Approach to equilibrium in translation-invariant quantum systems

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*"The zeroth law deals with the observed fact that a large system seems to normally have "states" described by a few macroscopic parameters like a temperature and density, and that any system not in one of these states, left alone, rapidly **approaches one of these states**. When Boltzmann and Gibbs tried to find a macroscopic basis for thermodynamics, they realized that the **approach to equilibrium** was the most puzzling and deepest problem in such a formalism."*

from Simon, B.: The Statistical Mechanics of Lattice Gasses.

Heuristic motivation

- Consider an infinite quantum system on a d-dimensional lattice
- At $t = 0$: consider a translation invariant state ρ_0.
  Ex: equilibrium state at inverse temperature β "$\rho_0 = \mathrm{e}^{-\beta H_1}$"
- Evolve it with a distinct dynamics for $t > 0$ "$\tau_t = \mathrm{e}^{\mathrm{i}tH_2} \cdot \mathrm{e}^{-\mathrm{i}tH_2}$"

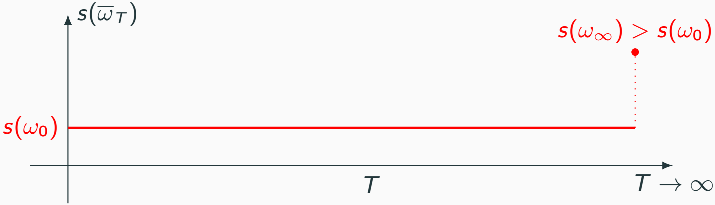
What happens to the state as $t \to \infty$?

- Equilibrium states for infinite systems are well known: KMS states
- A lot is known when H_2 is a local perturbation: return to equilibrium, non-equilibrium steady states, entropy production...

What happens when the perturbation is infinite, but translation invariant ?

Our contribution

1. Formulate the problem
2. Establish some structural results about entropy and irreversibility
3. In a nutshell:



4. Identify weak-Gibbs states ω and their related entropy balance equation:

$$s(\nu|\omega) = -s(\nu) + \nu(E_\Phi) + p(\Phi).$$

5. Infer some consequences for adiabatic theory (not in today's talk).

Quantum spins on a lattice

Statements from this section can be found in

Bratteli-Robinson I and II '79/81, Israel '79, Simon '93...

Consider the d-dimensional lattice $\mathbb{Z}^d$ (d arbitrary) and a Hilbert space $\mathcal{H}_0 \cong \mathbb{C}^2$.

Let $\mathcal{F} = \{X \subset \mathbb{Z}^d \,|\, |X| < \infty\}$.

- For $x \in \mathbb{Z}^d$, $\mathcal{H}_x = \mathcal{H}_0$
- For $X \in \mathcal{F}$, $\mathcal{H}_X = \bigotimes_{x \in X} \mathcal{H}_X$ and $\mathcal{U}_X = \mathcal{B}(\mathcal{H}_X)$
- For $X \subset X'$, $\mathcal{U}_X \subset \mathcal{U}_{X'}$ via $A \mapsto A \otimes 1_{X' \setminus X}$
- $\mathcal{U}_{\mathrm{loc}} = \bigcup_{X \in \mathcal{F}} \mathcal{U}_X$ and $\mathcal{U} = \overline{\mathcal{U}_{\mathrm{loc}}}$ (in norm) is the spin C^*-algebra.

Remark: what follows also works for the (even) fermionic CAR algebra (the proof are very similar, see Araki-Moriya '03)

States and entropy

For $x \in \mathbb{Z}^d$ translation automorphism τ_x sending $\mathcal{U}_X$ to $\mathcal{U}_{X+x}$.

Translation invariant states

$\mathcal{S}_{\mathrm{I}} = \{\nu : \mathcal{U} \to \mathbb{C} \,|\, \nu \,\mathrm{positive, linear,}\, \nu(1) = 1 \,\mathrm{and}\, \nu \circ \tau_x = \nu, \, \forall x \in \mathbb{Z}^d\}$

For $\nu \in \mathcal{S}_{\mathrm{I}}$ and $\Lambda \in \mathcal{F}$ a cube of $\mathbb{Z}^d$ centered at 0, let ν_Λ be the restriction of ν to $\mathcal{U}_\Lambda$, i.e. $\nu(A) = \mathrm{tr}(\nu_\Lambda A)$ for all $A \in \mathcal{U}_\Lambda$.

- Mean specific entropy: the limit

$$s(\nu) = -\lim_{\Lambda \to \mathbb{Z}^d} \frac{1}{|\Lambda|} \mathrm{tr}(\nu_\Lambda \log(\nu_\Lambda))$$

  exists and $s : \mathcal{S}_{\mathrm{I}} \to [0, \log(2)]$ is affine and upper-semicontinuous.

- Mean relative entropy:

$$s(\nu|\omega) = -\lim_{\Lambda \to \mathbb{Z}^d} \frac{1}{|\Lambda|} \mathrm{tr}\big(\nu_\Lambda(\log(\omega_\Lambda) - \log(\nu_\Lambda))\big)$$

  If such a limit exists, then $s(\nu|\omega) \geq 0$.

Interactions, local Hamiltonian

An interaction is a family $\{\Phi(X)\}_{X\in\mathcal{F}}$ such that $\Phi(X) \in \mathcal{U}_X$ is a self-adjoint. Moreover we assume:

1. Translation invariance: $\tau_x(\Phi(X)) = \Phi(X+x)$
2. Short range:
$$\|\Phi\|_r = \sum_{X\ni 0} \mathrm{e}^{r(|X|-1)} \|\Phi(X)\| < \infty$$
(think finite range: $\Phi(X) = 0$ if $\mathrm{diam}(X) > r$)

The set of interactions is denoted by $\mathcal{B}_r$.

For $\Lambda \in \mathcal{F}$ a cube of $\mathbb{Z}^d$ centered at 0, the local Hamiltonian is $H_\Phi(\Lambda) = \sum_{X\subset\Lambda} \Phi(X)$

Examples:

- Ising model for spins $\Phi(X) = -\sigma_i^z\sigma_j^z$ if $X = \{i,j\}, |i-j| = 1$ and 0 otherwise.
- Quasi-free Fermions $\Phi(X) = a_i^* a_j + a_j^* a_i$ if $X = \{i,j\}, |i-j| = 1$ and 0 otherwise.

$$H_\Phi(\Lambda) = -\sum_{\substack{i,j\in\Lambda\\|i-j|=1}} \sigma_i^z\sigma_j^z, \qquad H_\Phi(\Lambda) = \sum_{\substack{i,j\in\Lambda\\|i-j|=1}} a_i^* a_j + a_j^* a_i$$

Pressure and mean energy

The local Hamiltonian

$$H_\Phi(\Lambda) = \sum_{X \subset \Lambda} \Phi(X)$$

has no limit in $\mathcal{U}$ as $\Lambda \to \mathbb{Z}^d$. However we can define the following quantities.

- Pressure (or Helmholtz free energy) of interaction $\Phi \in \mathcal{B}_r$:

$$p(\Phi) = \lim_{\Lambda \to \mathbb{Z}^d} \frac{1}{|\Lambda|} \log(\mathrm{tr}(\mathrm{e}^{-\beta H_\Phi(\Lambda)})) < \infty$$

- Mean energy of interaction $\Phi \in \mathcal{B}_r$ in a state $\nu \in \mathcal{S}_\mathrm{I}$:

$$\lim_{\Lambda \to \mathbb{Z}^d} \frac{1}{|\Lambda|} \nu(H_\Phi(\Lambda)) = \nu(E_\Phi), \qquad E_\Phi = \sum_{X \ni 0} \frac{\Phi(X)}{|X|}$$

Notice: entropy of <u>state</u>, pressure of <u>interaction</u> and energy of <u>interaction and state</u>.

Equilibrium states

Gibbs variational principle

For any $\Phi \in \mathcal{B}_r$

$$p(\Phi) = \sup_{\nu \in \mathcal{S}_{\mathrm{I}}} \big(s(\nu) - \beta\nu(E_\Phi)\big)$$

Equilibrium states:

$$\mathcal{S}_{\mathrm{eq}}(\Phi) = \{\nu \in \mathcal{S}_{\mathrm{I}} \mid p(\Phi) = s(\nu) - \beta\nu(E_\Phi)\}$$

In the following we set $\beta = 1$, so that high-temperature = small interaction.

Dynamics

Let $\Phi \in \mathcal{B}_r$ and $t \in \mathbb{R}$. For $A \in \mathcal{U}_X$ with $X \in \mathcal{F}$ the limit

$$\tau_t^\Phi(A) = \lim_{\Lambda \to \mathbb{Z}^d} \mathrm{e}^{\mathrm{i}tH_\Phi(\Lambda)} A \mathrm{e}^{-\mathrm{i}tH_\Phi(\Lambda)}$$

exists, is uniform for t in compact sets and extends to a strongly continuous one-parameter group of *-automorphism on $\mathcal{U}$, that we denote by τ_t^Φ.

Prop: If $\nu \in \mathcal{S}_{\mathrm{eq}}(\Phi)$ then $\nu \circ \tau_t^\Phi = \nu$.

(the reciprocal statement involves the KMS condition.)

Back to our question:

Take two distinct interactions Ψ and Φ on $\mathcal{U}$ (spins or fermions), let $\nu_0 \in \mathcal{S}_{\mathrm{eq}}(\Psi)$ and define $\nu_t = \nu_0 \circ \tau_t^\Phi$.

What can we say about ν_t as $t \to \infty$?

Approach to equilibrium

Let $\omega_0 \in \mathcal{S}_{\mathrm{I}}$ and $\Phi \in \mathcal{B}_r$. For $T > 0$ define

$$\overline{\omega}_T = \frac{1}{T} \int_0^T \omega_0 \circ \tau_t^{\Phi} \mathrm{d}t$$

Consider the set of Equilibrium Steady States (ESS) (in contrast to NESS):

$$\mathcal{S}_\infty(\omega_0, \Phi) = \{\mathrm{weak} * - \lim(\overline{\omega}_T)_{T>0},\ T \to \infty\}.$$

One has:

$$\omega_\infty \in \mathcal{S}_\infty(\omega_0, \Phi) \qquad \Leftrightarrow \qquad \exists (T_n)_n, T_n \to \infty, \forall A \in \mathcal{U}, \quad \omega_\infty(A) = \lim_{n \to \infty} \overline{\omega}_{T_n}(A).$$

Moreover, $\omega_\infty \in \mathcal{S}_{\mathrm{I}}$ and $\omega_\infty \circ \tau_t^{\Phi} = \omega_\infty$ (stationary states).

Approach to equilibrium

We speak about approach to equilibrium when $\omega_\infty \in S_\infty(\omega_0, \Phi) \cap S_{\mathrm{eq}}(\Phi)$

1. When do we have it? When do we have $S_\infty(\omega_0, \Phi) \subset S_{\mathrm{eq}}(\Phi)$? When is $S_\infty(\omega_0, \Phi)$ a singleton? (mostly open)
2. What are the general properties of $S_\infty(\omega_0, \Phi)$, independent from the details of ω_0 and Φ? (structural theory)

Very little is known about $S_\infty(\omega_0, \Phi)$, except in a few cases, such as:

- Lanford-Robinson '71 (see also Haag-Radison-Kastler '73 and Sukhov '83): quasi-free interaction for fermions

$$H_\Phi(\Lambda) = \sum_{i,j \in \Lambda} h(i-j)(a_i^* a_j + a_j^* a_i)$$

- Radin '70: generalized Ising model for spins

$$H_\Phi(\Lambda) = \sum_{i,j \in \Lambda} h(i-j)\sigma_i^z \sigma_j^z$$

In both cases, the dynamics is exactly solvable. However, $\omega_\infty \notin S_{\mathrm{eq}}(\Phi)$.

A question of Ruelle

For finite and closed systems, it is a well known fact that the von Neumann entropy $S(\rho) = \mathrm{tr}(\rho \ln \rho)$ is preserved by unitary evolution.

> *"It is unclear to the author whether the evolution of an infinite system should increase its entropy per unit volume. Another possibility is that, when the time tends to ∞, a state has a limit with strictly larger entropy."*

Ruelle, D.: States of classical statistical mechanics. J. Math. Phys. 1967

Conservation laws at finite time

Recall

$$\omega(E_\Phi) = \lim_{\Lambda\to\mathbb{Z}^d} \frac{1}{|\Lambda|}\omega(H_\Phi(\Lambda)), \qquad s(\omega) = -\lim_{\Lambda\to\mathbb{Z}^d} \frac{1}{|\Lambda|}\mathrm{tr}(\omega_\Lambda \log(\omega_\Lambda)).$$

Proposition

For any $\omega \in \mathcal{S}_{\mathrm{I}}$, $\Phi \in \mathcal{B}_r$ and $t \in \mathbb{R}$

1. $\omega \circ \tau_t^\Phi(E_\Phi) = \omega(E_\Phi)$ [Jaksic-Pillet-T. '24]
2. $s(\omega \circ \tau_t^\Phi) = s(\omega)$ [Lanford-Robinson '68]

The first result is expected but not straightforward. The second one is more surprising and goes back to Lanford and Robinson '68 in the case of spin systems. Intuitively, the von Neumann entropy $S(\rho) = \mathrm{tr}(\rho \ln \rho)$ of a closed finite system cannot change by a unitary evolution. The argument extends to the mean entropy of infinite systems thank to the $|\Lambda|^{-1}$-term which cancels possible boundary effects.

The last results naturally extend to the averaged dynamics $\overline{\omega}_T = \dfrac{1}{T}\displaystyle\int_0^T \omega_0 \circ \tau_t^{\Phi} \mathrm{d}t$

Proposition

For all $T \in \mathbb{R}$, $\overline{\omega}_T(E_\Phi) = \omega_0(E_\Phi)$ and $s(\overline{\omega}_T) = s(\omega_0)$.

By (semi-) continuity:

Proposition

For $\omega_\infty \in \mathcal{S}_\infty(\omega_0, \Phi)$

- $s(\omega_0) \leq s(\omega_\infty)$
- $\omega_0(E_\Phi) = \omega_\infty(E_\Phi)$

When do we have strict increase of entropy ?

Weak Gibbs states

Defintion

A state $\omega \in \mathcal{S}_I(\mathcal{U})$ is called weak Gibbs for interaction $\Phi \in \mathcal{B}^r$ if there are constants $C_\Lambda > 0$ satisfying

$$C_\Lambda^{-1} \frac{\mathrm{e}^{-H_\Phi(\Lambda)}}{\mathrm{tr}(\mathrm{e}^{-H_\Phi(\Lambda)})} \leq \omega_\Lambda \leq C_\Lambda \frac{\mathrm{e}^{-H_\Phi(\Lambda)}}{\mathrm{tr}(\mathrm{e}^{-H_\Phi(\Lambda)})}, \qquad \lim_{\Lambda \uparrow \mathbb{Z}^d} \frac{1}{|\Lambda|} \log C_\Lambda = 0$$

The set of weak Gibbs states for Φ is denoted by $\mathcal{S}_{\mathrm{wg}}(\Phi)$.

This is the quantum analogue of classical weak Gibbs states [Yuri '02].

Prop: In particular, $\mathcal{S}_{\mathrm{wg}}(\Phi) \subset \mathcal{S}_{\mathrm{eq}}(\Phi)$

Theorem [Jaksic-Pillet-T. '24]

Suppose that either $d = 1$ and $\Phi \in \mathcal{B}_{\mathrm{f}}$ or $d \geq 1$, $\Phi \in \mathcal{B}^r$, and $\|\Phi\|_r < r$.

Then $\mathcal{S}_{\mathrm{eq}}(\Phi) = \mathcal{S}_{\mathrm{wg}}(\Phi)$.

Conjecture: For any $r > 0$ and $\Phi \in \mathcal{B}_r$, $\mathcal{S}_{\mathrm{eq}}(\Phi) = \mathcal{S}_{\mathrm{wg}}(\Phi)$.

This is true in classical systems for any $\Phi \in \mathcal{B}_f$ [Pfister-Sullivan '19].

Entropy balance equation

Proposition

Let $\Phi \in \mathcal{B}^r$ and $\omega \in \mathcal{S}_{\mathrm{wg}}(\Phi)$. Then for any $\nu \in \mathcal{S}_I(\mathcal{U})$,

$$s(\nu|\omega) = \lim_{\Lambda \uparrow \mathbb{Z}^d} \frac{S(\nu_\Lambda|\omega_\Lambda)}{|\Lambda|} = -s(\nu) + \nu(E_\Phi) + p(\Phi). \tag{1}$$

In particular $\nu \mapsto s(\nu|\omega)$ is lower-semicontinuous and $s(\nu|\omega) \geq 0$ with equality iff $\nu \in \mathcal{S}_{\mathrm{eq}}(\Phi)$.

Summarizing: At high temperature, all equilibrium states are weak Gibbs and hence satisfy (1).

Increase of entropy

$$\mathrm{I}: \quad \mathcal{S}_I(\mathcal{U}) \ni \omega \xrightarrow[\Phi]{} \omega_\infty \in \mathcal{S}_{\mathrm{eq}}(\Phi), \qquad \mathrm{II}: \quad \mathcal{S}_{\mathrm{eq}}(\Psi_0) \ni \omega_0 \xrightarrow[\Phi]{} \omega_\infty$$

Theorem I [Jaksic-Pillet-T. '24]

Let $\omega \in \mathcal{S}_I(\mathcal{U})$, $\Phi \in \mathcal{B}^r$ such that ω is not τ^Φ-invariant. Let $\omega_\infty \in \mathcal{S}_\infty(\omega, \Phi)$, and **suppose** that $\omega_\infty \in \mathcal{S}_{\mathrm{wg}}(\Phi)$. Then

$$s(\omega_\infty) > s(\omega).$$

This is a conditional result, in particular it requires $\omega_\infty \in \mathcal{S}_{\mathrm{eq}}(\Phi)$.

Proof: (!) The entropy balance equation gives

$$\begin{aligned} 0 \leq s(\omega|\omega_\infty) &= -s(\omega) + \omega(E_\Phi) + p(\Phi) \\ &= -s(\omega) + \omega_\infty(E_\Phi) + p(\Phi) \\ &= s(\omega_\infty) - s(\omega). \end{aligned}$$

Thus if $s(\omega_\infty) = s(\omega)$ then $\omega \in \mathcal{S}_{\mathrm{eq}}(\Phi)$, and so ω is τ^Φ-invariant. $\square$

Irreversibility

$$\mathrm{I}: \quad \mathcal{S}_I(\mathcal{U}) \ni \omega \xrightarrow[\Phi]{} \omega_\infty \in \mathcal{S}_{\mathrm{eq}}(\Phi), \qquad \mathrm{II}: \quad \mathcal{S}_{\mathrm{eq}}(\Psi_0) \ni \omega_0 \xrightarrow[\Phi]{} \omega_\infty$$

Theorem II [Jaksic-Pillet-T. '24]

Let $\Phi, \Psi_0 \in \mathcal{B}^r$. Suppose that $\mathcal{S}_{\mathrm{eq}}(\Psi_0) = \{\omega_0\}$ with ω_0 weak Gibbs and not τ^Φ-invariant. Then for any $\omega_\infty \in \mathcal{S}_\infty(\omega_0, \Phi)$,

$$\omega_\infty(E_{\Psi_0}) > \omega_0(E_{\Psi_0}).$$

This one is unconditional on ω_∞. Its proof is similar.

<u>Cor:</u> Assume the setting to be reversible: $\omega_0 \in \mathcal{S}_\infty(\omega_\infty, \Psi_0)$.

$$\omega_0 \xrightarrow[\Phi]{} \omega_\infty, \qquad \omega_0 \xleftarrow[\Psi_0]{} \omega_\infty$$

Then $\omega_\infty = \omega_0$ and ω_0 is τ^Φ-invariant.

Approach to equilibrium is irreversible (in contrast to return to equilibrium).

Elements of proof

Back to weak Gibbs states

Let $\Phi \in \mathcal{B}_r$ with $\|\Phi\|_r < r$ and $\omega \in \mathcal{S}_{\mathrm{eq}}(\Phi)$. Let's prove that $\omega \in \mathcal{S}_{\mathrm{wg}}(\Phi)$, namely find C_Λ such that

$$C_\Lambda^{-1} \frac{\mathrm{e}^{-H_\Phi(\Lambda)}}{\mathrm{tr}(\mathrm{e}^{-H_\Phi(\Lambda)})} \leq \omega_\Lambda \leq C_\Lambda \frac{\mathrm{e}^{-H_\Phi(\Lambda)}}{\mathrm{tr}(\mathrm{e}^{-H_\Phi(\Lambda)})}, \qquad \lim_{\Lambda \uparrow \mathbb{Z}^d} \frac{1}{|\Lambda|} \log C_\Lambda = 0.$$

Proposition **[Israel '79]**

For any $A \in \mathcal{U}_{\mathrm{loc}}$ the map $\mathbb{R} \ni t \mapsto \tau_\Phi^t(A)$ has an analytic extension to the strip

$$|\mathrm{Im}(z)| < \frac{r}{2\|\Phi\|_r}$$

where we have $\|\tau_\Phi^z(A)\| \leq \|A\| \, \mathrm{e}^{r\,\mathrm{supp}(A)} \frac{r}{r - 2\|\Phi\|_r |\mathrm{Im}(z)|}$.

Recall that $\|\Phi\|_r = \sum_{X \ni 0} \mathrm{e}^{r(|X|-1)} \|\Phi(X)\|$.

Definition

Let $\Phi \in \mathcal{B}_r$. A state $\omega \in \mathcal{S}_I(\mathcal{U})$ is (τ_Φ, β)-KMS if for all $A, B \in \mathcal{U}$, the function

$$\mathbb{R} \ni t \mapsto F_{AB}(t) = \omega(\tau_\Phi^t(A)B)$$

has an extension analytic in the strip $\{0 < \mathrm{Im}(z) < \beta\}$, bounded and continuous on its closure, and satisfying the Kubo-Martin-Schwinger boundary condition $F_{AB}(t + \mathrm{i}\beta) = \omega(B\tau_\Phi^t(A))$

Prop: Any (τ_Φ, β)-KMS is τ_Φ-invariant. Moreover $\omega \in \mathcal{S}_{\mathrm{eq}}(\Phi) \Leftrightarrow \omega$ is (τ_Φ, β)-KMS.

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Prop: Any (τ_Φ, β)-KMS is τ_Φ-invariant. Moreover $\omega \in \mathcal{S}_{\mathrm{eq}}(\Phi) \Leftrightarrow \omega$ is (τ_Φ, β)-KMS.

Proposition

For any $V = V^* \in \mathcal{U}$ there exist a unique perturbed dynamics $(\tau_\Phi)_V$, generated by $(\delta_\Phi)_V(A) = \delta_\Phi(A) + [V, A]$ for all $A \in \mathcal{U}_{\mathrm{loc}}$. For ω a (β, Φ)-KMS state, we denote by ω_V the corresponding perturbed KMS state.

[Simon '93, Araki-Moriya '03]

Surface energies and Gibbs condition

For $\Phi \in \mathcal{B}_r$ and $\Lambda \subset \mathbb{Z}^d$ finite consider the surface energies

$$W_\Lambda = \sum_{\substack{X \cap \Lambda \neq \emptyset \\ X \cap \Lambda^c \neq \emptyset}} \Phi(X)$$

Prop: For $\Phi \in \mathcal{B}_r$, $\lim_{\Lambda \to \mathbb{Z}^d} \dfrac{1}{|\Lambda|} \|W_\Lambda\| = 0$

Removing W_Λ in the dynamics disconnects the box Λ from the rest of $\mathbb{Z}^d$.

Gibbs condition **[Simon '93, Araki-Moriya '03]**

For fixed Λ, let $(\tau_\Phi)_{-W_\Lambda}$ be the dynamics perturbed by $-W_\Lambda$ and let ω_{-W_Λ} be the corresponding KMS state. Then for any $A \in \mathcal{U}(\Lambda)$ one has

$$\omega_{-W_\Lambda}(A) = \frac{\mathrm{tr}(\mathrm{e}^{-\beta H_\Phi(\Lambda)} A)}{\mathrm{tr}(\mathrm{e}^{-\beta H_\Phi(\Lambda)})}$$

Comparing perturbed and original state

Theorem **[Lenci-Rey-Bellet '05]**

Let ω be a $(\tau, 1)$-KMS state and let ω_V be a $(\tau_V, 1)$-KMS state associated to the perturbed dynamics τ_V for some $V = V^* \in \mathcal{U}$. Assume that $\mathbb{R} \ni t \mapsto \tau^t(V)$ has an analytic extension to the strip $\{0 < |\mathrm{Im}(z)| < \frac{1}{2}\}$, bounded and continuous on its closure. Then

$$\omega_V \leq \mathrm{e}^{\|V\| + \|\tau^{\mathrm{i}/2}(V)\|}\omega, \qquad \omega_V \geq \mathrm{e}^{-\|V\| - \|\tau_{-V}^{\mathrm{i}/2}(V)\|}\omega$$

In our case, since $\|\Phi\|_r < r$, $z \mapsto \tau_\Phi^z(W_\Lambda)$ is analytic on $\mathrm{Im}(z) < a$ for some $a > 1/2$ which implies

$$\mathrm{e}^{-\|W_\Lambda\| - \|\tau_\Phi^{\mathrm{i}/2}(W_\Lambda)\|} \frac{\mathrm{e}^{-H_\Phi(\Lambda)}}{\mathrm{tr}(\mathrm{e}^{-H_\Phi(\Lambda)})} \leq \omega_\Lambda \leq \mathrm{e}^{\|W_\Lambda\| + \|(\tau_\Phi^{\mathrm{i}/2})_{-W_\Lambda}(W_\Lambda)\|} \frac{\mathrm{e}^{-H_\Phi(\Lambda)}}{\mathrm{tr}(\mathrm{e}^{-H_\Phi(\Lambda)})}$$

It remains to show that

$$\lim_{\Lambda \to \mathbb{Z}^d} \frac{1}{|\Lambda|} \|\tau_\Phi^{\mathrm{i}/2}(W_\Lambda)\| = 0$$

and similarly for $(\tau_\Phi^{\mathrm{i}/2})_{-W_\Lambda}(W_\Lambda)$.

End of proof

For $X \in \mathcal{F}$ we have $\|\tau_\Phi^{\mathrm{i}/2}(\Phi(X))\| \leq \|\Phi(X)\| \frac{\mathrm{e}^{r|X|}}{1-\|\Phi\|_r/r}$, so that

$$\|\tau_\Phi^{\mathrm{i}/2}(W_\Lambda)\| \leq \sum_{\substack{X\cap\Lambda\neq\emptyset \\ X\cap\Lambda^c\neq\emptyset}} \|\Phi(X)\| \frac{\mathrm{e}^{r|X|}}{1-\|\Phi\|_r/r}.$$

We conclude the proof by showing that

$$\lim_{\Lambda\uparrow\mathbb{Z}^d} \frac{1}{|\Lambda|} \sum_{\substack{X\cap\Lambda\neq\emptyset \\ X\cap\Lambda^c\neq\emptyset}} \|\Phi(X)\|\mathrm{e}^{r|X|} = 0.$$

This relation is immediate for Φ finite range, and the general case follows by density in $\mathcal{B}_r$ and the bound

$$\frac{1}{|\Lambda|} \sum_{X\cap\Lambda\neq\emptyset} \|\Psi(X)\|\mathrm{e}^{r|X|} \leq \mathrm{e}^r \|\Psi\|_r$$

that holds for all $\Psi \in \mathcal{B}_r$.

The argument for $(\tau_\Phi^{\mathrm{i}/2})_{-W_\Lambda}(W_\Lambda)$ is similar.

Recent progress and perspectives

Stability of weak Gibbs states

Is weak Gibbs property or entropy balance equation somehow preserved along a state trajectory?

(work in progress with Vojkan Jaksic, Anna Szczepanek and Claude-Alain Pillet)

Theorem

*Let Ψ_0 and Φ be **finite range** interactions. Assume that $\mathcal{S}_{\mathrm{wg}}(\Psi_0) = \mathcal{S}_{\mathrm{eq}}(\Psi_0) = \{\omega_0\}$ and consider $\omega_t = \omega_0 \circ \tau_t^{\Phi}$. Then it exists $T_0 > 0$ such that for all $|t| < T_0$*

$$\omega_t \in \mathcal{S}_{\mathrm{wg}}(\Phi_t)$$

for some interaction Φ_t. Moreover $T_0 = +\infty$ if $d = 1$.

The proof relies on :

1. Lieb-Robinson bounds
2. Existence and vanishing of surface energy for Φ_t
3. Araki's estimate for $d = 1$

Conservation of relative entropy

Recall that for $\omega \in \mathcal{S}_I$ and $\Phi \in \mathcal{B}^r$ we have $s(\omega \circ \tau_t^\Phi) = s(\omega)$. What about relative entropy?

(work in progress with Vojkan Jaksic, Anna Szczepanek and Claude-Alain Pillet)

Corollary

Let Ψ_0 and Φ be ***finite range*** *interactions. Assume that $\mathcal{S}_{\mathrm{wg}}(\Psi_0) = \mathcal{S}_{\mathrm{eq}}(\Psi_0) = \{\omega_0\}$ and consider $\omega_t = \omega_0 \circ \tau_t^\Phi$. Then it exists $T_0 > 0$ such that for all $|t| < T_0$ and any $\nu \in \mathcal{S}_I(\mathcal{U})$, with $\nu_t = \nu \circ \tau_t^\Phi$,*

$$s(\nu_t|\omega_t) = s(\nu|\omega_0) = -s(\nu_0) + \nu(E_{\Psi_0}) + p(\Psi_0)$$

for some interaction Φ_t. Moreover $T_0 = +\infty$ if $d = 1$.

Conclusion

Formulation of Approach to Equilibrium for infinite and translation-invariant quantum spin or fermionic systems

- Entropy is constant for finite time
- Strict increase of entropy at $t = \infty$
- Approach to Equilibrium is not compatible with reversibility or adiabatic theorem
- Central tool: weak Gibbs states and entropy balance equation
- At least at high temperature, all equilibrium states are always weak Gibbs

Perspectives

The problem of approach to equilibrium remains mostly open

- Prove approach to equilibrium in specific models, beyond quasi-free or exactly solvable cases.
- A possible choice is the canonical two-body interaction fermionic model, related to the Quantum Boltzmann equation
  [Hugenholz '83, Erdös-Salmhofer-Yau '04, Benedetto-Castella-Esposito-Pulvirenti '04-'08]

Thank you for your attention!

- Jaksic, V., Pillet, C. A., & Tauber, C. (2024). Approach to equilibrium in translation-invariant quantum systems: some structural results. Annales Henri Poincaré (Vol. 25, No. 1, pp. 715-749)
- Jaksic, V., Pillet, C. A., & Tauber, C. (2024). A note on adiabatic time evolution and quasi-static processes in translation-invariant quantum systems. Annales Henri Poincaré (Vol. 25, No. 1, pp. 751-771).