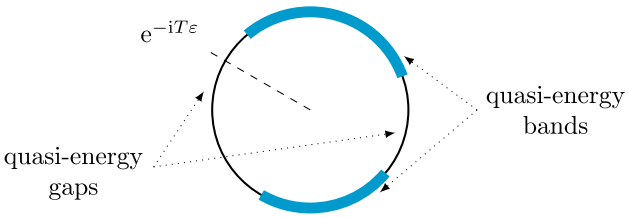

FIGURE 1. Example of spectrum for $U_{\rm B}(T)$ with two quasi-energy bands and gaps

$U_B(T)$, is sufficient to define the topological indices. The gap assumption may fail at some intermediate times, i.e., for $U_{\rm B}(t)$, $0 < t < T$.

Remark 2.7. The term "insulator" is somewhat misleading here as its meaning is purely mathematical: The existence of a spectral gap. The physical interpretation is not obvious since the spectrum of a unitary operator lives on a circle, so there is no ground state (in fact energy is not even conserved) and thus no notion of Fermi energy. The analogy with (time independent) topological insulators should then be used with care. Some attempt of interpretation is given in Sect. 3.3 below.

3. Bulk–Edge Correspondence

The main result of this paper is to define a bulk and edge index and to show that they coincide, for each spectral gap of $U_{\rm B}(T)$. The indices are, respectively, defined in terms of the bulk and edge propagators $U_{\rm B}$ and $U_{\rm E}$, generated by the corresponding Hamiltonians. To do so, the first thing to establish is that $U_{\rm B}$ and $U_{\rm E}$ are both local when $H_{\rm B}$ is, see Sect. 4.2 below. The operations of truncating space and generating time evolution do not commute, so that the truncated bulk propagator does not equal that of the edge. The important point however is that

$$|D(t)_{\mathbf{m},\mathbf{n}}| \leq C\mathrm{e}^{-\lambda|m_2-n_2|}\mathrm{e}^{-\lambda|n_1|}, \qquad D(t) \equiv U_E(t) - \iota^* U_{\rm B}(t)\iota \tag{3.1}$$

for some $C > 0$ and $0 < \lambda < \mu$, see Proposition 4.10. Namely the difference D is confined near the edge since it is exponentially decaying in direction 1, compare with (2.1). The bulk and edge indices are then defined using switch functions [2].

Definition 3.1. A switch function $\Lambda : \mathbb{Z} \to \mathbb{R}$ is a function so that $\Lambda(n) = 1$ (resp. 0) for n large and positive (resp. negative). We also call switch function and denote by Λ the multiplicative operator acting on $\ell^2(\mathbb{Z})$, and by Λ_i a switch function $\Lambda_i(\mathbf{n}) = \Lambda(n_i)$ in direction i acting on $\ell^2(\mathbb{Z}^2)$ or $\ell^2(\mathbb{N} \times \mathbb{Z})$.

For instance Λ can be a step function, in which case it is a projection, such as P_1 in (2.4). The commutator with a switch function allows to confine a local operator in a particular direction and is a powerful tool to eventually end up with trace class expressions. This is detailed in Sect. 4.1.

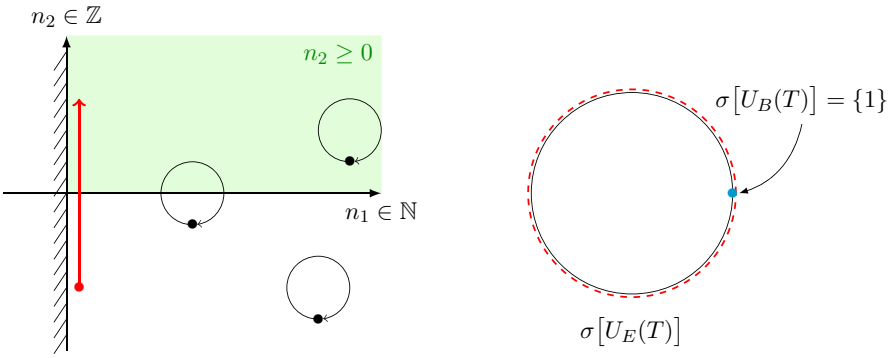

FIGURE 2. (Left) $\mathcal{I}_{\mathrm{E}}$ compares the density in the upper right quadrant between times $t = 0$ and $t = T$. Only the electrons localized at the edge contribute since $U_{\mathrm{B}}(T) = I$. (Right) Bulk and edge spectra. The latter might be gapless but only with extra states confined at the boundary

3.1. The Case of a Time-Periodic Propagator

The definition and properties of the indices, as well as the bulk–edge correspondence, are first established under the auxiliary assumption that the bulk propagator satisfies:

$$U_{\mathrm{B}}(T) = I, \tag{3.2}$$

where I is the identity. Although not really physical, this situation still belongs to the Floquet Topological Insulators in the sense of Definition 2.6 since the spectrum of $U_{\mathrm{B}}(T)$ is degenerated to $\{1\}$ so that $\mathcal{S}^1\backslash\{1\}$ constitutes a canonical spectral gap (see Fig. 2 right). The general case, treated in the next section, is nothing but a reduction to this particular one.

Proposition 3.2 (Edge index definition). *Let H_{B} be a bulk Hamiltonian so that $U_{\mathrm{B}}(T) = I$. Let H_{E} and U_{E} the associated edge Hamiltonian and propagator, and Λ_2 a switch function in direction 2 on $\mathcal{H}_{\mathrm{E}}$. The edge index*

$$\mathcal{I}_{\mathrm{E}} \equiv \mathrm{Tr}_{\mathcal{H}_{\mathrm{E}}}\Big(U_E^*(T)[\Lambda_2, U_E(T)]\Big) \tag{3.3}$$

is well defined and integer valued, independent of the choice of Λ_2, and continuous in H_{B} (in local norm) as long as $U_{\mathrm{B}}(T) = I$.

In that case $U_{\mathrm{E}}(T) = I + D(T)$ so that U_{E} is time-periodic up to a correction confined at the edge. The index has the interpretation of a *non-adiabatic quantized charge pumping* [21]: It counts the net number of particles that have moved into the upper half-plane within a period. In fact, by the independence on Λ_2, we may pick $\Lambda_2 = P_2$, the projection associated with that half-plane, so that

$$\mathcal{I}_{\mathrm{E}} = \mathrm{Tr}_{\mathcal{H}_{\mathrm{E}}}\Big(U_E^*(T)P_2U_E(T) - P_2\Big) \tag{3.4}$$

indeed computes the difference in the number of particles therein at times separated by a period. Moreover, the net transport takes place near the edge because far away from it we may pretend $U_{\mathrm{E}}(T) = 1$ by (3.1, 3.2). See Fig. 2. As we shall see (3.4) is the index of a pair of projection [2] and hence an integer.

Proposition 3.3 (Bulk index definition). *Let H_{B} be a bulk Hamiltonian and U_{B} the corresponding propagator such that $U_{\mathrm{B}}(T) = I$. The bulk index*

$$\mathcal{I}_{\mathrm{B}} \equiv \frac{1}{2}\int_0^T \mathrm{d}t\, \mathrm{Tr}_{\mathcal{H}_{\mathrm{B}}}\Big(U_{\mathrm{B}}^*\partial_t U_{\mathrm{B}}\Big[U_{\mathrm{B}}^*[\Lambda_1, U_{\mathrm{B}}],\, U_{\mathrm{B}}^*[\Lambda_2, U_{\mathrm{B}}]\Big]\Big) \tag{3.5}$$

is well defined and independent of the choice of the switch function Λ_i in direction $i = 1, 2$. It is moreover an integer depending continuously on H_{B} (in local norm) as long as $U_{\mathrm{B}}(T) = I$.

The bulk–edge correspondence then provides a physical interpretation of the bulk index[1] through the edge picture. The main result, proved in Sect. 5.1, is indeed

Theorem 3.4 *(Bulk–edge correspondence). Let H_{B} be a bulk Hamiltonian so that $U_{\mathrm{B}}(T) = I$. Then*

$$\mathcal{I}_{\mathrm{B}} = \mathcal{I}_{\mathrm{E}}. \tag{3.6}$$

Finally note that this approach generalizes the one from [19] where translation invariance in space is assumed, namely

$$H_{\mathrm{B}}(t)_{\mathbf{m},\mathbf{n}} = H_{\mathrm{B}}(t)_{0,\mathbf{n}-\mathbf{m}} \qquad \forall \mathbf{m}, \mathbf{n} \in \mathbb{Z}^2, \tag{3.7}$$

which is then also true for $U_{\mathrm{B}}(t)$, $H_{\mathrm{E}}(t)$ and $U_{\mathrm{E}}(t)$ (only in direction 2 for the edge operators). We denote by $\widehat{U_{\mathrm{B}}}(t, k_1, k_2) : \mathbb{T}^3 \mapsto \mathcal{U}(\mathbb{C}^N)$ and $\widehat{U_{\mathrm{E}}}(t, k_2) : \mathbb{T}^2 \mapsto \mathcal{U}(\ell^2(\mathbb{N}) \otimes \mathbb{C}^N)$ their corresponding Fourier transform.

Proposition 3.5 (Translation-invariant case). *Let H_{B} be a bulk Hamiltonian that is translation invariant, then $\mathcal{I}_{\mathrm{E}}$ is the winding number of $U_{\mathrm{E}}(T)$ along k_2, i.e.,*

$$\mathcal{I}_{\mathrm{E}} = \frac{\mathrm{i}}{2\pi}\int_0^{2\pi} \mathrm{d}k_2\, \mathrm{Tr}_{\ell^2(\mathbb{N})\otimes\mathbb{C}^N}\big(\widehat{U_{\mathrm{E}}}^*(T, k_2)\partial_{k_2}\widehat{U_{\mathrm{E}}}(T, k_2)\big) \tag{3.8}$$

and $\mathcal{I}_{\mathrm{B}}$ is the 3d-winding number of $\widehat{U_{\mathrm{B}}}$, namely

$$\mathcal{I}_{\mathrm{B}} = -\frac{1}{8\pi^2}\int_{\mathbb{T}^3} \mathrm{d}t\mathrm{d}k_1\mathrm{d}k_2\, \mathrm{tr}_{\mathbb{C}^N}\big(\widehat{U_{\mathrm{B}}}^*\partial_t\widehat{U_{\mathrm{B}}}\big[\widehat{U_{\mathrm{B}}}^*\partial_{k_1}\widehat{U_{\mathrm{B}}},\, \widehat{U_{\mathrm{B}}}^*\partial_{k_2}\widehat{U_{\mathrm{B}}}\big]\big). \tag{3.9}$$

Note that a more geometric way to write (3.9) is to use the language of differential forms, namely

$$\mathcal{I}_{\mathrm{B}} = -\frac{1}{24\pi^2}\int_{\mathbb{T}^3} \mathrm{tr}_{\mathbb{C}^N}\big((\widehat{U_{\mathrm{B}}}^*\,\mathrm{d}\widehat{U_{\mathrm{B}}})^{\wedge 3}\big), \tag{3.10}$$

which is the degree or odd Chern number [4,17]. Finally a disordered system has been considered in [21] where a finite sample is threaded by fluxes whose

[1] It was recently identified with a magnetization density in a particular system [14].

parameter space is a torus. That torus replaces the Brillouin zone of the space-periodic case. Thus, the expression of the bulk invariant there is analogue to (3.9) by averaging over those fluxes, even though it is evident, at least heuristically, that there is no dependence on them in the thermodynamic limit.

3.2. The General Case

In the general case, $U_{\mathrm{B}}(T) \neq I$, we shall define a bulk and edge index for each spectral gap of $U_{\mathrm{B}}(T)$ by deforming the latter to I and therefore come back to the previous case. Before doing that we establish the bulk–edge correspondence in a more general context. Consider two bulk Hamiltonians $H_{\mathrm{B},1}$ and $H_{\mathrm{B},2}$ together with their respective propagators $U_{\mathrm{B},1}$ and $U_{\mathrm{B},2}$ which are assumed to satisfy

$$U_{\mathrm{B},1}(T) = U_{\mathrm{B},2}(T). \tag{3.11}$$

We join the two Hamiltonians to a single one by placing their times intervals back to back, so to speak with opposite arrow of time. Explicitly, we define the relative Hamiltonian as

$$H_{\mathrm{B,rel}}(t) = \begin{cases} 2H_{\mathrm{B},1}(2t), & (0 < t < T/2) \\ -2H_{\mathrm{B},2}(2(T-t)), & (T/2 < t < T) \end{cases} \tag{3.12}$$

where the rescaling allows to keep the period T. By periodicity the second entry can be written more symmetrically to the first one as $-2H_{\mathrm{B},2}(-t)$ for $-T/2 < t < 0$. The Hamiltonians comply with Definition 2.1 despite jump discontinuities at $t = T/2$ and T. The corresponding evolution is

$$U_{\mathrm{B,rel}}(t) = \begin{cases} U_{\mathrm{B},1}(2t), & (0 \leq t \leq T/2) \\ U_{\mathrm{B},2}(2(T-t)), & (T/2 \leq t \leq T) \end{cases} \tag{3.13}$$

with continuity at $t = T/2$ by (3.11). It satisfies $U_{\mathrm{B,rel}}(T) = 1$ as intended. Indeed, the construction from the previous section applies.

Corollary 3.6 *(Relative bulk–edge correspondence). Let $H_{\mathrm{B},1}$ and $H_{\mathrm{B},2}$ be two bulk Hamiltonians such that $U_{\mathrm{B},1}(T) = U_{\mathrm{B},2}(T)$. Consider the relative Hamiltonian $H_{\mathrm{B,rel}}$, cf. (3.12), and the associated propagator $U_{\mathrm{B,rel}}$, as well as $H_{\mathrm{E,rel}} = \iota^* H_{\mathrm{B,rel}} \iota$ and $U_{\mathrm{E,rel}}$. The relative bulk and edge indices, defined by*

$$\mathcal{I}_{\mathrm{B}}^{\mathrm{rel}} = \mathcal{I}_{\mathrm{B}}[U_{\mathrm{B,rel}}] \qquad \mathcal{I}_{\mathrm{E}}^{\mathrm{rel}} = \mathcal{I}_{\mathrm{E}}[U_{\mathrm{E,rel}}(T)], \tag{3.14}$$

satisfy all the properties of Propositions 3.2 and 3.5, and moreover Theorem 3.4 applies, namely

$$\mathcal{I}_{\mathrm{B}}^{\mathrm{rel}} = \mathcal{I}_{\mathrm{E}}^{\mathrm{rel}}. \tag{3.15}$$

Given a single bulk Hamiltonian H_{B}, it is still possible to define bulk and edge indices through this relative construction. The required second Hamiltonian H_0 will be chosen as time independent and in such a way that $U_{\mathrm{B}}(T) = \mathrm{e}^{-\mathrm{i}TH_0}$, i.e., as a logarithm of $U_{\mathrm{B}}(T)$.

Definition 3.7 (*Effective Hamiltonian*). Let H_{B} be a bulk Hamiltonian and pick ε so that $\mathrm{e}^{-\mathrm{i}T\varepsilon}$ belongs to a gap of $U_{\mathrm{B}}(T)$. The effective Hamiltonian is defined on $\mathcal{H}_{\mathrm{B}}$ by

$$H_{\mathrm{B}}^{\varepsilon} = \frac{\mathrm{i}}{T} \log_{-T\varepsilon} U_{\mathrm{B}}(T) \tag{3.16}$$

through spectral decomposition of $U_{\mathrm{B}}(T)$, where $-T\epsilon$ is the branch cut of the logarithm, defined by $\log_\alpha(\mathrm{e}^{\mathrm{i}\phi}) = \mathrm{i}\phi$ for $\alpha - 2\pi < \phi < \alpha$.

It will be shown in Proposition 5.6 that $H_{\mathrm{B}}^{\varepsilon}$ is local. It thus conforms with Definition 2.1, since its other conditions hold true obviously. The pair H_{B}, $H_{\mathrm{B}}^{\varepsilon}$ satisfy (3.11), so that we have the general result:

Theorem 3.8 (Bulk–edge correspondence). *Let H_{B} be a bulk Hamiltonian and ε so that $\mathrm{e}^{-\mathrm{i}T\varepsilon}$ belongs to a gap of $U_{\mathrm{B}}(T)$. Consider the relative Hamiltonian $H_{\mathrm{B,rel}}^{\varepsilon}$, defined by (3.12) with $H_{\mathrm{B},1} = H_{\mathrm{B}}$ and $H_{\mathrm{B},2} = H_{\mathrm{B}}^{\varepsilon}$ from Definition 3.7, and the associated relative operators $U_{\mathrm{B,rel}}^{\varepsilon}$, $H_{\mathrm{E,rel}}^{\varepsilon}$ and $U_{\mathrm{E,rel}}^{\varepsilon}$. The bulk and edge indices*

$$\mathcal{I}_{\mathrm{B}}(\varepsilon) = \mathcal{I}_{\mathrm{B}}[U_{\mathrm{B,rel}}^{\varepsilon}], \qquad \mathcal{I}_{\mathrm{E}}(\varepsilon) = \mathcal{I}_{\mathrm{E}}[U_{\mathrm{E,rel}}^{\varepsilon}(T)] \tag{3.17}$$

satisfy all the properties of Proposition 3.2 and 3.5, and moreover Theorem 3.4 applies, namely

$$\mathcal{I}_{\mathrm{B}}(\varepsilon) = \mathcal{I}_{\mathrm{E}}(\varepsilon). \tag{3.18}$$

This is nothing but a specific case of Corollary 3.6: We constructed a relative evolution that fulfills the assumption of the previous section, namely $H_{\mathrm{B,rel}}^{\varepsilon}$ is a bulk Hamiltonian and $U_{\mathrm{B,rel}}^{\varepsilon}(T) = I$. The influence of the choice of ε is summarized by the next two statements:

Lemma 3.9. *Let H_{B} be a bulk Hamiltonian and ε, ε' so that $\mathrm{e}^{-\mathrm{i}T\varepsilon}$ and $\mathrm{e}^{-\mathrm{i}T\varepsilon'}$ belong to a gap of $U_{\mathrm{B}}(T)$. Then*

$$H_{\mathrm{B}}^{\varepsilon+2\pi/T} - H_{B}^{\varepsilon} = \frac{2\pi}{T} I \tag{3.19}$$

and for $0 \leq \varepsilon' - \varepsilon < 2\pi/T$

$$H_{\mathrm{B}}^{\varepsilon'} - H_{\mathrm{B}}^{\varepsilon} = \frac{2\pi}{T} P_{\varepsilon,\varepsilon'}, \tag{3.20}$$

where $P_{\varepsilon,\varepsilon'}$ is the spectral projection of $U_B(T)$ associated with the spectrum between $\mathrm{e}^{-\mathrm{i}T\varepsilon}$ and $\mathrm{e}^{-\mathrm{i}T\varepsilon'}$ clockwise.

Proposition 3.10 (Influence of ε). *Let H_{B} be a bulk Hamiltonian and ε, ε' so that $\mathrm{e}^{-\mathrm{i}T\varepsilon}$ and $\mathrm{e}^{-\mathrm{i}T\varepsilon'}$ belong to a gap of $U_{\mathrm{B}}(T)$. Then*

$$\mathcal{I}_{\mathrm{B}}(\varepsilon + 2\pi/T) = \mathcal{I}_{\mathrm{B}}(\varepsilon) \tag{3.21}$$

and for $0 \leq \varepsilon' - \varepsilon < 2\pi/T$

$$\mathcal{I}_{\mathrm{B}}(\varepsilon') - \mathcal{I}_{\mathrm{B}}(\varepsilon) = c(P_{\varepsilon,\varepsilon'}), \tag{3.22}$$

where

$$c(P) = -2\pi\mathrm{i}\,\mathrm{Tr}\Big(P\big[[\Lambda_1, P], [\Lambda_2, P]\big]\Big) \in \mathbb{Z} \tag{3.23}$$

is the non-commutative Chern number (or Kubo-Středa formula [2]) of P.

We have $\mathcal{I}_{\mathrm{B}}(\varepsilon) = \mathcal{I}_{\mathrm{B}}(\varepsilon')$ when ε and ε' belong to the same gap, by $P_{\varepsilon,\varepsilon'} = 0$. Similarly (3.22) implies (3.21) by $P_{\varepsilon,\varepsilon'} = I$ if $\varepsilon' \nearrow \varepsilon + 2\pi$. Note that (3.19) also implies (3.21) through $U_{\mathrm{B}}^{\varepsilon+2\pi/T}(t) = U_{\mathrm{B}}^{\varepsilon}(t)\mathrm{e}^{-2\pi\mathrm{i}t/T}$. In regard to

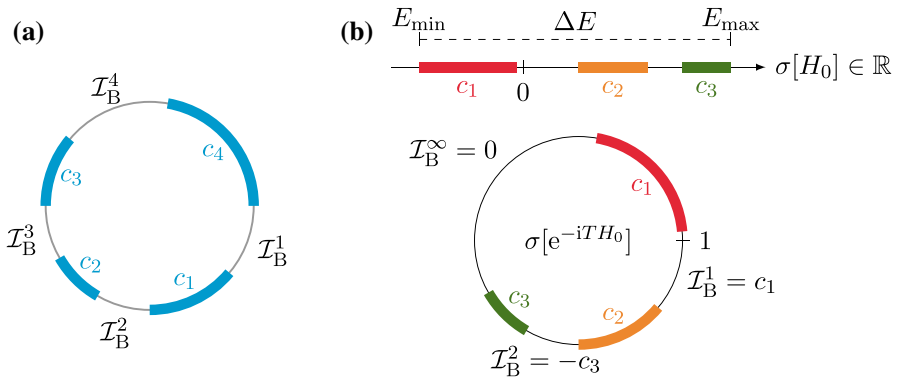

FIGURE 3. **a** Example of spectrum of $U_{\mathrm{B}}(T)$ with gap indices $\mathcal{I}_{\mathrm{B}}^i$ related by Chern numbers c_i of the bands through $\mathcal{I}_{\mathrm{B}}^{i+1} - \mathcal{I}_{\mathrm{B}}^i = c_i$. **b** For a time-independent Hamiltonian, the index of the "gap at infinity" always vanishes so that gap indices and Chern numbers are equivalent here

the operator seen in (3.5) that change contributes a commutator, which leaves the trace and hence the index unaffected.

A typical situation is illustrated in Fig. 3: To each gap of $U_{\mathrm{B}}(T)$ one associates a single index $\mathcal{I}_{\mathrm{B}}$, and indices between two distinct gaps are related through the Chern number of the band in between, so that the set of Chern numbers only gives the relative value of the gap indices. Finally note that Theorem 3.8 generalizes 3.4 since when $U_{\mathrm{B}}(T) = I$ then $H_{\mathrm{B}}^{\varepsilon} = 0$ for every $0 < \varepsilon < 2\pi$ so that $\mathcal{I}_{\mathrm{B}}(\varepsilon)$ coincides with $\mathcal{I}_{\mathrm{B}}$ from the previous section.

Remark 3.11. For a time-independent bulk Hamiltonian H_0, the set of gap invariants is equivalent to the set of Chern numbers, as illustrated in Fig. 3b. Indeed the spectrum of $\mathrm{e}^{-\mathrm{i}TH_0}$ consists in winding the spectrum of H_0 around the unit circle. Here T is arbitrary but as long as $T < (2\pi)/\Delta E$ where ΔE is the bandwidth of H_0, the propagator $\mathrm{e}^{-\mathrm{i}TH_0}$ possesses a "gap at infinity" coming from the gluing of the trivial gaps of H_0 at $\pm\infty$. When taking the branch cut in this gap (e.g., at $\varepsilon = E_{\min} - \eta$ for η small enough), one has $H_{\mathrm{B}}^{\varepsilon} = H_0$, so that $U_{\mathrm{B},1} = U_{\mathrm{B},2}$ in the relative evolution. A direct computation shows that $\mathcal{I}_{\mathrm{B}}^{\infty} = 0$ which sets a reference value for $\mathcal{I}_{\mathrm{B}}$. Thus, the other gap indices are in one-to-one correspondence with the set of Chern numbers by (3.22).

Finally, this construction is stable under continuous deformations.

Corollary 3.12 *(Homotopy invariance). Let $H_{\mathrm{B},0}$ and $H_{\mathrm{B},1}$ be two bulk Hamiltonians related by a homotopy $H_{\mathrm{B},s}$ of bulk Hamiltonians for $s \in [0,1]$. Assume the existence of ε so that for every s, $\mathrm{e}^{-\mathrm{i}T\varepsilon}$ belongs to a gap of $U_{\mathrm{B},s}(T)$. Then*

$$\mathcal{I}_{\mathrm{B},0}(\varepsilon) = \mathcal{I}_{\mathrm{B},1}(\varepsilon) \qquad \text{and} \qquad \mathcal{I}_{\mathrm{E},0}(\varepsilon) = \mathcal{I}_{\mathrm{E},1}(\varepsilon) \tag{3.24}$$

The proof of it follows from the continuity of the indices, and the fact that $U_{\mathrm{B}} \mapsto H_{\mathrm{B}}^{\varepsilon}$ is continuous, see Proposition 5.7.

Remark 3.13 (Weak disorder). Any disordered configuration can be implemented through H_{B} and the construction works as long as a spectral gap is open. Moreover, Theorem 3.8 is deterministic in the sense that the definition of $\mathcal{I}_{\mathrm{B}}$ and $\mathcal{I}_{\mathrm{E}}$ and the bulk–edge correspondence are valid for any configuration and do not rely on ergodicity or average computation. Finally, the indices are continuous in H_{B} in the sense of Remark 2.3, so that they coincide for two close configurations.

For example this covers the model developed in [21], but more generally take $H_{\mathrm{B}}^{\omega} = H_0 + \lambda V^{\omega}$ with H_0 translation invariant, $\{V^{\omega}\}_{\omega\in\Omega}$ a random potential and small λ so that, from 0 to λ, $\mathrm{e}^{-\mathrm{i}T\varepsilon}$ is in a spectral gap of $U_{\mathrm{B}}^{\omega}(T)$. Then $\mathcal{I}_{\mathrm{B}}^{\omega}(\varepsilon) = \mathcal{I}_{\mathrm{E}}^{\omega}(\varepsilon)$ for any $\omega \in \Omega$ and $\mathcal{I}_{\mathrm{B}}^{\omega}(\varepsilon) = \mathcal{I}_{\mathrm{B}}(\varepsilon)$, the latter corresponding to $\lambda = 0$.

3.3. Index of an Interface and Space–Time Duality

Though the invariants $\mathcal{I}_{\mathrm{B}}(\varepsilon)$ and $\mathcal{I}_{\mathrm{E}}(\varepsilon)$ of Theorem 3.8 are mathematically well defined and coincide, the physical interpretation of the relative evolution and effective Hamiltonian are not obvious. Here we propose a more intuitive reformulation by replacing the edge with an interface.

Consider again the general relative evolution of Corollary 3.6. For two bulk Hamiltonians $H_{\mathrm{B},1}$ and $H_{\mathrm{B},2}$ such that $U_{\mathrm{B},1}(T) = U_{\mathrm{B},2}(T)$ we define the relative Hamiltonian $H_{\mathrm{B,rel}}$ and deduce $U_{\mathrm{B,rel}}$, $H_{\mathrm{E,rel}} = \iota^* H_{\mathrm{B,rel}} \iota$ and $U_{\mathrm{E,rel}}$. In particular,

$$U_{\mathrm{E,rel}}(t) = \begin{cases} U_{\mathrm{E},1}(2t), & (0 \le t \le T/2) \\ U_{\mathrm{E},2}(2(T-t)) U_{\mathrm{E},2}^*(T) U_{\mathrm{E},1}(T), & (T/2 \le t \le T) \end{cases} \tag{3.25}$$

with $U_{\mathrm{E,rel}}(T) = U_{\mathrm{E},2}^*(T) U_{\mathrm{E},1}(T)$, so that the edge index of Corollary 3.6 can be reformulated as, *cf.* (3.3),

$$\mathcal{I}_{\mathrm{E}}^{\mathrm{rel}} = \mathrm{Tr}_{\mathcal{H}_{\mathrm{E}}} \Big(\big[\Lambda_2, U_{\mathrm{E},1}(T)\big] U_{\mathrm{E},1}^*(T) - \big[\Lambda_2, U_{\mathrm{E},2}(T)\big] U_{\mathrm{E},2}^*(T) \Big). \tag{3.26}$$

The expression looks like the difference of two edge indices from Proposition 3.2 except that the trace cannot be split since $U_{\mathrm{B},1}(T) = U_{\mathrm{B},2}(T)$ differ from I. However that suggests:

Definition 3.14. (*Interface Hamiltonian*) Let $H_{\mathrm{B},1}$ and $H_{\mathrm{B},2}$ be two bulk Hamiltonians and H_{int} be a bulk Hamiltonian that also satisfies $|H_{\mathrm{int}}(t)_{\mathbf{m},\mathbf{n}}| \le C\mathrm{e}^{-\mu|n_1|}\mathrm{e}^{-\mu|m_2-n_2|}$. Then define

$$H_{\mathrm{I}}(t) = P_1 H_{\mathrm{B},1}(t) P_1 + (1-P_1) H_{\mathrm{B},2}(t)(1-P_1) + H_{\mathrm{int}}(t). \tag{3.27}$$

This interface Hamiltonian is a bulk Hamiltonian acting on $\mathcal{H}_{\mathrm{B}}$ and gluing $H_{\mathrm{B},1}$ and $H_{\mathrm{B},2}$ on each half of the sample through a perturbation H_{int} confined to the interface, as illustrated in Fig. 4.

Proposition 3.15 (Interface index). *Let $H_{\mathrm{B},1}$ and $H_{\mathrm{B},2}$ be two bulk Hamiltonians such that $U_{\mathrm{B},1}(T) = U_{\mathrm{B},2}(T) \equiv U_{\mathrm{B}}(T)$. Consider the interface Hamiltonian H_{I} from Definition 3.14 and its evolution U_{I}. Then the interface index, defined by*

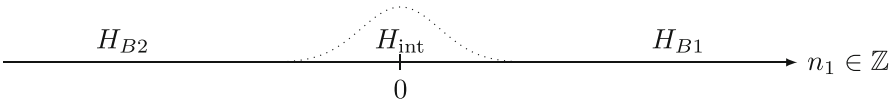

FIGURE 4. Interface between two samples

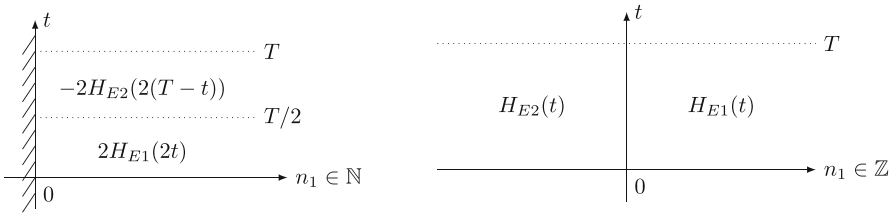

FIGURE 5. Duality of space and time. Left: relative evolution on a sample with an edge. Right: evolution on the interface between two samples

$$\mathcal{I}_{\rm I} = {\rm Tr}_{\mathcal{H}_{\rm B}}\Big(U_{\rm I}^* U_{\rm B}(T)\Big[\Lambda_2, U_{\rm B}^* U_{\rm I}(T)\Big]\Big), \tag{3.28}$$

is well defined, integer valued, independent of the choice of Λ_2 *and independent of* $H_{\rm int}$*. Moreover,*

$$\mathcal{I}_{\rm I} = \mathcal{I}_{\rm E}^{\rm rel}, \tag{3.29}$$

where $\mathcal{I}_{\rm E}^{\rm rel}$ *is the relative edge index* (3.17) *associated with* $H_{\rm B,1}$ *and* $H_{\rm B,2}$*.*

Proposition 3.15 establishes a duality between space and time. It tells that the relative index $\mathcal{I}_{\rm E}^{\rm rel}$ (and consequently $\mathcal{I}_{\rm B}^{\rm rel}$ through the bulk–edge correspondence) is nothing but the index $\mathcal{I}_{\rm I}$ of a sharp interface between two samples ruled by $H_{\rm E,1}$ and $H_{\rm E,2}$, as illustrated in Fig. 5. Moreover, a smooth gluing through $H_{\rm int}$ confined around the interface leads to the same index.

Remark 3.16 (Effective vacua). In the context of Theorem 3.8, where $H_{\rm B,1} = H_{\rm B}$ and $H_{\rm B,2} = H_{\rm B}^{\varepsilon}$, we deduce $\mathcal{I}_{\rm B}(\varepsilon) = \mathcal{I}_{\rm E}(\varepsilon) = \mathcal{I}_{\rm I}(\varepsilon)$. So the bulk index counts the number of topological edge modes appearing at the interface between the original and an effective sample ruled by $H_{\rm B}^{\varepsilon}$. Hence, the latter plays the role of a vacuum that selects the gap of $U_{\rm B}(T)$ around $\mathrm{e}^{-{\rm i}T\varepsilon}$, in analogy with the choice of Fermi energy. This vacuum depends on the system but is described by a time-independent and local dynamics, and there are as many distinct vacua as gaps in $U_{\rm B}(T)$. By expanding the commutators in (3.26), we have

$$\mathcal{I}_{\rm E}(\varepsilon) = {\rm Tr}_{\mathcal{H}_{\rm E}}\Big(U_{\rm E}(T)\Lambda_2 U_{\rm E}^*(T) - \mathrm{e}^{-{\rm i}TH_{\rm E}^{\varepsilon}}\Lambda_2 \mathrm{e}^{{\rm i}TH_{\rm E}^{\varepsilon}}\Big), \tag{3.30}$$

so that the interpretation of Fig. 2 still holds: $\mathcal{I}_{\rm E}(\varepsilon)$ measures the charge pumped in the upper quadrant, but relatively to the dynamics of the effective vacuum. This ensures that the index is well defined and the pumping

remains quantized. Finally if $U_{\rm B}(T) = I$ then $H^{\varepsilon}_{\rm B} = 0$ for every $0 < \varepsilon < 2\pi$ so the only effective vacuum is the usual one.

4. Properties of Bulk and Edge Propagator

The indices are defined through trace expressions involving $U_{\rm B}$ or $U_{\rm E}$. In this section, we study their properties and compare them. Before we recall a series of lemmas relating local and trace class operators through the notion of confinement [5].

4.1. Locality, Confinement and Switch Functions

In the following, we say that $f : \mathbb{Z}^2$ or $\mathbb{N} \times \mathbb{Z} \to \mathbb{R}$ is a Lipschitz function (of constant 1) if it satisfies

$$|f(\mathbf{m}) - f(\mathbf{n})| \leq |\mathbf{m} - \mathbf{n}|, \qquad \forall\, \mathbf{m}, \mathbf{n} \in \mathbb{Z}^2 \tag{4.1}$$

For $\lambda > 0$ we denote by $\mathrm{e}^{\lambda f}$ and $\mathrm{e}^{-\lambda f}$ the multiplicative operators on $\mathcal{H}_{\rm B}$ or $\mathcal{H}_{\rm E}$. We shall first rephrase the notion of locality appearing in Definition 2.1.

Lemma 4.1. *Let A be a local operator on $\mathcal{H}_{\rm B}$ or $\mathcal{H}_{\rm E}$ with locality exponent $\mu > 0$, then*

$$\|A\| \leq \|A\|_{\mu}\, c(\mu) \tag{4.2}$$

with $c(\mu) < \infty$ and

$$\forall\, 0 \leq \lambda < \mu \qquad \|\mathrm{e}^{-\lambda f} A \mathrm{e}^{\lambda f} - A\| \leq \|A\|_{\mu}\, b(\lambda) < \infty \tag{4.3}$$

for any Lipschitz function f where $b(\lambda) \to 0$, $(\lambda \to 0)$.

Proof. We apply the Holmgren-Schur estimate

$$\|A\| \leq |||A||| \equiv \max\Big(\sup_{\mathbf{m} \in \mathbb{Z}^2} \sum_{\mathbf{n} \in \mathbb{Z}^2} |A_{\mathbf{m},\mathbf{n}}|\,,\, \mathbf{m} \leftrightarrow \mathbf{n} \Big) \tag{4.4}$$

and estimate $|||A||| \leq C c(\mu)$ with $c(\mu) = \sum_{\mathbf{n} \in \mathbb{Z}^2} \mathrm{e}^{-\mu|\mathbf{n}|}$ for any C as in the definition (2.2) of the local norm. We then pass to the infimum over C. As for the second inequality we estimate

$$\begin{aligned} |(\mathrm{e}^{-\lambda f} A \mathrm{e}^{\lambda f} - A)_{\mathbf{m},\mathbf{n}}| &= \big|\mathrm{e}^{-\lambda f(\mathbf{m})} A_{\mathbf{m},\mathbf{n}} \mathrm{e}^{\lambda f(\mathbf{n})} - A_{\mathbf{m},\mathbf{n}}\big| \\ &\leq C \mathrm{e}^{-\mu|\mathbf{m}-\mathbf{n}|} \big(\mathrm{e}^{\lambda|\mathbf{m}-\mathbf{n}|} - 1\big) \end{aligned} \tag{4.5}$$

where we have used $|\mathrm{e}^{a} - 1| \leq \mathrm{e}^{|a|} - 1$ and (4.1), so as to obtain

$$|||\mathrm{e}^{-\lambda f} A \mathrm{e}^{\lambda f} - A||| \leq \|A\|_{\mu}\, b(\lambda) \tag{4.6}$$

with $b(\lambda) = \sum_{\mathbf{n} \in \mathbb{Z}^2} \mathrm{e}^{-\mu|\mathbf{n}|} \big(\mathrm{e}^{\lambda|\mathbf{n}|} - 1\big) < \infty$. Finally $b(\lambda) \to 0$, $(\lambda \to 0)$ by dominated convergence. □

Corollary 4.2. *Let A be a local operator on $\mathcal{H}_{\rm B}$ or $\mathcal{H}_{\rm E}$ with locality exponent μ, then*

$$\forall\, 0 \leq \lambda < \mu \qquad \big\|\mathrm{e}^{-\lambda f} A \mathrm{e}^{\lambda f}\big\| \leq B_{\lambda}\, \|A\|_{\mu} \tag{4.7}$$

with $B_\lambda > 0$ and for any Lipschitz function f. Conversely if $\mathrm{e}^{-\lambda f} A \mathrm{e}^{\lambda f}$ is bounded for some $\lambda > 0$ and any f Lipschitz then it is local, namely $\|A\|_\lambda \leq \sup_f \left\|\mathrm{e}^{-\lambda f} A \mathrm{e}^{\lambda f}\right\|$.

Proof. The first statement is an immediate consequence of the previous lemma and the triangle inequality. The second one is proved by taking for any $\mathbf{m}, \mathbf{n} \in \mathbb{Z}^2$, the Lipschitz function $f(\mathbf{p}) = |\mathbf{p} - \mathbf{m}|$ leading to

$$|A_{\mathbf{m},\mathbf{n}}| = |\langle \mathbf{m}|\mathrm{e}^{\lambda f}\mathrm{e}^{-\lambda f} A \mathrm{e}^{\lambda f}\mathrm{e}^{-\lambda f}|\mathbf{n}\rangle| \leq \left\|\mathrm{e}^{-\lambda f} A \mathrm{e}^{\lambda f}\right\| \mathrm{e}^{-\lambda|\mathbf{m}-\mathbf{n}|}, \tag{4.8}$$

where we have used Cauchy-Schwarz inequality and the fact that $|m\rangle$ and $|n\rangle$ are normalized. □

Thus, locality of A as by Remark 2.3 is equivalent to the boundedness of $\mathrm{e}^{-\lambda f} A \mathrm{e}^{\lambda f}$, up to a change from μ to $\lambda < \mu$. We then refine this notion by considering Lipschitz functions in direction i as Lipschitz functions $f_i(\mathbf{n}) = f(n_i)$. We observe that $f_1 + f_2$ is again Lipschitz, i.e., with constant 1.

Definition 4.3. A bounded operator A on $\mathcal{H}_\mathrm{B}$ or $\mathcal{H}_\mathrm{E}$ is called, for $i, j = 1, 2$, $i \neq j$:

- *i-local* (or *local in direction i*) if it exists $\lambda > 0$ so that $\mathrm{e}^{-\lambda f_i} A \mathrm{e}^{\lambda f_i}$ is bounded for any Lipschitz function f_i in direction i.
- *i-confined* (or *confined in direction i*) if it exists $\lambda > 0$ so that $A\mathrm{e}^{\lambda|n_i|}$ is bounded.
- simultaneously i-confined and j-local if it exists $\lambda > 0$ so that $\mathrm{e}^{-\lambda f_j} A \mathrm{e}^{\lambda f_j} \mathrm{e}^{\lambda|n_i|}$ is bounded.

The bounds are meant uniformly in f_i. The suprema provide norms associated with each property.

For example, in analogy with Lemma 4.1 and Corollary 4.2, Eq. (3.1) for D and Definition 3.14 of H_int means that these two operators are simultaneously 1-confined and 2-local. Another way to produce i-confined operators is to use switch functions from Definition 3.1.

Lemma 4.4. *Let A be a local operator and Λ_i a switch function in direction i. Then $[\Lambda_i, A]$ is simultaneously i-confined and j-local with corresponding norm bounded in terms of $\|A\|_\mu$.*

Proof. We rewrite

$$\mathrm{e}^{-\lambda f_j}[\Lambda_i, A]\mathrm{e}^{\lambda f_j}\mathrm{e}^{\lambda|n_i|} = \mathrm{e}^{-\lambda f_j}\big(\Lambda_i A(1-\Lambda_i) - (1-\Lambda_i)A\Lambda_i\big)\mathrm{e}^{\lambda f_j}\mathrm{e}^{\lambda|n_i|} \tag{4.9}$$

The first term reads

$$\begin{aligned} &\mathrm{e}^{-\lambda f_j}\Lambda_i A(1-\Lambda_i)\mathrm{e}^{\lambda f_j}\mathrm{e}^{\lambda|n_i|} \\ &\quad = \Lambda_i \mathrm{e}^{-\lambda n_i} \cdot \mathrm{e}^{-\lambda(f_j - n_i)} A \mathrm{e}^{\lambda(f_j - n_i)} \cdot (1-\Lambda_i)\mathrm{e}^{\lambda(n_i + |n_i|)} \end{aligned} \tag{4.10}$$

The middle factor is bounded by (4.7) and the other two are by the support property of the switch function. The second term in (4.9) is similarly bounded. □

Lemma 4.5. *For $i = 1, 2$ and $j \neq i$ let A_i be simultaneously i-confined and j-local, and Λ_j a switch function in direction j. Then $[\Lambda_j, A_i]$ and $A_i A_j$ are trace class with matching bonds on the norms.*

Proof. Similar to the previous proof we write

$$[\Lambda_j, A_i]\mathrm{e}^{\lambda|n_i|}\mathrm{e}^{\lambda|n_j|} = \Lambda_j A_i \mathrm{e}^{\lambda|n_i|}(1-\Lambda_j)\mathrm{e}^{\lambda|n_j|} - (1-\Lambda_j)A_i\mathrm{e}^{\lambda|n_i|}\Lambda_j\mathrm{e}^{\lambda|n_j|} \quad (4.11)$$

and see that each term is bounded by the same arguments. Then we write

$$[\Lambda_j, A_i] = [\Lambda_j, A_i]\mathrm{e}^{\lambda|n_i|}\mathrm{e}^{\lambda|n_j|} \cdot \mathrm{e}^{-\lambda|n_i|}\mathrm{e}^{-\lambda|n_j|}. \quad (4.12)$$

The second factor is obviously trace class, and so is $[\Lambda_j, A_i]$. Similarly

$$A_i A_j \mathrm{e}^{\lambda|n_i|}\mathrm{e}^{\lambda|n_j|} = A_i \mathrm{e}^{\lambda|n_i|} \cdot \mathrm{e}^{-\lambda|n_i|} A_j \mathrm{e}^{\lambda|n_i|}\mathrm{e}^{\lambda|n_j|} \quad (4.13)$$

is bounded, so that $A_i A_j$ is trace class. □

This lemma, combined with the previous one, will be of particular interest when $A_i = [\Lambda_i, A]$ for A local or $A_i = D$ from Proposition 4.10 below. We finally need:

Lemma 4.6. *Let Λ_i be a switch function in direction $i = 1, 2$ and A an operator such that $[\Lambda_i, A]$ is trace class. Then*

$$\mathrm{Tr}\big([\Lambda_i, A]\big) = 0. \quad (4.14)$$

Proof. Note that only $[\Lambda_i, A]$ is trace class so that we cannot open the commutator and separate the traces. However this allows to compute the trace through the diagonal kernel (take $\mathcal{H} = \ell^2(\mathbb{Z}^2)$ for concreteness)

$$\mathrm{Tr}\big([\Lambda_i, A]\big) = \sum_{\mathbf{n}\in\mathbb{Z}^2} ([\Lambda_i, A])_{\mathbf{n},\mathbf{n}} = \sum_{\mathbf{n}\in\mathbb{Z}^2} \Lambda(n_i)A_{\mathbf{n},\mathbf{n}} - A_{\mathbf{n},\mathbf{n}}\Lambda(n_i) = 0. \quad (4.15)$$

This result is the analogue of a vanishing integral of a total derivative. □

4.2. Locality and Continuity of Propagators

In this section, we show that the propagator is local when the Hamiltonian is, and that the propagator is continuous in the Hamiltonian with respect to the local norm.

Proposition 4.7. *Let H_B be a bulk Hamiltonian in the sense of Definition 2.1, μ its locality exponent, and U_B the corresponding unitary propagator. Then*

$$\forall\, t \in [0, T] \quad \forall\, 0 \leq \lambda < \mu \qquad \big\|\mathrm{e}^{-\lambda f} U_\mathrm{B}(t)\mathrm{e}^{\lambda f} - U_\mathrm{B}(t)\big\| \leq \alpha_\lambda \quad (4.16)$$

for any Lipschitz function f on $\mathbb{Z}^2$, and with α_λ independent of t and $\alpha_\lambda \to 0$, $(\lambda \to 0)$.

Proof. In order to work with bounded operators we define for a Lipschitz function f its bounded version $f_n = nf/(n + |f|)$ for $n \in \mathbb{N}$, which is again Lipschitz and so that $f_n \to f$ when $n \to \infty$. Define

$$V_n(t) = \mathrm{e}^{-\lambda f_n} U_\mathrm{B}(t)\mathrm{e}^{\lambda f_n} - U_\mathrm{B}(t), \quad (4.17)$$

which is bounded for every $n \in \mathbb{N}$, with $V_n(0) = 0$. Thus,

$$\|V_n(t)\| \leq \int_0^t \left\| \frac{\mathrm{d}V_n(s)}{\mathrm{d}s} \right\| \mathrm{d}s. \tag{4.18}$$

Using that $U_{\mathrm{B}}(t)\varphi$ satisfies Schrödinger equation (2.7) for every $\varphi \in \mathcal{H}_{\mathrm{B}}$, we deduce

$$\begin{aligned} \left\| \frac{\mathrm{d}V_n(s)}{\mathrm{d}s} \right\| &= \left\| \mathrm{e}^{-\lambda f_n} H_{\mathrm{B}}(s) U_{\mathrm{B}}(s) \mathrm{e}^{\lambda f_n} - H_{\mathrm{B}}(s) U_{\mathrm{B}}(s) \right\| \\ &= \left\| \left(\mathrm{e}^{-\lambda f_n} H_{\mathrm{B}}(s) \mathrm{e}^{\lambda f_n} - H_{\mathrm{B}}(s) \right) \mathrm{e}^{-\lambda f_n} U_{\mathrm{B}}(s) \mathrm{e}^{\lambda f_n} \right. \\ &\quad \left. + H_{\mathrm{B}}(s) \left(\mathrm{e}^{-\lambda f_n} U_{\mathrm{B}}(s) \mathrm{e}^{\lambda f_n} - U_{\mathrm{B}}(s) \right) \right\|. \end{aligned} \tag{4.19}$$

Since H_{B} is local it is bounded uniformly in time by Lemma 4.1 and $\|H_{\mathrm{B}}(s)\| \leq \|H_{\mathrm{B}}\|_\mu c(\mu)$, we furthermore deduce

$$\|V_n(t)\| \leq \int_0^t \left(\|H_{\mathrm{B}}\|_\mu \left(c(\mu) + b(\lambda) \right) \|V_n(s)\| + \|H_{\mathrm{B}}\|_\mu b(\lambda) \right) \mathrm{d}s, \tag{4.20}$$

where $c(\mu)$ and $b(\lambda)$ are independent of t and n and $b(\lambda) \to 0$ when $\lambda \to 0$. We then get by application of Grönwall Lemma and the fact that $e^x - 1 \leq x\mathrm{e}^x$ for $x > 0$

$$\begin{aligned} \|V_n(t)\| &\leq \|H_{\mathrm{B}}\|_\mu b(\lambda) t \mathrm{e}^{\|H_{\mathrm{B}}\|_\mu (c(\mu)+b(\lambda))t} \\ &\leq \|H_{\mathrm{B}}\|_\mu b(\lambda) T \mathrm{e}^{\|H_{\mathrm{B}}\|_\mu (c(\mu)+b(\lambda))T} \equiv \alpha_\lambda \end{aligned} \tag{4.21}$$

uniformly in $t \in [0, T]$ and $n \in \mathbb{N}$. Thus, $V = \mathrm{e}^{-\lambda f} U_{\mathrm{B}} \mathrm{e}^{\lambda f} - U_{\mathrm{B}}$ is well defined and $V_n(t)$ converges strongly to $V(t)$. In particular, $\|V(t)\| \leq \alpha_\lambda$, and from the previous expression we see that $\alpha_\lambda \to 0$ when $\lambda \to 0$. □

By Corollary 4.2, this last proposition shows that U_{B} is local for any exponent $\lambda < \mu$. Furthermore we have:

Proposition 4.8. *Let H_{B} be a bulk Hamiltonian in the sense of Definition 2.1, with locality exponent μ. Then the map $H_{\mathrm{B}} \mapsto U_{\mathrm{B}}$ is continuous (uniformly in time) with the respective norms $\|\cdot\|_\mu$ and $\|\cdot\|_\lambda$ for $\lambda < \mu$.*

Proof. Let $H_{\mathrm{B},1}$ and $H_{\mathrm{B},2}$ be two bulk Hamiltonians and $U_{\mathrm{B},1}$, $U_{\mathrm{B},2}$ their propagators. For $\lambda < \mu$ and f a Lipschitz function we compute

$$\begin{aligned} &\left\| \mathrm{e}^{-\lambda f} (U_{\mathrm{B},1} - U_{\mathrm{B},2})(t) \mathrm{e}^{\lambda f} \right\| \\ &\leq \int_0^t \mathrm{d}s \left\| \mathrm{e}^{-\lambda f} (H_{\mathrm{B},1} U_{\mathrm{B},1} - H_{\mathrm{B},2} U_{\mathrm{B},2})(s) \mathrm{e}^{\lambda f} \right\| \\ &\leq \int_0^t \mathrm{d}s \Big(B \|H_{\mathrm{B},1}\|_\mu \left\| \mathrm{e}^{-\lambda f} (U_{\mathrm{B},1} - U_{\mathrm{B},2})(s) \mathrm{e}^{\lambda f} \right\| \\ &\quad + B \|H_{\mathrm{B},1} - H_{\mathrm{B},2}\|_\mu \left\| \mathrm{e}^{-\lambda f} U_{\mathrm{B},2}(s) \mathrm{e}^{\lambda f} \right\| \Big) \end{aligned} \tag{4.22}$$

where we have used Corollary 4.2 for $H_{\mathrm{B},1}$ and $H_{\mathrm{B},1} - H_{\mathrm{B},2}$ with some constant $B > 0$. Adding $U_{\mathrm{B},2} - U_{\mathrm{B},2}$ in the last norm, by Proposition 4.7 and Grönwall

inequality as in the previous proof we deduce

$$\left\|\mathrm{e}^{-\lambda f}(U_{\mathrm{B},1}-U_{\mathrm{B},2})(t)\mathrm{e}^{-\lambda f}\right\| \leq B' \left\|H_{\mathrm{B},1}-H_{\mathrm{B},2}\right\|_{\mu} (1+\alpha_\lambda) T \mathrm{e}^{B\|H_{\mathrm{B},1}\|_\mu T} \tag{4.23}$$

uniformly in time so that $\|U_{\mathrm{B},1}-U_{\mathrm{B},2}\|_\lambda$ is similarly bounded. □

Finally note that all this discussion naturally extends to the edge Hilbert space $\mathcal{H}_{\mathrm{E}}$ since H_{E} satisfies similar properties of H_{B} by construction, thus all the proofs remain unchanged.

Corollary 4.9. *Let H_{B} be a bulk Hamiltonian in the sense of Definition 2.1, H_{E} the corresponding edge Hamiltonian defined in 2.4 and U_{E} the associated propagator. Then Lemma 4.1 holds for H_E and Proposition 4.7 and 4.8 hold for U_E.*

4.3. Comparing Bulk and Edge Propagator

As announced in the beginning of Sect. 3, we prove identity (3.1) that compares the edge propagator and the truncated bulk one and is crucial for the proof of bulk–edge correspondence.

Proposition 4.10. *Let H_{B} and H_{E} be a bulk and its corresponding edge Hamiltonian, and U_{B} and U_{E} the respective propagators. Define*

$$D(t) = U_E(t) - \iota^* U_B(t)\iota. \tag{4.24}$$

Then for $0 \leq \lambda < \mu$ where μ is the locality exponent of H_{B}, $D(t)$ is 1-confined and 2-local on $\mathcal{H}_{\mathrm{E}}$, uniformly in $t \in [0,T]$. Moreover, the map $H_{\mathrm{B}} \mapsto D(t)$ is continuous in the relevant norms.

Proof. As in proof of Proposition 4.7 we start with bounded version of Lipschitz functions, namely $f_2^p = pf_2/(p+|f_2|)$, and similarly $f_1^p = pn_1/(p+n_1)$. Then for $t \in [0,T]$ we compute

$$\begin{aligned}
&\mathrm{e}^{-\lambda f_2^p} D(t) \mathrm{e}^{\lambda f_2^p} \mathrm{e}^{\lambda f_1^p} \\
&\quad = \mathrm{e}^{-\lambda f_2^p}\big(U_{\mathrm{E}}(t) - \iota^* U_{\mathrm{B}}(t)\iota\big)\mathrm{e}^{\lambda f_2^p}\mathrm{e}^{\lambda f_1^p} \\
&\quad = -\mathrm{e}^{-\lambda f_2^p} U_{\mathrm{E}}(t) \int_0^t \mathrm{d}s\, \partial_s \Big(U_{\mathrm{E}}^*(s)\iota^* U_{\mathrm{B}}(s)\iota\, \mathrm{e}^{\lambda f_2^p}\mathrm{e}^{\lambda f_1^p}\Big) \\
&\quad = -\mathrm{e}^{-\lambda f_2^p} U_{\mathrm{E}}(t) \int_0^t \mathrm{d}s\, U_{\mathrm{E}}^*(s)\mathrm{i}\big(H_{\mathrm{E}}(s)\iota^* - \iota^* H_{\mathrm{B}}(s)\big) U_{\mathrm{B}}(s)\iota\, \mathrm{e}^{\lambda f_2^p}\mathrm{e}^{\lambda f_1^p} \\
&\quad = \mathrm{i}\int_0^t \mathrm{d}s\, \mathrm{e}^{-\lambda f_2^p} U_{\mathrm{E}}(t) U_{\mathrm{E}}^*(s)\iota^* H_{\mathrm{B}}(s)(1-P_1) U_{\mathrm{B}}(s)\iota\, \mathrm{e}^{\lambda f_2^p}\mathrm{e}^{\lambda f_1^p},
\end{aligned} \tag{4.25}$$

where we used $\partial_s U_{\mathrm{B}} = -\mathrm{i}H_{\mathrm{B}}U_{\mathrm{B}}$ and $\partial_s U_{\mathrm{E}}^* = \mathrm{i}U_{\mathrm{E}}^* H_{\mathrm{E}}$, Definition 2.4 of H_{E} and property (2.4). Then we write

$$\iota \mathrm{e}^{\pm\lambda f_2^p} = \mathrm{e}^{\pm\lambda f_2^p}\iota \qquad \mathrm{e}^{\pm\lambda f_2^p}\iota^* = \iota^* \mathrm{e}^{\pm\lambda f_2^p}, \tag{4.26}$$

where on the right-hand side each equation is the natural extension of $\mathrm{e}^{\pm\lambda f_2^p}$ on $\mathcal{H}_{\mathrm{B}}$, that we denote by the same symbol, and similarly for $\mathrm{e}^{\pm\lambda f_1^p}$ since $f_1^p \to n_1$ has also a natural extension on $\mathbb{Z}^2$. Thus, we can rewrite the integrand as

$$\begin{aligned}
&\mathrm{e}^{-\lambda f_2^p} U_{\mathrm{E}}(t) U_{\mathrm{E}}^*(s) \iota^* H_{\mathrm{B}}(s)(1-P_1) U_{\mathrm{B}}(s) \iota \mathrm{e}^{\lambda f_2^p} \mathrm{e}^{\lambda f_1^p} \\
&\quad = \mathrm{e}^{-\lambda f_2^p} U_{\mathrm{E}}(t) \mathrm{e}^{\lambda f_2^p} \cdot \mathrm{e}^{-\lambda f_2^p} U_{\mathrm{E}}^*(s) \mathrm{e}^{\lambda f_2^p} \cdot \iota^* \mathrm{e}^{-\lambda f_2^p} H_{\mathrm{B}}(s) \mathrm{e}^{\lambda f_2^p} \cdot (1-P_1) \mathrm{e}^{\lambda f_1^p} \\
&\qquad \cdot \mathrm{e}^{-\lambda f_1^p} \mathrm{e}^{-\lambda f_2^p} U_B(s) \mathrm{e}^{-\lambda f_2^p} \mathrm{e}^{\lambda f_1^p} \iota
\end{aligned} \tag{4.27}$$

Note that $(1-P_1)\mathrm{e}^{\lambda f_1^p}$ is bounded uniformly in $p \in \mathbb{N}$. By Lemma 4.1, Proposition 4.7 and Corollary 4.9, each one of the remaining factor is uniformly bounded in $s \in [0,T]$ and $p \in \mathbb{N}$. Then so is the operator appearing on the l.h.s. of (4.25). Since the estimate is independent of p, it strongly converges to $\mathrm{e}^{-\lambda f_2} D(t) \mathrm{e}^{\lambda f_2}$
$\mathrm{e}^{\lambda n_1}$ which is also bounded uniformly in $t \in [0,T]$. More precisely

$$\left\| \mathrm{e}^{-\lambda f_2} D(t) \mathrm{e}^{\lambda f_2} \mathrm{e}^{\lambda n_1} \right\| \leq BT \left\| U_{\mathrm{E}} \right\|_\mu^2 \left\| H_{\mathrm{B}} \right\|_\mu \left\| U_{\mathrm{B}} \right\|_\mu \tag{4.28}$$

with a constant $B > 0$. The continuity follows from (4.27) by Corollary 4.9. □

5. Proofs

We mostly follows the order of statements of Sect. 3. First we assume that $U_{\mathrm{B}}(T) = I$ and prove the bulk–edge correspondence, then check that it also applies to the relative evolution with effective Hamiltonian in the general case. In between some extra properties are established, such as the invariance under change in boundary condition and additivity property of the bulk index.

5.1. Periodic Unitary Propagator

Proof of Proposition 3.2. When $U_{\mathrm{B}}(T) = I$, Proposition 4.10 reduces to $U_{\mathrm{E}}(T) = I + D(T)$ with $D(T)$ simultaneously 1-confined and 2-local. In particular,

$$[\Lambda_2, U_E(T)] = [\Lambda_2, D(T)] \tag{5.1}$$

is trace class according to Lemma 4.5, so that $\mathcal{I}_{\mathrm{E}}$ is well defined. Then for two switch functions Λ_2 and $\widetilde{\Lambda}_2$, the difference in the corresponding indices reads

$$\mathcal{I}_{\mathrm{E}} - \widetilde{\mathcal{I}_{\mathrm{E}}} = \mathrm{Tr}_{\mathcal{H}_{\mathrm{E}}} \Big(U_E^*(T) \big((\Lambda_2 - \widetilde{\Lambda}_2) D(T) - D(T)(\Lambda_2 - \widetilde{\Lambda}_2) \big) \Big). \tag{5.2}$$

Since $\Lambda_2 - \widetilde{\Lambda}_2$ is compactly supported in direction 2, it is 2-confined, and even obviously simultaneously 1-local and 2-confined. Applying again Lemma 4.5, we deduce that $(\Lambda_2 - \widetilde{\Lambda}_2)D(T)$ and $D(T)(\Lambda_2 - \widetilde{\Lambda}_2)$ are separately trace class, so that we can split the trace into two parts that are actually equal by cyclicity and the fact that $U_{\mathrm{E}}^*(T)$ and $D(T) = I - U_{\mathrm{E}}(T)$ commute, so that $\mathcal{I}_{\mathrm{E}} = \widetilde{\mathcal{I}_{\mathrm{E}}}$.

In particular, we can compute $\mathcal{I}_{\mathrm{E}}$ with $\Lambda_2 = P_2$ that is also a projection. Rewriting

$$\mathcal{I}_{\mathrm{E}} = \mathrm{Tr}_{\mathcal{H}_{\mathrm{E}}} \big(U_E^*(T) P_2 U_E(T) - P_2 \big) = \mathrm{Ind}(U_E^*(T) P_2 U_E(T), P_2) \tag{5.3}$$

we recognize the index of a pair of projections (see [2]) defined by

$$\mathrm{Ind}(P,Q) = \dim \ker(P - Q - 1) - \dim \ker(P - Q + 1) \in \mathbb{Z}. \tag{5.4}$$

Indeed, when $(P-Q)^{2n+1}$ is trace class for some $n \geq 0$, then

$$\forall m \geq n \qquad \mathrm{Tr}((P-Q)^{2m+1}) = \mathrm{Tr}((P-Q)^{2n+1}) = \mathrm{Ind}(P,Q). \tag{5.5}$$

In our case, $P = U_E^*(T)P_2U_E(T)$, $Q = P_2$ and $n = 0$ gives (5.3) which is an integer by the definition above. The continuity of $H_\mathrm{B} \mapsto \mathrm{Tr}([P_2, D(T)])$ follows by Proposition 4.10 and Lemma 4.5. It implies that of $\mathcal{I}_\mathrm{E}$ by (5.1). □

Since $\mathcal{I}_\mathrm{E}$ is continuous and integer valued, it is constant. As a consequence, we have:

Proposition 5.1 (Influence of the boundary condition). *Let H_B be a bulk Hamiltonian so that $U_\mathrm{B}(T) = I$. Consider the alternative edge Hamiltonian*

$$\widetilde{H_\mathrm{E}}(t) = \iota^* H_\mathrm{B}(t)\iota + H_\mathrm{bc}(t) \tag{5.6}$$

with H_bc a self-adjoint operator on $\mathcal{H}_\mathrm{E}$ that is simultaneously 1-confined and 2-local. Let $\widetilde{U_\mathrm{E}}$ and $\widetilde{\mathcal{I}_\mathrm{E}}$ the associated propagator and edge index. Then $\widetilde{\mathcal{I}_\mathrm{E}} = \mathcal{I}_\mathrm{E}$ where $\mathcal{I}_\mathrm{E}$ corresponds to $H_\mathrm{bc} = 0$.

Proof. First note that $\widetilde{H_\mathrm{E}}$ is still local. Thus, adapting the proof of Proposition 4.10 where we replace H_E by $H_\mathrm{E} + H_\mathrm{bc}$ and using the fact that H_bc is 1-confined and 2-local, we end up with

$$\widetilde{U_\mathrm{E}}(T) = I + \widetilde{D}(T) \tag{5.7}$$

with $\widetilde{D}(T)$ 1-confined and 2-local, so that $\widetilde{\mathcal{I}_\mathrm{E}}$ is well defined and shares the properties of Proposition 3.2. In particular, it is now continuous in both H_B and H_bc in the relevant norms. Moreover, note that H_E and $\widetilde{H_\mathrm{E}}$ are homotopic through

$$H_s(t) \equiv \iota^* H_\mathrm{B}(t)\iota + sH_\mathrm{bc}(t) \tag{5.8}$$

for $s \in [0, 1]$. By Corollary 4.9 this induces a homotopy U_λ from U_E to $\widetilde{U}_E$ which preserves H_B and U_B. Thus, $\widetilde{\mathcal{I}_\mathrm{E}} = \mathcal{I}_\mathrm{E}$. □

In the translation-invariant case, the bulk Hamiltonian satisfies $(H_\mathrm{B})_{\mathbf{m},\mathbf{n}}(t) = (H_\mathrm{B})_{0,\mathbf{n}-\mathbf{m}}(t)$ and induces this property on U_B, but also on the edge operators H_E and U_E in direction 2, namely $(U_\mathrm{E})_{\mathbf{m},\mathbf{n}}(t) = (U_\mathrm{E})_{(m_1,0),(n_1,n_2-m_2)}(t)$. In that case, we define the Fourier transform of U_E in direction 2 by

$$\widehat{U_\mathrm{E}}(t, k_2) = \sum_{n_2 \in \mathbb{Z}} \mathrm{e}^{-\mathrm{i}k_2 n_2} (U_\mathrm{E})_{0,n_2}(t). \tag{5.9}$$

Noticing that $\ell^2(\mathbb{N} \times \mathbb{Z}) \cong \ell^2(\mathbb{N}) \otimes \ell^2(\mathbb{Z})$, this operator is defined for each pair $(m_1, n_1) \in \mathbb{N}^2$ so that $\widehat{U_\mathrm{E}}(t, k_2)$ acts on $\ell^2(\mathbb{N}) \otimes \mathbb{C}^N$. Since U_E is local from Corollary 4.9, this Fourier transform is well defined and even smooth in k_2.

Lemma 5.2. *Let A and B be bounded and translation-invariant operators on $\ell^2(\mathbb{Z})$ with $\mathcal{C}^1$-Fourier transform $\widehat{A}(k)$ and $\widehat{B}(k)$, and Λ a switch function. If $A[\Lambda, B]$ is trace class then*

$$\mathrm{Tr}_{\ell^2(\mathbb{Z})}\Big(A[\Lambda, B]\Big) = (AXB)_{00} = \mathrm{i}\int_0^{2\pi} \frac{\mathrm{d}k}{2\pi} \widehat{A}(k)\partial_k \widehat{B}(k), \tag{5.10}$$

where X is the position operator.

Proof. Since $A[\Lambda, B]$ is trace class, its trace can be computed through its diagonal kernel:

$$\mathrm{Tr}_{\ell^2(\mathbb{Z})}\Big(A[\Lambda,B]\Big) = \sum_{p,q\in\mathbb{Z}} A_{p,q}B_{q,p}(\Lambda(q)-\Lambda(p)) = \sum_{p'\in\mathbb{Z}} A_{0,p'}p'B_{p',0} \tag{5.11}$$

which gives the first equality. We have used $A_{p,q} = A_{0,q-p}$, similarly for B, the change in variables $q \mapsto p' = q - p$ and the fact that

$$\sum_{p\in\mathbb{Z}} (\Lambda(p'+p) - \Lambda(p)) = p' \tag{5.12}$$

for any $p' \in \mathbb{Z}$ and any switch Λ. The second equality is a standard Fourier computation. □

By Proposition 3.2, we know that $U_{\mathrm{E}}^*(T)[\Lambda_2, U_{\mathrm{E}}(T)]$ is trace class, so that we can apply the previous lemma in direction 2

$$\mathcal{I}_{\mathrm{E}} = \mathrm{i}\,\mathrm{Tr}_{\ell^2(\mathbb{N})\otimes\mathbb{C}^N} \int_0^{2\pi} \frac{\mathrm{d}k_2}{2\pi} \Big(\widehat{U_{\mathrm{E}}}^*(T,k_2)\partial_{k_2}\widehat{U_{\mathrm{E}}}(T,k_2)\Big) \tag{5.13}$$

Finally, since $U_{\mathrm{E}}(T) = I + D(T)$ where both D and U_{E} are 2-local, their respective Fourier transform $\widehat{D}(T,k_2)$ and $\widehat{U_{\mathrm{E}}}(T,k_2)$ are smooth in k_2, so that $\partial_{k_2}\widehat{U_{\mathrm{E}}}(t,k_2) = \partial_{k_2}\widehat{D}(T,k_2)$ is 1-confined on $\ell^2(\mathbb{N})\otimes\mathbb{C}^N$, namely it is trace class for each k_2. Hence, trace and integral can be exchanged in the last formula. This proves identity (3.8) of Proposition 3.5.

***Proof of Proposition** 3.3.* By Proposition 4.7 U_{B} is local, then by Lemma 4.4 $U_{\mathrm{B}}^*[\Lambda_i, U_{\mathrm{B}}]$ is i-confined and j-local so that the product of two such terms for $i \neq j$ is trace class according to Lemma 4.5, so $\mathcal{I}_{\mathrm{B}}$ is well defined. Then for two switch functions in direction 1, consider their difference $\Delta\Lambda_1$ and the corresponding difference of indices. We open the inner commutator and separate traces, as we can by Lemma 4.5. Then we conjugate by U_{B} the expression under the first one, so as to obtain

$$\begin{aligned}\Delta\mathcal{I}_{\mathrm{B}} = \frac{1}{2}\int_0^T \mathrm{d}t\,\mathrm{Tr}_{\mathcal{H}_{\mathrm{B}}}\Big((\partial_t U_{\mathrm{B}})U_{\mathrm{B}}^*\Big[\Delta\Lambda_1, [\Lambda_2, U_{\mathrm{B}}]U_{\mathrm{B}}^*\Big]\Big)\\ -\mathrm{Tr}_{\mathcal{H}_{\mathrm{B}}}\Big(U_{\mathrm{B}}^*\partial_t U_{\mathrm{B}}\Big[\Delta\Lambda_1,\, U_{\mathrm{B}}^*[\Lambda_2, U_{\mathrm{B}}]\Big]\Big).\end{aligned} \tag{5.14}$$

Each term is vanishing: Since $\Delta\Lambda_1$ is 1-confined, one can open the outer commutator in each one. Up to algebraic manipulation, we get for the first one

$$\begin{aligned}&\mathrm{Tr}_{\mathcal{H}_{\mathrm{B}}}\Big((\partial_t U_{\mathrm{B}})U_{\mathrm{B}}^*\Big[\Delta\Lambda_1, [\Lambda_2, U_{\mathrm{B}}]U_{\mathrm{B}}^*\Big]\Big)\\ &= \partial_t\mathrm{Tr}_{\mathcal{H}_{\mathrm{B}}}\Big([U_{\mathrm{B}},\Lambda_2]U_{\mathrm{B}}^*\Delta\Lambda_1\Big) + \mathrm{Tr}_{\mathcal{H}_{\mathrm{B}}}\Big([U_{\mathrm{B}}\Lambda_2 U_{\mathrm{B}}^*,\, (\partial_t U_{\mathrm{B}})U_{\mathrm{B}}^*\Delta\Lambda_1]\Big)\end{aligned} \tag{5.15}$$

The first term vanishes when integrated over t since U_{B} is t-periodic. As for the second

$$[U_{\mathrm{B}}\Lambda_2 U_{\mathrm{B}}^*,\, (\partial_t U_{\mathrm{B}})U_{\mathrm{B}}^*\Delta\Lambda_1] = U_{\mathrm{B}}[\Lambda_2,\, U_{\mathrm{B}}^*(\partial_t U_{\mathrm{B}})U_{\mathrm{B}}^*\Delta\Lambda_1 U_{\mathrm{B}}]U_{\mathrm{B}}^* \tag{5.16}$$

Since $\Delta\Lambda_1$ is compactly supported it is 1-confined, so that $U_{\mathrm{B}}^*(\partial_t U_{\mathrm{B}})$ $U_{\mathrm{B}}^*\Delta\Lambda_1 U_{\mathrm{B}}$ is simultaneously 1-confined and 2-local and the commutator of it with Λ_2 is trace class according to Lemma 4.5, and with vanishing trace according to Lemma 4.6. Thus, the first part of (5.14) vanishes, and similarly for the second, so that $\mathcal{I}_{\mathrm{B}}$ is independent of the choice of Λ_1. We proceed analogously for Λ_2.

To show that $\mathcal{I}_{\mathrm{B}}$ is an integer, we identify it with a non-commutative odd Chern number [17]. Since U_{B} is periodic in time, consider its inverse Fourier transform along the time direction. Namely for $p, q \in \mathbb{Z}$ define $\check{U}_{p,q} = \check{U}_{0,q-p}$ where

$$\check{U}_{0,p} = \frac{1}{T}\int_0^T U_{\mathrm{B}}(t)\mathrm{e}^{\mathrm{i}\frac{2\pi}{T}pt}\mathrm{d}t \tag{5.17}$$

that acts on $\mathcal{H}_{\mathrm{B}} \otimes \ell^2(\mathbb{Z})$. Then consider the following operator appearing in (3.5) up to cyclicity

$$O = \Big[U_{\mathrm{B}}^*[\Lambda_1, U_{\mathrm{B}}],\, U_{\mathrm{B}}^*[\Lambda_2, U_{\mathrm{B}}]\Big]U_{\mathrm{B}}^* \quad \Rightarrow \quad \check{O} = \Big[\check{U}^*[\Lambda_1, \check{U}],\, \check{U}^*[\Lambda_2, \check{U}]\Big]\check{U}^* \tag{5.18}$$

since U_{B} and U_{B}^* are t-periodic and Λ_1, Λ_2 naturally extends to $\mathcal{H}_{\mathrm{B}} \otimes \ell^2(\mathbb{Z})$. Hence, by Lemma 5.2 in direction t for $A = O$, $B = U_{\mathrm{B}}$ and Λ_t a switch function in direction t, we finally get

$$\mathcal{I}_{\mathrm{B}} = -\mathrm{i}\pi\mathrm{Tr}_{H_{\mathrm{B}}\otimes\ell^2(\mathbb{Z})}\Big(\check{U}^*[\Lambda_t, \check{U}]\Big[\check{U}^*[\Lambda_1, \check{U}],\, \check{U}^*[\Lambda_2, \check{U}]\Big]\Big) = C_3. \tag{5.19}$$

This identifies $\mathcal{I}_{\mathrm{B}}$ with C_3, the non-commutative version of the odd Chern number in dimension 3, see [17]. In particular, $\mathcal{I}_{\mathrm{B}} \in \mathbb{Z}$. Finally the continuity is given by opening the double commutator in expression (3.5) of $\mathcal{I}_{\mathrm{B}}$, and noticing that

$$U_{\mathrm{B}}^*(t)[\Lambda_1, U_{\mathrm{B}}(t)]U_{\mathrm{B}}^*(t)[\Lambda_2, U_{\mathrm{B}}(t)] = -[\Lambda_1, U_{\mathrm{B}}^*(t)][\Lambda_2, U_{\mathrm{B}}(t)] \tag{5.20}$$

is trace class by Lemma 4.5, and similarly for the second term where $1 \leftrightarrow 2$. Then consider $U_{\mathrm{B},1}$ and $U_{\mathrm{B},2}$ so that $U_{\mathrm{B},1}(T) = U_{\mathrm{B},2}(T) = I$ and denote by ν one of their common locality exponent. By introducing a mixed term, and inspecting the proof of Lemma 4.4

$$\begin{aligned}\Big\|\big([\Lambda_1, U_{\mathrm{B},1}^*(t)][\Lambda_2, U_{\mathrm{B},1}(t)] - [\Lambda_1, U_{\mathrm{B},2}^*(t)][\Lambda_2, U_{\mathrm{B},2}(t)]\big)\mathrm{e}^{\lambda|n_1|}\mathrm{e}^{\lambda|n_2|}\Big\| \\ \leq B\big(\,\|U_{\mathrm{B},1}\|_\nu + \|U_{\mathrm{B},2}\|_\nu\,\big)\,\|U_{\mathrm{B},1} - U_{\mathrm{B},2}\|_\nu\end{aligned} \tag{5.21}$$

for $\lambda < \nu$, uniformly in time, so that $U_{\mathrm{B}} \mapsto [\Lambda_1, U_{\mathrm{B}}^*][\Lambda_2, U_{\mathrm{B}}]$ is continuous with respect to $\|\cdot\|_\nu$ and trace norm $\|\cdot\|_1$. By composition with continuous functions, we deduce that $U_{\mathrm{B}} \mapsto \mathcal{I}_{\mathrm{B}}$ is continuous in $\|\cdot\|_\nu$, and by Proposition 4.8 that $H_{\mathrm{B}} \mapsto \mathcal{I}_{\mathrm{B}}$ is continuous in $\|\cdot\|_\mu$ as long as $U_{\mathrm{B}}(T) = I$. □

Before proving the bulk–edge correspondence, we establish another property of the bulk index that will be used in the general case when U_{B} is not anymore time-periodic. The proof of this proposition is purely algebraic but quite tedious, we postpone it to "Appendix A.1," so as not to overburden the reading.

Proposition 5.3 (Additivity of the bulk index). *Consider U and V two unitary propagators satisfying $U(T) = V(T) = I$. Then*

$$\mathcal{I}_{\mathrm{B}}[UV] = \mathcal{I}_{\mathrm{B}}[U] + \mathcal{I}_{\mathrm{B}}[V] \tag{5.22}$$

where $UV(t) = U(t)V(t)$ on $\mathcal{H}_{\mathrm{B}}$.

As we did for the edge index, we can also at the expression of $\mathcal{I}_{\mathrm{B}}$ when the bulk Hamiltonian is translation invariant in space. In that case $(U_{\mathrm{B}})_{\mathbf{m},\mathbf{n}}(t) = (U_{\mathrm{B}})_{0,\mathbf{n}-\mathbf{m}}(t)$ and we define its Fourier transform

$$\widehat{U_{\mathrm{B}}}(t, k_1, k_2) = \sum_{\mathbf{n}\in\mathbb{Z}^2} \mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{n}} (U_{\mathrm{B}})_{0,n}(t) \tag{5.23}$$

that defines $\widehat{U_{\mathrm{B}}} : \mathbb{T}^3 \to \mathcal{U}(N)$, where $\mathbb{T}^3 = [0,T] \times [0,2\pi]^2$, namely a matrix valued function periodic in time and quasi-momentum. In analogy with Lemma 5.2, we have

Lemma 5.4. *Let A, B and C be three bounded and translation-invariant operators on $\ell^2(\mathbb{Z}^2)$ with $\mathcal{C}^2$ Fourier transform denoted by $\widehat{A}$, $\widehat{B}$ and $\widehat{C}$. Let Λ_1 and Λ_2 be two switch functions in direction 1 and 2. If $A[\Lambda_1, B[\Lambda_2, C]]$ is trace class then*

$$\mathrm{Tr}_{\ell^2(\mathbb{Z}^2)}\Big(A[\Lambda_1, B[\Lambda_2, C]]\Big) = (AX_1BX_2C)_{00} = \mathrm{i}^2 \int \frac{\mathrm{d}^2k}{(2\pi)^2} \widehat{A}(k)\partial_{k_1}\big(\widehat{B}\,\partial_{k_2}\widehat{C}\big). \tag{5.24}$$

The proof is completely similar to the one of Lemma 5.2, one dimension higher. If we rewrite the operator appearing in the definition (3.5) of the bulk index as

$$\Big[U_{\mathrm{B}}^*[\Lambda_1, U_{\mathrm{B}}],\, U_{\mathrm{B}}^*[\Lambda_2, U_{\mathrm{B}}]\Big] = -\big[\Lambda_1, U_{\mathrm{B}}^*[\Lambda_2, U_{\mathrm{B}}]\big] + \big[\Lambda_2, U_{\mathrm{B}}^*[\Lambda_1, U_{\mathrm{B}}]\big], \tag{5.25}$$

each term is separately trace class by Lemma 4.5 since U_{B} is local. The locality also implies that $\widehat{U_{\mathrm{B}}}$ is smooth in k_1 and k_2. We then apply Lemma 5.4 to each part of (5.25) to end up with identity (3.9) of Proposition 3.5.

Finally, the proof of the bulk–edge correspondence is based on a partial result that improves Proposition 4.10.

Lemma 5.5. *Let H_{B} be a bulk Hamiltonian and H_{E}, $U_{\mathrm{B}}, U_{\mathrm{E}}$ the corresponding edge Hamiltonian and bulk and edge propagator. For any switch function in direction 2*

$$\Delta(t) \equiv [\Lambda_2, U_{\mathrm{E}}(t)]U_{\mathrm{E}}^*(t) - \iota^*[\Lambda_2, U_{\mathrm{B}}(t)]U_{\mathrm{B}}^*(t)\iota \tag{5.26}$$

is trace class on $\mathcal{H}_{\mathrm{E}}$ for every $t \in [0,T]$.

Proof. From Proposition 4.10, we have that $U_{\mathrm{E}}(t) = \iota^* U_{\mathrm{B}}(t)\iota + D(t)$. We put this expression of U_{E} in the definition of Δ, use the fact that $\iota^*\Lambda_2 = \Lambda_2\iota^*$ and $\Lambda_2\iota = \iota\Lambda_2$ where on the right-hand side we mean the extension of Λ_2 on $\mathcal{H}_{\mathrm{B}}$, and that $\iota\iota^* = P_1$. We end up with

$$\Delta = \iota^*[\Lambda_2, U_{\mathrm{B}}][P_1, U_{\mathrm{B}}^*]\iota + \iota^*[\Lambda_2, U_{\mathrm{B}}]\iota D^* + [\Lambda_2, D](\iota^* U_{\mathrm{B}}^*\iota + D^*) \tag{5.27}$$

where each term is separately trace class by using that U_{B} is local, D is 1-confined and 2-local, P_1 is also a switch function and by applying Lemmas 4.4 and 4.5. □

***Proof of Theorem** 3.4.* We start by the edge index that we rewrite for convenience

$$\mathcal{I}_{\mathrm{E}} = \mathrm{Tr}_{\mathcal{H}_{\mathrm{E}}}\Big([\Lambda_2, U_{\mathrm{E}}(T)]U_{\mathrm{E}}^*(T)\Big). \tag{5.28}$$

In order to restore a time dependence we introduce a cutoff in direction 1. For $r \in \mathbb{N}$ take $Q_{1,r} = \chi_{n_1<r}$ on $\mathcal{H}_{\mathrm{E}}$ and note that $Q_{1,r} = I - P_{1,r}$ where $P_{1,r}$ is a also a switch function. Since $U_{\mathrm{B}}(T) = I$ the operator in the previous expression of $\mathcal{I}_{\mathrm{E}}$ is nothing but $\Delta(T)$ which is trace class. Moreover, $Q_{1,r} \to I$ strongly when $r \to \infty$, so that

$$\mathcal{I}_{\mathrm{E}} = \lim_{r\to\infty} \mathcal{I}_{\mathrm{E}}^r \qquad \mathcal{I}_{\mathrm{E}}^r = \mathrm{Tr}_{\mathcal{H}_{\mathrm{E}}}\Big([\Lambda_2, U_{\mathrm{E}}(T)]U_{\mathrm{E}}^*(T)Q_{1,r}\Big) \tag{5.29}$$

Then we can rewrite

$$\mathcal{I}_{\mathrm{E}}^r = \int_0^T \mathrm{d}t\, \partial_t \mathrm{Tr}_{\mathcal{H}_{\mathrm{E}}}\Big([\Lambda_2, U_{\mathrm{E}}(t)]U_{\mathrm{E}}^*(t)Q_{1,r}\Big) \equiv \int_0^T \mathrm{d}t\, I_r(t) \tag{5.30}$$

Indeed

$$\begin{aligned} I_r(t) = {} & \mathrm{Tr}_{\mathcal{H}_{\mathrm{E}}}\Big([\Lambda_2, \partial_t U_{\mathrm{E}}(t)]U_{\mathrm{E}}^*(t)Q_{1,r}\Big) \\ & -\mathrm{Tr}_{\mathcal{H}_{\mathrm{E}}}\Big([\Lambda_2, U_{\mathrm{E}}(t)]U_{\mathrm{E}}(t)^*(-\mathrm{i}H_{\mathrm{E}})(t)Q_{1,r}\Big) \end{aligned} \tag{5.31}$$

is trace class by Lemmas 4.4 and 4.5 since U_{E}, H_{E} and $\partial_t U_{\mathrm{E}} = -\mathrm{i}H_{\mathrm{E}}U_{\mathrm{E}}$ are local and $Q_{1,r}$ is trivially 1-confined and 2-local simultaneously. From now on we drop the time dependence in $I_r(t)$. By Lemma 4.6, we have

$$\begin{aligned} 0 &= \mathrm{Tr}_{\mathcal{H}_{\mathrm{E}}}\Big(\big[\Lambda_2, (\partial_t U_{\mathrm{E}})U_{\mathrm{E}}^*Q_{1,r}\big]\Big) \\ &= \mathrm{Tr}_{\mathcal{H}_{\mathrm{E}}}\Big([\Lambda_2, \partial_t U_{\mathrm{E}}]U_{\mathrm{E}}^*Q_{1,r}\Big) + \mathrm{Tr}_{\mathcal{H}_{\mathrm{E}}}\Big(\partial_t U_{\mathrm{E}}[\Lambda_2, U_{\mathrm{E}}^*Q_{1,r}]\Big) \end{aligned} \tag{5.32}$$

where on the r.h.s the first term appears in $I_r(t)$ and the second one can be expanded by using that $[\Lambda_2, Q_{1,r}] = 0$, $[\Lambda_2, U_{\mathrm{E}}^*] = -U_{\mathrm{E}}^*[\Lambda_2, U_{\mathrm{E}}]U_{\mathrm{E}}$ and $(\partial_t U_{\mathrm{E}})U_{\mathrm{E}}^* = -\mathrm{i}H_{\mathrm{E}}$. We end up with

$$I_r(t) = \mathrm{Tr}_{\mathcal{H}_{\mathrm{E}}}\Big([\Lambda_2, U_{\mathrm{E}}]U_{\mathrm{E}}^*\,[P_{1,r}\,, \mathrm{i}H_{\mathrm{E}}]\Big) \tag{5.33}$$

where we have also used that $[H_{\mathrm{E}}, Q_{1,r}] = [P_{1,r}, H_{\mathrm{E}}]$. This expression can now be recast as a bulk expression. By Lemma 5.5, Definition 2.4 of H_{E}, and by denoting $\iota^* P_{1,r} = P_{1,r}\iota^*$ where on the right-hand side $P_{1,r} = \chi_{n_1\geq r}$ on $\mathcal{H}_{\mathrm{B}}$, and similarly with ι, we get

$$\begin{aligned} I_r(t) = \mathrm{Tr}_{\mathcal{H}_{\mathrm{E}}}\Big(& \iota^*[\Lambda_2, U_{\mathrm{B}}]U_{\mathrm{B}}^*\,[P_{1,r}\,, \mathrm{i}H_{\mathrm{B}}]\iota + \iota^*\big[[\Lambda_2, U_{\mathrm{B}}]U_{\mathrm{B}}^*, P_1\big] \\ & [P_{1,r}\,, \mathrm{i}H_B]\iota + \Delta[P_{1,r}\,, \mathrm{i}H_{\mathrm{E}}]\Big) \end{aligned} \tag{5.34}$$

where we have used that $\iota\iota^* = P_1$ and $\iota^*\iota = I$. The traces of the last two terms vanish in the limit $r \to \infty$. Indeed both $[[\Lambda_2, U_\mathrm{B}]U_\mathrm{B}^*, P_1]$ and Δ are trace class according to Lemmas 4.5 and 5.5, respectively, and $P_{1,r} \to 0$ strongly, so that

$$\mathrm{Tr}_{\mathcal{H}_\mathrm{B}}\Big(\iota^*\big[[\Lambda_2, U_\mathrm{B}]U_\mathrm{B}^*, P_1\big][P_{1,r}\,, \mathrm{i}H_B]\iota + \Delta[P_{1,r}\,, \mathrm{i}H_\mathrm{E}]\Big) \underset{r\to\infty}{\longrightarrow} 0. \tag{5.35}$$

Finally note that for any trace class operator O on $\mathcal{H}_\mathrm{E}$ one has $\mathrm{Tr}_{\mathcal{H}_\mathrm{E}}(O) = \mathrm{Tr}_{\mathcal{H}_\mathrm{B}}(\iota O \iota^*)$, so that

$$I_r(t) = \mathrm{Tr}_{\mathcal{H}_\mathrm{B}}\Big([\Lambda_2, U_\mathrm{B}]U_\mathrm{B}^*[P_{1,r}\,, \mathrm{i}H_\mathrm{B}]P_1\Big) + o(1) \tag{5.36}$$

where we have used again that $\iota\iota^* = P_1$, $P_1^2 = P_1$ and the cyclicity of trace.

The next step is to show that P_1 can be omitted in the previous expression. Intuitively, $[P_{1,r}\,, H_\mathrm{B}]$ is confined along $n_1 = r$ so that its contribution for $n_1 < 0$ vanishes exponentially when r is big enough. More explicitly we compute

$$\begin{aligned}&\mathrm{Tr}_{\mathcal{H}_\mathrm{B}}\Big([\Lambda_2, U_\mathrm{B}]U_\mathrm{B}^*[P_{1,r}\,, \mathrm{i}H_\mathrm{B}](1-P_1)\Big)\\ &\quad = -\mathrm{Tr}_{\mathcal{H}_\mathrm{B}}\Big(P_{1,r}[(1-P_1), \mathrm{i}H_\mathrm{B}][\Lambda_2, U_\mathrm{B}]U_\mathrm{B}^*\Big) \to 0\end{aligned} \tag{5.37}$$

where we used that $P_{1,r}(1-P_1) = 0$ and the cyclicity of the trace. Since $[(1-P_1), \mathrm{i}H_\mathrm{B}][\Lambda_2, U_\mathrm{B}]U_\mathrm{B}^*$ is trace class and $P_{1,r} \to 0$ strongly then the previous expression vanishes in the limit $r \to \infty$. Moreover, the trace on the l.h.s. can be split into two traces so that

$$I_r(t) = \mathrm{Tr}_{\mathcal{H}_\mathrm{B}}\Big([\Lambda_2, U_\mathrm{B}]U_\mathrm{B}^*[P_{1,r}\,, \mathrm{i}H_\mathrm{B}]\Big) + o(1) \tag{5.38}$$

as announced. This expression is nothing but (5.33) where we have replaced every E by B and up to corrections vanishing in the limit $r \to \infty$, even when integrated over the compact interval $[0, T]$.

The final step is to get back the expression of the bulk index as in (3.5). First we rewrite $\mathrm{i}H_\mathrm{B} = -(\partial_t U_\mathrm{B})U_\mathrm{B}^*$. Then we have the following identity

$$\begin{aligned}&\mathrm{Tr}_{\mathcal{H}_\mathrm{B}}\Big([\Lambda_2, U_\mathrm{B}]U_\mathrm{B}^*[(\partial_t U_\mathrm{B})U_\mathrm{B}^*, P_{1,r}]\Big)\\ &\quad = \frac{1}{2}\mathrm{Tr}_{\mathcal{H}_B}\Big((\partial_t U_\mathrm{B})U_\mathrm{B}^*\Big[[P_{1,r},\, U_\mathrm{B}]U_\mathrm{B}^*,\, [\Lambda_2, U_\mathrm{B}]U_\mathrm{B}^*\Big]\Big)\\ &\qquad + \frac{1}{2}\partial_t \mathrm{Tr}_{\mathcal{H}_\mathrm{B}}\Big(\Big[[\Lambda_2, U_\mathrm{B}], P_{1,r}\Big]U_\mathrm{B}^*\Big).\end{aligned} \tag{5.39}$$

This identity is purely algebraic but quite tedious to show so we postpone the computation to "Appendix A.2." Since U_B is periodic, the second term vanishes when integrated over time. Conjugating the first one by U_B^* and U_B and putting all together, we get

$$\mathcal{I}_\mathrm{E} = \lim_{r\to\infty}\frac{1}{2}\int_0^T \mathrm{d}t\, \mathrm{Tr}_{\mathcal{H}_B}\Big(U_\mathrm{B}^*\partial_t U_\mathrm{B}\Big[U_\mathrm{B}^*[P_{1,r}\,, U_\mathrm{B}],\, U_\mathrm{B}^*[\Lambda_2, U_\mathrm{B}]\Big]\Big) \tag{5.40}$$

but on the right-hand side we recognize the bulk index expression, that is independent of the choice of switch function. In particular, $P_{1,r}$ can be replaced by P_1 or any Λ_1, so that the limit is trivial and we get $\mathcal{I}_\mathrm{E} = \mathcal{I}_\mathrm{B}$. □

5.2. General Case

In the general case, the bulk–edge correspondence is a corollary of Theorem 3.4, so we only need to check that this theorem applies, namely that the effective Hamiltonian $H_{\mathrm{B}}^{\varepsilon}$ from Definition 3.7 has the required properties, in particular that it is local. By spectral decomposition

$$U_{\mathrm{B}}(T) = \int_{\mathcal{S}^1} \lambda \mathrm{d}P(\lambda) \quad \Rightarrow \quad H_{\mathrm{B}}^{\varepsilon} = \frac{\mathrm{i}}{T} \int_{\mathcal{S}^1} \log_{-T\varepsilon}(\lambda) \mathrm{d}P(\lambda) \tag{5.41}$$

where the integration is done over the unit circle and $\mathrm{d}P(\lambda)$ is the spectral measure of $U_B(T)$.

Proposition 5.6. *Let $H_{\mathrm{B}}^{\varepsilon}$ be an effective Hamiltonian constructed from a bulk Hamiltonian H_{B}. Then $H_{\mathrm{B}}^{\varepsilon}$ is local, namely it exists $\lambda^* > 0$ such that for $0 \leq \lambda < \lambda^*$*

$$\|\mathrm{e}^{-\lambda f} H_{\mathrm{B}}^{\varepsilon} \mathrm{e}^{\lambda f} - H_B^{\varepsilon}\| \leq \beta_\lambda \tag{5.42}$$

for any Lipschitz function f. Moreover, $\beta_\lambda \to 0$ for $\lambda \to 0$.

Proof. As in the proof of Proposition 4.7, we consider bounded f_n instead of f to work with bounded operators, get a uniform estimate independent of n allowing to consider the $n \to \infty$ limit. We compute $H_{\mathrm{B}}^{\varepsilon}$ through the resolvent formula

$$H_{\mathrm{B}}^{\varepsilon} = -\frac{1}{2\pi\mathrm{i}} \frac{\mathrm{i}}{T} \int_{\Gamma} \mathrm{d}z \log_{-T\varepsilon}(z) R_{U_{\mathrm{B}}}(z) \tag{5.43}$$

where $R_{U_{\mathrm{B}}}(z) \equiv (U_{\mathrm{B}}(T) - z)^{-1}$ and Γ is illustrated in Fig. 6. In particular, one has

$$\mathrm{e}^{-\lambda f_n} H_{\mathrm{B}}^{\varepsilon} \mathrm{e}^{\lambda f_n} = -\frac{1}{2\pi\mathrm{i}} \frac{\mathrm{i}}{T} \int_{\Gamma} \mathrm{d}z \log_{-T\varepsilon}(z) R_{U_n}(z), \tag{5.44}$$

where we defined $U_n(T) \equiv \mathrm{e}^{-\lambda f_n} U_{\mathrm{B}}(T) \mathrm{e}^{\lambda f_n}$.

The usual resolvent identity leads to

$$R_{U_n}(z)\Big(1 + (U_n(T) - U_{\mathrm{B}}(T)))R_{U_{\mathrm{B}}}(z)\Big) = R_{U_{\mathrm{B}}}(z). \tag{5.45}$$

First we have

$$\|R_{U_{\mathrm{B}}}(z)\| \leq \frac{1}{\mathrm{dist}(z, \sigma(U(T)))} \leq \frac{1}{\eta} \tag{5.46}$$

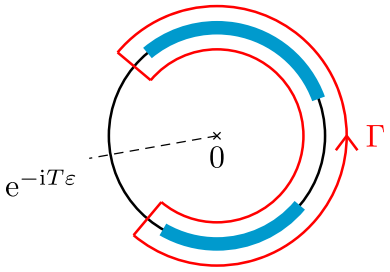

FIGURE 6. Contour Γ to compute the logarithm with branch cut in the spectral gap of $U_{\mathrm{B}}(T)$

where $\eta = \inf_{z\in\Gamma}\big(\mathrm{dist}(z,\sigma(U(T)))\big) > 0$. Then from Proposition 4.7, we know that for λ sufficiently small, let say $\lambda < \lambda^*$ one has

$$\|U_n(T) - U_{\mathrm{B}}(T)\| \leq \alpha_\lambda < \eta \tag{5.47}$$

independently from n. This implies that $1 + (U_n(T) - U_{\mathrm{B}}(T))R_{U_{\mathrm{B}}}(z)$ is invertible for $z \in \Gamma$. Thus,

$$\|R_{U_n}(z)\| = \Big\| R_{U_{\mathrm{B}}}(z)\Big(1 + (U_n(T) - U_{\mathrm{B}}(T))R_{U_{\mathrm{B}}}(z)\Big)^{-1}\Big\| \leq \frac{1}{\eta - \alpha_\lambda}, \tag{5.48}$$

so that $R_{U_n}(z)$ is bounded independently from n. We compute

$$\mathrm{e}^{-\lambda f_n} H_{\mathrm{B}}^\varepsilon \mathrm{e}^{\lambda f_n} - H_{\mathrm{B}}^\varepsilon = -\frac{1}{2\pi\mathrm{i}}\frac{\mathrm{i}}{T}\int_\Gamma \mathrm{d}z \log_{-T\varepsilon}(z)(R_{U_n}(z) - R_{U_{\mathrm{B}}}(z)) \tag{5.49}$$

then again by the resolvent identity and the previous estimates

$$\|R_{U_n}(z) - R_{U_{\mathrm{B}}}(z)\| = \|R_{U_n}(z)(U_{\mathrm{B}}(T) - U_n(T))R_{U_{\mathrm{B}}}(z)\| \leq \frac{1}{\eta - \alpha_\lambda}\alpha_\lambda\frac{1}{\eta}. \tag{5.50}$$

Finally

$$\big\|\mathrm{e}^{-\lambda f_n} H_{\mathrm{B}}^\varepsilon \mathrm{e}^{\lambda f_n} - H_{\mathrm{B}}^\varepsilon\big\| \leq \frac{|\Gamma|}{2\pi T}\sup_{z\in\Gamma}\{|\log(z)|\}\frac{\alpha_\lambda}{\eta(\eta - \alpha_\lambda)} \qquad \equiv \beta_\lambda \tag{5.51}$$

for $\lambda < \lambda^*$ such that $\alpha_\lambda < \eta$. The term on the r.h.s. is finite, independent of n and goes to 0 when $\lambda \to 0$. Thus, we have the same when $n \to \infty$, leading to the result. □

We then study the influence on the choice of ε, first on $H_{\mathrm{B}}^\varepsilon$ then on the bulk index. The proof of Lemma 3.9 is straightforward. Both identities come from the spectral decomposition (5.41) and the properties of the logarithm. The first one from the fact that $\log_{\alpha+2\pi} = \log_\alpha + 2\pi\mathrm{i}$ and the second from

$$\log_{\alpha'}(\mathrm{e}^{\mathrm{i}\phi}) - \log_\alpha(\mathrm{e}^{\mathrm{i}\phi}) = \begin{cases} 0, & (0 \leq \phi < \alpha) \\ 2\pi\mathrm{i}, & (\alpha < \phi < \alpha') \\ 0, & (\alpha' < \phi < 2\pi) \end{cases}. \tag{5.52}$$

As we shall see, (3.19) tells us that we can restrict ε to any interval of length $2\pi/T$ and (3.20) compares two effective Hamiltonians in that interval. In particular, they coincide when $\mathrm{e}^{-\mathrm{i}T\varepsilon}$ and $\mathrm{e}^{-\mathrm{i}T\varepsilon'}$ belong to the same gap.

Proof of Proposition 3.10. By construction, $H_{\mathrm{B}}^\varepsilon$ is time independent, so that the relative evolution is

$$U_{\mathrm{B,rel}}^\varepsilon(t) = \begin{cases} U_{\mathrm{B}}(2t), & (0 \leq t \leq T/2) \\ \exp\big(-\mathrm{i}2(T-t)H_{\mathrm{B}}^\varepsilon\big), & (T/2 \leq t \leq T) \end{cases}. \tag{5.53}$$

From (3.19), we deduce $U_{\mathrm{B,rel}}^{\varepsilon+2\pi/T} = U_{\mathrm{B,rel}}^\varepsilon U_{\mathrm{Id}}$ where

$$U_{\mathrm{Id}}(t) = \begin{cases} I, & (0 \leq t \leq T/2) \\ \exp\Big(-\mathrm{i}2(T-t)\dfrac{2\pi}{T}I\Big), & (T/2 \leq t \leq T) \end{cases} \tag{5.54}$$

that also satisfy $U_{\mathrm{Id}}(T) = I$. Moreover, $\mathcal{I}_{\mathrm{B}}[U_{\mathrm{Id}}] = 0$ since U_{Id} acts trivially on $\mathcal{H}_{\mathrm{B}}$, so that by the additivity from Proposition 5.3 we deduce $\mathcal{I}_{\mathrm{B}}(\varepsilon + 2\pi/T) = \mathcal{I}_{\mathrm{B}}(\varepsilon)$.

Similarly, for $0 \leq \varepsilon' - \varepsilon < 2\pi$ we get from (3.20) that $U^{\varepsilon'}_{\mathrm{B,rel}} = U^{\varepsilon}_{\mathrm{B,rel}} U_{P_{\varepsilon,\varepsilon'}}$ where $U_{P_{\varepsilon,\varepsilon'}}$ is similar to U_{Id} but with $P_{\varepsilon,\varepsilon'}$ instead of I in (5.54). It is then shown in "Appendix A.3" that

$$\mathcal{I}_{\mathrm{B}}[U_{P,\varepsilon,\varepsilon'}] = -2\pi\mathrm{i}\,\mathrm{Tr}\Big(P_{\varepsilon,\varepsilon'}\Big[[\Lambda_1, P_{\varepsilon,\varepsilon'}], [\Lambda_2, P_{\varepsilon,\varepsilon'}]\Big] P_{\varepsilon,\varepsilon'}\Big) = c(P_{\varepsilon,\varepsilon'}) \in \mathbb{Z} \tag{5.55}$$

which is the Kubo-Středa formula or non-commutative Chern number of $P_{\varepsilon,\varepsilon'}$ from the Quantum Hall effect [2]. We conclude by the additivity property of $\mathcal{I}_{\mathrm{B}}$ from Proposition 5.3. □

We finally deal with continuity properties.

Proposition 5.7. *It exists $\lambda, \nu > 0$ such that $U_{\mathrm{B}} \mapsto H^{\varepsilon}_{\mathrm{B}}$ is continuous with respect to $\|\cdot\|_{\nu}$ and $\|\cdot\|_{\lambda}$ as long as $\mathrm{e}^{-\mathrm{i}T\varepsilon}$ lies in a spectral gap of $U_{\mathrm{B}}(T)$.*

Proof. Let $U_{\mathrm{B},1}(T)$ and $U_{\mathrm{B},2}(T)$ with $\mathrm{e}^{-\mathrm{i}T\varepsilon}$ belonging to a common spectral gap. Take $0 < \lambda < \lambda^*$ from Proposition 5.6 so that $H^{\varepsilon}_{\mathrm{B},1}$ and $H^{\varepsilon}_{\mathrm{B},2}$ are both local with common exponent λ. Similarly to the proof of Proposition 5.6

$$\begin{aligned} &\mathrm{e}^{-\lambda f}(H^{\varepsilon}_{\mathrm{B},1} - H^{\varepsilon}_{\mathrm{B},2})\mathrm{e}^{\lambda f} \\ &\quad = -\frac{1}{2\pi\mathrm{i}}\frac{\mathrm{i}}{T}\int_{\Gamma} \mathrm{d}z \log_{-T\varepsilon}(z) R_{U_{1f}}(z)\mathrm{e}^{-\lambda f}(U_{\mathrm{B},1} - U_{\mathrm{B},2})(T)\mathrm{e}^{\lambda f} R_{U_{2f}}(z) \end{aligned} \tag{5.56}$$

where $U_{if} = \mathrm{e}^{-\lambda f} U_{\mathrm{B},i}(T)\mathrm{e}^{\lambda f}$, Γ is a contour common to $U_{\mathrm{B},1}(T)$ and $U_{\mathrm{B},2}(T)$, and where we have used the resolvent identity. We know from the previous proof that $R_{U_{1f}}$ and $R_{U_{2f}}$ are both bounded for $z \in \Gamma$. By Proposition 4.7 and Corollary 4.2, we know that $\mathrm{e}^{-\lambda f}(U_{\mathrm{B},1} - U_{\mathrm{B},2})(T)\mathrm{e}^{\lambda f}$ is bounded by $\|(U_{\mathrm{B},1} - U_{\mathrm{B},2})\|_{\nu}$ for some $\nu > \lambda$. Thus,

$$\big\|\mathrm{e}^{-\lambda f}(H^{\varepsilon}_{\mathrm{B},1} - H^{\varepsilon}_{\mathrm{B},2})\mathrm{e}^{\lambda f}\big\| \leq B\,\|(U_{\mathrm{B},1} - U_{\mathrm{B},2})\|_{\nu} \tag{5.57}$$

and consequently we have a similar estimate for $\|(H^{\varepsilon}_{\mathrm{B},1} - H^{\varepsilon}_{\mathrm{B},2})\|_{\lambda}$. □

Together with Proposition 4.8, we deduce that $H_{\mathrm{B}} \mapsto U^{\varepsilon}_{\mathrm{B,rel}}$ is continuous, respectively, with $\|\cdot\|_{\mu}$ and $\|\cdot\|_{\lambda}$. This proves Corollary 3.12 on homotopy invariance of $\mathcal{I}_{\mathrm{B}}$.

5.3. Interface Index Properties

Note that one can also embed the edge Hamiltonians instead of gluing the bulk ones. Namely by considering $\mathbb{N}_- = \mathbb{Z}\backslash\mathbb{N}$, $\mathcal{H}^-_{\mathrm{E}} = \ell^2(\mathbb{N}_- \times \mathbb{Z}) \otimes \mathbb{C}^N$, $\iota_- : \mathcal{H}^-_{\mathrm{E}} \to \mathcal{H}_{\mathrm{B}}$ and $\iota_-^* : \mathcal{H}_{\mathrm{B}} \to \mathcal{H}^-_{\mathrm{E}}$, one has

$$H_{\mathrm{I}} = \iota H_{\mathrm{E},1}\iota^* + \iota_- H^-_{\mathrm{E},2}\iota_-^* + H_{\mathrm{int}} \tag{5.58}$$

where we have defined $H^-_{\mathrm{E}} = \iota_-^* H_{\mathrm{B}}\iota$, namely the edge Hamiltonian on the other half space, and used (2.4) and similarly $\iota_-^*\iota_- = \mathrm{Id}_{\mathcal{H}^-_{\mathrm{E}}}$ and $\iota_-\iota_-^* = 1 - P_1$.

Lemma 5.8. *Let H_{I} be the interface Hamiltonian from Definition 3.14. Then the corresponding propagator satisfies*

$$U_{\mathrm{I}}(t) = \iota U_{\mathrm{E},1}(t)\iota^* + \iota_- U^-_{\mathrm{E},2}(t)\iota^*_- + D_{\mathrm{I}}(t) \tag{5.59}$$

where $U^-_{\mathrm{E},2}$ is generated by H^-_{E} and $D_{\mathrm{I}}(t)$ is simultaneously 1-confined and 2-local.

Proof. From (5.58) and Definition 3.14

$$\iota^* H_I \iota = H_{\mathrm{E},1} + \iota^* H_{\mathrm{int}} \iota, \tag{5.60}$$

where $\iota^* H_{\mathrm{int}} \iota$ being simultaneously 1-confined and 2-local on $\mathcal{H}_{\mathrm{E}}$ plays the role of a boundary condition as in Proposition 5.1, and H_{I} that of the bulk Hamiltonian for $H_{\mathrm{E},1}$. In particular, by adapting the proof of Proposition 4.10

$$U_{\mathrm{E},1}(t) = \iota^* U_{\mathrm{I}}(t)\iota + D_{\mathrm{I},1}(t) \quad \Rightarrow \quad P_1 U_I P_1 = \iota(U_{\mathrm{E},1} - D_{\mathrm{I},1})\iota^* \tag{5.61}$$

where $D_{\mathrm{I},1}$ is simultaneously 1-confined and 2-local and where we have used (2.4). Similarly:

$$(1-P_1)U_I(1-P_1) = \iota_-(U^-_{\mathrm{E},2} - D_{\mathrm{I},2})\iota^*_- \tag{5.62}$$

on the other half space. By decomposing U_{I} over subspaces associated with P_1 and $1-P_1$ we get

$$U_{\mathrm{I}} = \iota U_{\mathrm{E},1}\iota^* + \iota_- U^-_{\mathrm{E},2}\iota^*_- - \iota D_{\mathrm{I},1}\iota^* - \iota_- D_{\mathrm{I},2}\iota^*_- + P_1 U_{\mathrm{I}}(1-P_1) + (1-P_1)U_{\mathrm{I}}P_1 \tag{5.63}$$

Each of the last four terms is simultaneously 1-confined and 2-local from the properties of $D_{\mathrm{I},1}$, $D_{\mathrm{I},2}$ and the fact that U_{I} is local. Together they define D_{I}. □

***Proof of Proposition** 3.15.* From Lemma 5.8 we get $U_{\mathrm{I}}(T)$ in terms of $U_{\mathrm{E},1}(T)$ and $U_{\mathrm{E},2}(T)$, but since the corresponding bulk propagator are not I, we need to normalize $U_{\mathrm{I}}(T)$ as in (3.28). In particular, consider the special interface

$$H_{\mathrm{B},2}(t) = \iota H_{\mathrm{E},2}\iota^* + \iota_- H^-_{\mathrm{E},2}\iota^*_- + P_1 H_{\mathrm{B},2}(1-P_1) + (1-P_1)H_{\mathrm{B},2}P_1 \tag{5.64}$$

which is nothing but an interface decomposition of $H_{\mathrm{B},2}$. Lemma 5.8 gives

$$U_{\mathrm{B},2}(T) = \iota U_{\mathrm{E},2}(T)\iota^* + \iota_- U^-_{\mathrm{E},2}(T)\iota^*_- + \widetilde{D_{\mathrm{I}}}(T) \tag{5.65}$$

Hence, by Lemma 5.8 applied for U_{I} and $U_{\mathrm{B},2}$ we deduce after some algebra

$$U^*_{\mathrm{B},2}U_{\mathrm{I}}(T) = \iota U^*_{\mathrm{E},2}U_{\mathrm{E},1}(T)\iota^* + \iota_- \mathrm{Id}_{\mathcal{H}^-_{\mathrm{E}}}\iota^*_- + \widehat{D}(T) \tag{5.66}$$

where we have used $\iota^*_-\iota = 0$ and $\iota^*\iota_- = 0$ and where $\widehat{D}(T)$ is simultaneously 1-confined and 2-local. Finally, from Proposition 4.10 and the fact that $U_{\mathrm{B},1}(T) = U_{\mathrm{B},2}(T)$ we deduce at $t = T$

$$U^*_{\mathrm{E},2}U_{\mathrm{E},1} = I + \iota^* U^*_{\mathrm{B},2}[P_1, U_{\mathrm{B},1}]\iota + \iota^* U^*_{\mathrm{B},2}\iota D_1 + D^*_2(\iota^* U_{\mathrm{B},1}\iota + D_1) \tag{5.67}$$

where each term except I is simultaneously 1-confined and 2-local. Putting all together, we deduce that $[\Lambda_2, U^*_{\mathrm{B},2}U_{\mathrm{I}}(T)]$ is trace class so that $\mathcal{I}_{\mathrm{I}}$ is finite. Similarly to the proof of Proposition 3.5, $\mathcal{I}_{\mathrm{I}}$ can be identified with an index of a pair of projections so it is integer valued, independent of Λ_2 and continuous

(with the local norm) in $U_{\mathrm{B},2}$ and U_{I}. In particular, consider the deformation of the previous derivation to the sharp interface where the two halves are disconnected

$$H_{\mathrm{I}} = \iota H_{\mathrm{E},1}\iota^* + \iota_- H^-_{\mathrm{E},2}\iota_-^*, \qquad \widetilde{H}_{\mathrm{B},2} = \iota H_{\mathrm{E},2}\iota^* + \iota_- H^-_{\mathrm{E},2}\iota_-^* \tag{5.68}$$

In that case the corresponding evolutions are also disconnected so that in (5.66) $\widehat{D}(T) = 0$, and we deduce $\mathcal{I}_{\mathrm{I}} = \mathcal{I}^{\mathrm{rel}}_{\mathrm{E}}$ from expression (3.26) and $\mathrm{Tr}_{\mathcal{H}_{\mathrm{B}}}(\iota O\iota^*) = \mathrm{Tr}_{\mathcal{H}_{\mathrm{E}}}(O)$. □

Appendix A. Some Algebraic Computations

A.1. Additivity of the Bulk Index

Here we prove Proposition 5.3. It is purely algebraic but quite tedious. From the definition (3.5) of $\mathcal{I}_{\mathrm{B}}$, we compute

$$\begin{aligned}
&\mathrm{Tr}\,(UV)^*\partial_t(UV)\Big[(UV)^*\big[\Lambda_1, UV\big], (UV)^*\big[\Lambda_2, UV\big]\Big] \\
&\quad = \mathrm{Tr}\,U^*\partial_t U\Big[U^*\big[\Lambda_1, U\big], U^*\big[\Lambda_2, U\big]\Big] + \mathrm{Tr}\,V^*\partial_t V\Big[V^*\big[\Lambda_1, V\big], V^*\big[\Lambda_2, V\big]\Big] \\
&\qquad \left.\begin{aligned}
&+\mathrm{Tr}\,U^*\partial_t U\Big[\big[\Lambda_1, V\big]V^*, U^*\big[\Lambda_2, U\big]\Big] + \mathrm{Tr}\,U^*\partial_t U\Big[U^*\big[\Lambda_1, U\big], \big[\Lambda_2, V\big]V^*\Big] \\
&+\mathrm{Tr}\,U^*\partial_t U\Big[\big[\Lambda_1, V\big]V^*, \big[\Lambda_2, V\big]V^*\Big] + \mathrm{Tr}\,(\partial_t V)V^*\Big[U^*\big[\Lambda_1, U\big], U^*\big[\Lambda_2, U\big]\Big] \\
&+\mathrm{Tr}\,(\partial_t V)V^*\Big[\big[\Lambda_1, V\big]V^*, U^*\big[\Lambda_2, U\big]\Big] + \mathrm{Tr}\,(\partial_t V)V^*\Big[U^*\big[\Lambda_1, U\big], \big[\Lambda_2, V\big]V^*\Big]
\end{aligned}\right\} \equiv R
\end{aligned} \tag{A.1}$$

where we have used Leibniz rule for ∂_t and $[\Lambda_i, \cdot\,]$ and the cyclicity of trace (note that each written term is trace class by Lemma 4.5 as long as U and V are local). The two first terms in the latter equation correspond to the index of U and V when integrated over time. After a bit of algebra, one can check that the remaining last three lines are actually equal to

$$\begin{aligned}
R = &-\mathrm{Tr}\Big[\Lambda_1, U^*\big[\Lambda_2, U\big](\partial_t V)V^* - U^*\partial_t U\big[\Lambda_2, V\big]V^*\Big] \\
&-\mathrm{Tr}\Big[\Lambda_2, U^*\partial_t U\big[\Lambda_2, V\big]V^* - U^*\big[\Lambda_1, U\big](\partial_t V)V^*\Big] \\
&-\mathrm{Tr}\,\partial_t\Big(U^*\big[\Lambda_1, U\big]\big[\Lambda_2, V\big]V^* - U^*\big[\Lambda_2, U\big]\big[\Lambda_1, V\big]V^*\Big).
\end{aligned} \tag{A.2}$$

The first two terms are trace class with vanishing trace according to Lemma 4.6, and the last one is a total time derivative that vanishes when integrated over time since U and V are periodic by assumption. Thus, R vanishes when integrated from 0 to T so that (A.1) leads to the expected result. Note that this proof is nothing but the one given in [4] in the periodic case adapted to the derivatives $[\Lambda_i, \cdot\,]$ for the space directions.

A.2. Proof of Identity (5.39)

In the following, all the traces involved are finite using that U_{B} is local and Lemma 4.4 and 4.5. On the one hand, we can expand

$$\mathrm{Tr}_{\mathcal{H}_\mathrm{B}}\Big([\Lambda_2,U_\mathrm{B}]U_\mathrm{B}^*\,[(\partial_t U_\mathrm{B})U_\mathrm{B}^*,P_{1,r}]\Big)=\mathrm{Tr}_{\mathcal{H}_\mathrm{B}}\Big([\Lambda_2,U_\mathrm{B}]U_\mathrm{B}^*\,[\partial_t U_\mathrm{B},P_{1,r}]U_\mathrm{B}^*\Big)$$
$$-\mathrm{Tr}_{\mathcal{H}_\mathrm{B}}\Big([\Lambda_2,U_\mathrm{B}]U_\mathrm{B}^*\,\partial_t U_\mathrm{B}U_\mathrm{B}^*\,[U_\mathrm{B},P_{1,r}]U_\mathrm{B}^*\Big) \tag{A.3}$$

and on the other hand we notice that, due to Lemma 4.6

$$0=\mathrm{Tr}_{\mathcal{H}_\mathrm{B}}\Big[[\Lambda_2,U_\mathrm{B}]U_\mathrm{B}^*\,(\partial_t U_\mathrm{B})U_\mathrm{B}^*,P_{1,r}\Big]=\mathrm{Tr}_{\mathcal{H}_\mathrm{B}}\Big([\Lambda_2,U_\mathrm{B}]U_\mathrm{B}^*\,[(\partial_t U_\mathrm{B})U_\mathrm{B}^*,P_{1,r}]\Big)$$
$$+\mathrm{Tr}_{\mathcal{H}_\mathrm{B}}\Big(\Big[[\Lambda_2,U_\mathrm{B}]U_\mathrm{B}^*\,,P_{1,r}\Big](\partial_t U_\mathrm{B})U_\mathrm{B}^*\Big). \tag{A.4}$$

The first term is the one of interest, and the second can be expanded

$$\mathrm{Tr}_{\mathcal{H}_\mathrm{B}}\Big(\Big[[\Lambda_2,U_\mathrm{B}]U_\mathrm{B}^*\,,P_{1,r}\Big](\partial_t U_\mathrm{B})U_\mathrm{B}^*\Big)=\mathrm{Tr}_{\mathcal{H}_\mathrm{B}}\Big(\Big[[\Lambda_2,U_\mathrm{B}],P_{1,r}\Big]U_\mathrm{B}^*\,(\partial_t U_\mathrm{B})U_\mathrm{B}^*\Big)$$
$$-\mathrm{Tr}_{\mathcal{H}_\mathrm{B}}\Big([\Lambda_2,U_\mathrm{B}]U_\mathrm{B}^*\,[U_\mathrm{B},P_{1,r}]U_\mathrm{B}^*\,\partial_t U_\mathrm{B}U_\mathrm{B}^*\Big). \tag{A.5}$$

Then we rewrite the first term appearing here using an integration by parts, namely

$$\mathrm{Tr}_{\mathcal{H}_\mathrm{B}}\Big(\Big[[\Lambda_2,U_\mathrm{B}],P_{1,r}\Big]U_\mathrm{B}^*\,(\partial_t U_\mathrm{B})U_\mathrm{B}^*\Big)$$
$$=-\partial_t\mathrm{Tr}_{\mathcal{H}_\mathrm{B}}\Big(\Big[[\Lambda_2,U_\mathrm{B}],P_{1,r}\Big]U_\mathrm{B}^*\Big)+\mathrm{Tr}_{\mathcal{H}_\mathrm{B}}\Big(\Big[[\Lambda_2,\partial_t U_\mathrm{B}],P_{1,r}\Big]U_\mathrm{B}^*\Big). \tag{A.6}$$

Finally, similarly as before,

$$\mathrm{Tr}_{\mathcal{H}_\mathrm{B}}\Big(\Big[[\Lambda_2,\partial_t U_\mathrm{B}],P_{1,r}\Big]U_\mathrm{B}^*\Big)$$
$$=-\mathrm{Tr}_{\mathcal{H}_\mathrm{B}}\Big(\Big[[\partial_t U_\mathrm{B},P_{1,r}],\Lambda_2\Big]U_\mathrm{B}^*\Big)$$
$$=-\mathrm{Tr}_{\mathcal{H}_\mathrm{B}}\Big(\Big[[\partial_t U_\mathrm{B},P_{1,r}]U_\mathrm{B}^*,\Lambda_2\Big]\Big)+\mathrm{Tr}_{\mathcal{H}_\mathrm{B}}\Big([\partial_t U_\mathrm{B},P_{1,r}][U_\mathrm{B}^*,\Lambda_2]\Big) \tag{A.7}$$

where the first term vanishes by Lemma 4.6. Putting together the last three equations, we deduce

$$\mathrm{Tr}_{\mathcal{H}_\mathrm{B}}\Big([\Lambda_2,U_\mathrm{B}]U_\mathrm{B}^*\,[(\partial_t U_\mathrm{B})U_\mathrm{B}^*,P_{1,r}]\Big)$$
$$=\partial_t\mathrm{Tr}_{\mathcal{H}_\mathrm{B}}\Big(\Big[[\Lambda_2,U_\mathrm{B}],P_{1,r}\Big]U_\mathrm{B}^*\Big)-\mathrm{Tr}_{\mathcal{H}_\mathrm{B}}\Big([U_\mathrm{B}^*,\Lambda_2][\partial_t U_\mathrm{B},P_{1,r}]\Big)$$
$$+\mathrm{Tr}_{\mathcal{H}_\mathrm{B}}\Big([\Lambda_2,U_\mathrm{B}]U_\mathrm{B}^*\,[U_\mathrm{B},P_{1,r}]U_\mathrm{B}^*\,\partial_t U_\mathrm{B}U_\mathrm{B}^*\Big). \tag{A.8}$$

Noticing that $[U_\mathrm{B}^*,\Lambda_2]=-U_\mathrm{B}^*[U_\mathrm{B},\Lambda_2]U_\mathrm{B}^*$ and summing (A.3) and (A.8) we get identity (5.39).

A.3. Proof of Identity (5.55)

We first rewrite $P=P_{\varepsilon,\varepsilon'}$ and $\mathrm{Tr}=\mathrm{Tr}_{\mathcal{H}_\mathrm{B}}$. Note that P is a spectral projector of $U_\mathrm{B}(T)$ so it is also local and all the following traces are finite. By definition the first half of the time integral is trivial for $U_{P,\varepsilon,\varepsilon'}$ (defined similarly to (5.54)). So that up to a change in variables

$$\mathcal{I}_\mathrm{B}[U_{P,\varepsilon,\varepsilon'}]=\frac{\mathrm{i}\pi}{T}\int_0^T \mathrm{d}t\mathrm{Tr}\Big(P\Big[\mathrm{e}^{2\pi\mathrm{i}\frac{t}{T}P}\big[\Lambda_1,\mathrm{e}^{-2\pi\mathrm{i}\frac{t}{T}P}\big],\mathrm{e}^{2\pi\mathrm{i}\frac{t}{T}P}\big[\Lambda_2,\mathrm{e}^{-2\pi\mathrm{i}\frac{t}{T}P}\big]\Big]\Big), \tag{A.9}$$

where we have used the fact that $\mathrm{e}^{-2\pi\mathrm{i}\frac{t}{T}P} = \mathrm{e}^{-2\pi\mathrm{i}\frac{t}{T}}P + I - P$. Then we notice that, since $P^2 = P$

$$P\Big[\big[\Lambda_1, P\big], \big[\Lambda_2, P\big]\Big]P = -P\Lambda_1(I-P)\Lambda_2 P + P\Lambda_2(I-P)\Lambda_1 P$$
$$= P\Big[\big[\Lambda_1, P\big], P\big[\Lambda_2, P\big]\Big]P + P\Big[P\big[\Lambda_1, P\big], \big[\Lambda_2, P\big]\Big]P \tag{A.10}$$

and

$$\mathrm{Tr}\Big(P\Big[P\big[\Lambda_1, P\big], P\big[\Lambda_2, P\big]\Big]P\Big) = 0 \tag{A.11}$$

Then expanding $\mathrm{e}^{-2\pi\mathrm{i}\frac{t}{T}P} = \mathrm{e}^{-2\pi\mathrm{i}\frac{t}{T}}P + I - P$ in the trace of the previous integral we are left after some algebra with

$$\mathrm{Tr}\Big(P\Big[\mathrm{e}^{2\pi\mathrm{i}\frac{t}{T}P}\big[\Lambda_1, \mathrm{e}^{-2\pi\mathrm{i}\frac{t}{T}P}\big], \mathrm{e}^{2\pi\mathrm{i}\frac{t}{T}P}\big[\Lambda_2, \mathrm{e}^{-2\pi\mathrm{i}\frac{t}{T}P}\big]\Big]\Big)$$
$$= 2\Big(\cos\Big(\frac{2\pi t}{T}\Big) - 1\Big)\mathrm{Tr}\Big(P\Big[\big[\Lambda_1, P\big], \big[\Lambda_2, P\big]\Big]P\Big), \tag{A.12}$$

which leads to (5.55) after integration over t.

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Communicated by Vieri Mastropietro.

Received: September 11, 2017.

Accepted: January 11, 2018.