Many-body spectral flow index and the Quantum Hall Effect

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joint work with Sven Bachmann and Jacob Shapiro

Topological Aspects of Condensed Matter Theory
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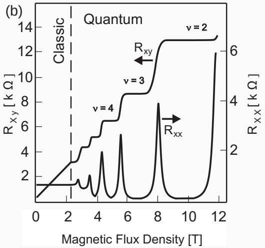

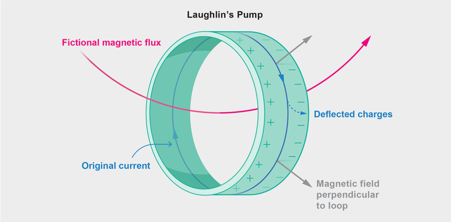

- Two-dimensional electron gas with transverse magnetic field
- The transverse conductivity is quantized and related to a topological index: $\sigma_{xy} = \frac{e^2}{h}\mathcal{N}$, $\mathcal{N} \in \mathbb{Z}$.
- Equivalently: quantization of charge deficiency with respect to magnetic flux insertion at the origin (Laughlin argument)
- Non-interacting setting: many models, many indices from the last 40 years

Stability against interactions?

A lot of progress in the last decade

- Giuliani, Mastropietro, Porta '15: Quantization of σ_{xy} in Haldane-Hubbard model (constructive QFT techniques)
- Hasting, Michalachis '15: quantization via quasi-adiabatic evolution
- Bachamnn, Bols, De Roeck, Fraas et. al, from '16: A many-body index via flux insertion through a finite torus in the large L limit
- Marcelli, Miyao, Monaco, Teufel, Wesle et al. from '17: Quantized linear response from NEASS
- Kapustin, Sopenko '20 (+ Kitaev) Flux insertion in the infinite setting
- Bachman, Bols, Rahnama '24: $\mathbb{Z}_2$ Many Body index, slightly simpler (!)

Our goal: functional analysis approach to define a many-body index directly in the infinite setting, whose construction connects well with physical intuition, and covers known (non-)interacting models.

Main result

Theorem **Bachmann, Shapiro, T. '25**

Let ω be an invertible and $U(1)$-symmetric state over the CAR algebra $\mathcal{A}$ of $\ell^2(\mathbb{Z}^2) \otimes \mathbb{C}^N$. Then it exists $\hat{\mathcal{A}}$ and a unitary $\hat{U} \in \hat{\mathcal{A}}$ such that the quantity:

$$\mathcal{I}(\omega) := \hat{\omega}\left(\hat{U}^* \delta^{Q \otimes \mathbb{1}}(\hat{U})\right) \in \mathbb{Z}$$

is a topological index. If ω_1 and ω_2 are continuously related by a $U(1)$-symmetric locally generated automorphism, then $\mathcal{I}(\omega_1) = \mathcal{I}(\omega_2)$.

- All those terms, including hats, need to be properly defined.
- In a nutshell, we gradually insert a unit of magnetic flux at the origin, leading to a defect state $\hat{\omega}_{2\pi} = \hat{\omega} \circ \mathrm{Ad}(\hat{U})$ with $\hat{U} \in \hat{\mathcal{A}}$ unitary.
- For a second-quantized non-interacting model for quantum Hall effect, $\mathcal{I}$ coincides with the spectral flow associated to the insertion of a unit of magnetic flux through the sample, starting with the Fermi projection (charge deficiency).
- Towards an abstract version: the index of a pair of pure states

Outline

Symmetric SRE states

Magnetic flux insertion

A many-body index for invertible states

Symmetric SRE states

The C^*-alegbra of fermionic observables

Consider $\mathcal{H} = \ell^2(\mathbb{Z}^2) \otimes \mathbb{C}^n$ Canonical Anti-commutation Relation (CAR) algebra $\mathcal{A}$, generated by $\mathbb{1}$, $a(h)$ for $h \in \mathcal{H}$, and

$$\{a(f), a^*(g)\} = \langle g, f \rangle \mathbb{1}, \qquad \{a(f), a(g)\} = \{a^*(f), a^*(g)\} = 0,$$

with $a^*(f) = (a(f))^*$.

For $\Lambda \subset \mathbb{Z}^d$ finite, $\mathcal{A}_\Lambda$ is the C^*-algebra generated by $\mathbb{1}$ and $a_{x,i} := a(\delta_x \otimes e_i)$ for $x \in \mathbb{Z}^d$ and $i \in \{1, \dots, n\}$. The $*$-algebra of local observables is

$$\mathcal{A}_{\mathrm{loc}} := \bigcup_{\Lambda \subset \mathbb{Z}^d,\, \Lambda \text{ finite}} \mathcal{A}_\Lambda$$

One has $\mathcal{A} \equiv \overline{\mathcal{A}_{\mathrm{loc}}}$, in the $\| \cdot \|$-topology.

Every $A \in \mathcal{A}$ is ϵ-close to a compactly supported observable.

States

A state

$$\omega : \mathcal{A} \to \mathbb{C}$$

is a positive linear functional on $\mathcal{A}$ such that $\omega(\mathbb{1}) = 1$ (hence continuous). The set of states is a weakly*-compact and convex subset of $\mathcal{A}^*$. Its extreme points are called pure states.

We say that ω_0 is a product state if for $X \cap Y = \emptyset$

$$\forall A_X, B_Y \in \mathcal{A}_X \times \mathcal{A}_Y, \qquad \omega_0(A_X B_Y) = \omega_0(A_X)\omega_0(B_Y).$$

Pure product states factorize completely over each lattice site x. Analog of the atomic limit or trivial phase, so we expect $\mathcal{I}(\omega_0) = 0$.

We need to implement the following notions:

- locality
- symmetry
- (gaped ground state of a parent H)

A pure, local and symmetric gaped ground state is the analog of a Fermi projection.

We classify states, not Hamiltonians.

0-chains

Remark: Objects like $\sum_{x \in \mathbb{Z}^2} h_{x,y} a_x^* a_y$ do not belong to $\mathcal{A}$.

Let $f : \mathbb{R}_+ \to \mathbb{R}_+^*$ be a rapidly decreasing function. We say that $A \in \mathcal{A}$ is f-localized near $x \in \mathbb{Z}^d$ if

$$\|A - A_n\| \leq f(n) \|A\|$$

for all $n \in \mathbb{N}$, with $A_n \in \mathcal{A}_{B_x(n)}$. $\mathcal{A}^{\mathrm{al}}$ = set of almost local observables (A f-localized near x). One has $\mathcal{A}^{\mathrm{loc}} \subset \mathcal{A}^{\mathrm{al}} \subset \mathcal{A}$.

Definition

A 0-chain F is a sequence $(F_x)_{x \in \mathbb{Z}^2}$ of elements of $\mathcal{A}^{\mathrm{al}}$ such that

- $\forall x \in \mathbb{Z}^2, F_x^* = F_x$ and F_x if f-localized near x.
- $\sup_{x \in \mathbb{Z}^2} \|F_x\| < \infty$

Ex: $H_x = \sum_{|x-y| \leq 1} a_x^* a_y + a_y^* a_x$ defines a 0-chain $H = (H_x)_{x \in \mathbb{Z}^2}$.

Dynamics and LGA

Proposition

Let F be a 0-chain. Then $\delta^F : \mathcal{A}^{\mathrm{al}} \to \mathcal{A}^{\mathrm{al}}$ given by

$$A \mapsto \delta^F(A) = [F, A] := \sum_{x \in \mathbb{Z}^2} [F_x, A]$$

is a $*$-derivation on $\mathcal{A}$. It generates a dynamics $\mathbb{R} \ni t \mapsto \alpha_t^F$ on $\mathcal{A}$, defined by

$$\alpha_0^F(A) = \mathbb{1}, \qquad \frac{\mathrm{d}}{\mathrm{d}t}\alpha_t^F(A) = \mathrm{i}\alpha_t^F(\delta^F(A))$$

on $\mathcal{A}^{\mathrm{al}}$ and extends by continuity to all of $\mathcal{A}$.

A 0-chain can also be time-dependent: $s \mapsto F(s)$. It generates a non-autonomous dynamics $\alpha_{s \to t}^F$.

Definition

An automorphism α on $\mathcal{A}$ is a called a locally generated automorphism (LGA) if there exists a 0-chain F and some $s \in [0, \infty)$ such that $\alpha = \alpha_{0 \to s}^F$.

Short-range entanglement (SRE)

Definition

A state ω is called Short Ranged Entangled (or SRE) if

$$\omega = \omega_0 \circ \alpha.$$

where α is a locally generated automorphism (LGA) and ω_0 is apure product state.

Recall that $\alpha = \alpha_t^F$ with some f-local 0-chain F, but not a physical Hamiltonian.

SRE should be taken as a definition for a state to be local in space: an f-local (finite) time evolution of a pure product state ω_0.

It appears naturally in the classification of symmetry-protected topological phases.

Charge and $U(1)$-symmetry

For each $x \in \mathbb{Z}^d$,

$$Q_x = \sum_{i=1}^{n} a_{x,i}^* a_{x,i} \in \mathcal{A}_x$$

is trivially an f-local 0-chain. For $\phi \in \mathbb{R}$ let $\rho_\phi := \alpha_\phi^Q$ be its associated LGA.

For $\Lambda \subset \mathbb{Z}^d$ finite the local charge $Q^\Lambda = \sum_{x\in\Lambda} Q_x \in \mathcal{A}_\Lambda$ has integer spectrum, and

$$\rho_\phi(A) = \lim_{\Lambda \to \mathbb{Z}^2} \mathrm{e}^{\mathrm{i}\phi Q^\Lambda} A \mathrm{e}^{-\mathrm{i}\phi Q^\Lambda} \qquad (\phi \in \mathbb{R}).$$

Moreover one has $\rho_{\phi+2\pi} = \rho_\phi$ so that ρ_ϕ is called the $U(1)$ gauge transformation associated to Q.

Definition

A state ω is symmetric if $\omega \circ \rho_\phi = \omega$ for all $\phi \in \mathbb{R}$.

Parent Hamiltonian

Proposition

Let ω be a symmetric and SRE state. Then it is the unique gaped ground state of some symmetric parent Hamiltonian: it exists a 0-chain H and $\Delta > 0$ such that:

- $\omega(A^*[H, A]) \geq \Delta\omega(A^* A)$ for all $A \in \mathcal{A}^{\mathrm{loc}}$ with $\omega(A) = 0$.
- $\rho_\phi(H_x) = H_x$ for all $\phi \in \mathbb{R}$ and $x \in \mathbb{Z}^d$

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Single-particle finite dimensional case: let $H = H^* \geq 0$ with $H\psi_0 = 0$. Gap condition for H is equivalent to

$$\langle\psi_0, A^*HA\psi_0\rangle - \underbrace{\langle\psi_0, A^*AH\psi_0\rangle}_{=0} \geq \Delta\langle\psi_0, A^*A\psi_0\rangle$$

for all A such that $\langle\psi_0, A\psi_0\rangle = 0$ (recall $\omega(A) = \langle\psi_0, A\psi_0\rangle$ in that case).

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Proof. $\omega = \omega_0 \circ \alpha$ with ω_0 pure product then it exist H_x^0 on each $\mathcal{A}_x$ such that ω_0 is a unique gaped ground state of ω_0 with gap Δ. Take $H_x = \alpha^{-1}(H_x^0)$. Since α is an LGA then H_x defines a 0-chain, and ω is gaped ground state of H_x. Take $H_x' = (2\pi)^{-1}\int_0^{2\pi} \rho_\phi(H_x)$. It is symmetric by construction and ω is again a gaped ground state of H_x'.

Magnetic flux insertion

A 3-step procedure

Summary: the state ω is

- SRE: $\omega = \omega_0 \circ \alpha$
- $U(1)$-symmetric: $\omega \circ \rho_\phi = \omega$
- the unique gaped ground state of some $U(1)$-symmetric 0-chain H

Goal: construct a defect state $\omega_{2\pi}^D$ corresponding to a 2π-magnetic flux insertion through the sample.

1. Half-plane gauge transformation
2. Quasi-adiabatic flow localized along a line
3. Restriction to the half-line

The defect state

1. Upper half-plane gauge transformation:

$$\omega_\phi^\uparrow = \omega \circ \rho_\phi^{Q_\uparrow}, \qquad (Q_\uparrow)_x = Q_x \chi_{\{x_2 \geq 0\}}.$$

is the gaped ground state of the parent Hamiltonian given by $(H_\phi^\uparrow)_x := \rho_{-\phi}^{Q_\uparrow}(H_x)$.

2. There exists a ϕ-dependent 0-chain $\phi \mapsto K(\phi)$ such that $\omega_\phi^\uparrow = \omega \circ \alpha_\phi^K$ and

$$\|K(\Phi)_x\| \leq g(|x_2|)$$

for some rapidly decreasing g. K is the quasi-adiabatic flow (Hastings '04).

3. Restriction to the half-line:

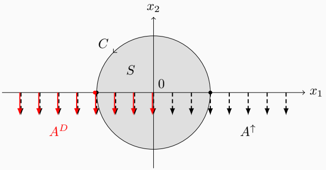
$$D(\phi)_x := K(\phi)_x \chi_{\{x_1 \leq 0\}}.$$

Let α_ϕ^D be its associated LGA. Flux insertion: $\omega \circ \alpha_\phi^D$ from 0 to 2π.

$\omega_{2\pi}^D := \omega \circ \alpha_{2\pi}^D$ is the defect state.

Prop: $\omega_{2\pi}^D$ is $U(1)$-symmetric.

Single-particle analogy: gauge transformation on $\mathcal{H} = \ell^2(\mathbb{Z}^2)$.



- Gauge transformation $\psi \mapsto \mathrm{e}^{-\mathrm{i}\phi\chi^\uparrow}\psi$ with $\chi^\uparrow(x) = \mathbb{1}_{\{x_2 \geq 0\}}$ generates a vector potential

$$A^\uparrow(x) = -\phi\nabla\chi(x) = \phi \begin{pmatrix} 0 \\ -1 \end{pmatrix} \mathbb{1}_{\{x_2 = 0\}},$$

which is localized along $x_2 = 0$. It satisfies $\nabla \times A^\uparrow = 0$.

- Truncate to the left: $A^D = A^\uparrow \mathbb{1}_{\{x_1 \leq 0\}}$. Not a total gradient anymore. For any surface S with boundary C one has

$$\int_S B^D \mathrm{d}s = \int_C A \mathrm{d}l = \begin{cases} \phi, & 0 \in S \\ 0 & 0 \notin S. \end{cases}$$

A^D corresponds to a magnetic ϕ-flux insertion at the origin.

Equality at infinity

Proposition

It exits $U \in \mathcal{A}^{\mathrm{al}}$ such that $\omega_{2\pi}^{D} = \omega \circ \mathrm{Ad}[U]$ with $\mathrm{Ad}[U](A) = U^{*}AU$.

Highly non trivial: existence $U \in \mathcal{A}$ is very strong. This property does not hold at the level of LGA $\alpha_{2\pi}^{D}$, and U depends on ω. Heavily relies on SRE property.

The proof is adapted from Bachmann, Bols, Rahnama '24 to our setting. It relies on the fact that $\omega_{2\pi}^{D}$ and ω are "equal at infinity":

$$|\omega_{2\pi}^{D}(A) - \omega(A)| \leq f(r)\,\|A\|\,, \qquad \forall A \in B(r)^{c}$$

for some rapidly decreasing f.

An integer quantity

Proposition

The quantity

$$i(\omega) := \omega(U^* \delta^Q(U))$$

is integer valued. Moreover, if α is an LGA which perserves the $U(1)$-symmetry, then $i(\omega \circ \alpha) = i(\omega)$.

Proof: Let $\mathcal{H}, \Pi, \Omega$ be the GNS representation of ω.

- ω is pure and $U(1)$-symmetric so Π induces an irreducible representation of the $U(1)$-symmetry ρ^Q_ϕ. Thus $\Pi(\rho^Q_\phi(A)) = \mathrm{e}^{\mathrm{i}\tilde{Q}} \Pi(A) \mathrm{e}^{-\mathrm{i}\tilde{Q}}$ with $\sigma(\tilde{Q}) = \mathbb{Z}$ and $\tilde{Q}\Omega = 0$.
- The index becomes

$$\omega(U^* \delta^Q(U)) = \omega(U^*[Q, U]) = \langle \Omega, \Pi(U)^* \tilde{Q} \Pi(U) - \tilde{Q}\Omega \rangle = \langle \Omega^D_{2\pi}, \tilde{Q}\Omega^D_{2\pi} \rangle$$

  where $\Omega^D_{2\pi} := \Pi(U)\Omega$ represents $\omega^D_{2\pi}$ in $\mathcal{H}$.
- $\omega^D_{2\pi}$ is $U(1)$-symmetric implies that $\Omega^D_{2\pi}$ is an eigenstate of $\tilde{Q}$, hence $i(\omega) \in \mathbb{Z}$.
- (LGA-invariance goes along the same lines in the GNS representation.)

A small caveat

Can we show that $i(\omega) \neq 0$ on some explicit model? Actually not...

Consider a single particle model on $\mathcal{H} = \ell^2(\mathbb{Z}^2) \otimes \mathbb{C}^n$ with Fermi projection P. It generates a quasi-free state

$$\omega_P(a_{x,i}^* a_{y,j}) = \langle (\delta_x \otimes e_i), P(\delta_y \otimes e_j) \rangle$$

(higher correlation functions are obtained by Wick contraction/Slater determinant). If the Hamiltonian is local, then $|\omega_P(a_{x,i}^* a_{y,j})| \leq C\mathrm{e}^{-\mu|x-y|}$.

Q: Can we continuously deform ω_P to a product state? Can we continuously deform P so that its matrix elements are diagonal in space?

Not true in general!

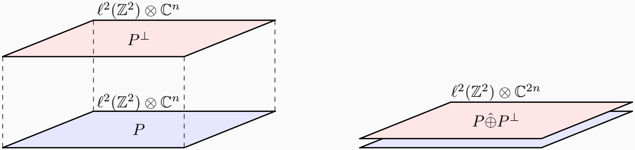
Proposition

If $c(P) = 0$ then ω_P is SRE.

Proved in Bachmann, Bols, Rahnama '24 for translation invariant systems, could be generalized beyond using Chung, Shapiro '25. The converse statement is a conjecture.

Slightly problematic...

A many-body index for invertible states



Consider the projection $P \oplus P^\perp$ of the external direct sum $\mathcal{H} \oplus \mathcal{H}$. Then

$$c(P \oplus P^\perp) = c(P) + c(P^\perp) = 0$$

.

Stacking

Let $\mathcal{A}^1$ and $\mathcal{A}^2$ respectively be the CAR algebras over $\ell^2(\mathbb{Z}^2) \otimes \mathbb{C}^n$ and $\ell^2(\mathbb{Z}^2) \otimes \mathbb{C}^m$ for $n, m \in \mathbb{N}$. We denote by $\mathcal{A}^1 \hat{\otimes} \mathcal{A}^2$ the stacked CAR algebra over $\ell^2(\mathbb{Z}^2) \otimes \mathbb{C}^{n+m}$.

For states we define:

$$\omega_1 \hat{\otimes} \omega_2 (A_1 \hat{\otimes} A_2) := \omega_1(A_1)\omega_2(A_2)$$

and extend it on $\mathcal{A}^1 \hat{\otimes} \mathcal{A}^2$ by linearly.

Invertible states

Definition

We say that a state ω of some CAR algebra $\mathcal{A}$ is invertible if it exist a state ω' over a CAR algebra $\mathcal{A}'$ such that $\omega \hat{\otimes} \omega'$ is SRE in $\mathcal{A} \hat{\otimes} \mathcal{A}'$.

We say that ω is symmetric invertible if moreover $\omega \hat{\otimes} \omega'$ is both $U(1)$-invariant with respect to $Q \hat{\otimes} \mathbb{1}$ and $\mathbb{1} \hat{\otimes} Q$.

Main motivation: any quasi-free state ω_P is symmetric invertible.

We are back to SRE states! but on $\mathcal{A} \hat{\otimes} \mathcal{A}'$. Index i applies to $\hat{\omega}$. Main difference: we have the full charge $\hat{Q} = Q \hat{\otimes} \mathbb{1} + 1 \hat{\otimes} Q$ and the charges $Q \hat{\otimes} \mathbb{1}$, $\mathbb{1} \hat{\otimes} Q$ on each layer.

A many-body index

Theorem — **Bachmann, Shapiro, T. '25**

Let ω be a symmetric invertible state over the CAR algebra $\mathcal{A}$ of $\ell^2(\mathbb{Z}^2) \otimes \mathbb{C}^N$. Then it exists a unitary $\hat{U} \in \hat{\mathcal{A}} = \mathcal{A} \hat{\otimes} \mathcal{A}'$ such that the quantity:

$$\mathcal{I}(\omega) := \hat{\omega}\left(\hat{U}^* \delta^{Q \otimes \mathbb{1}}(\hat{U})\right) \in \mathbb{Z}$$

is a topological index. If ω_1 and ω_2 are continuously related by a $U(1)$-symmetric LGA, then $\mathcal{I}(\omega_1) = \mathcal{I}(\omega_2)$.

$$\hat{\omega} \mapsto \hat{\omega}_{2\pi}^{D,\hat{Q}} \mapsto \hat{U} \in \mathcal{A} \hat{\otimes} \mathcal{A}'$$

But we measure the charge in one layer only.

Lemma: The $Q \otimes \mathbb{1}$ symmetry is preserved along the construction of $\hat{\omega}_{2\pi}^{D,\hat{Q}}$.

Same proof applies as well.

Back to the single particle case

The quasi-free state ω_P is symmetric invertible since $\omega_{P\oplus P^\perp} = \omega_P \hat{\otimes} \omega_{P^\perp}$ is *SRE* on $\mathcal{A}\hat{\otimes}\mathcal{A}'$.

Flux insertion in the single particle model:

1. Half-plane gauge transformation $P^\uparrow(\phi) = \mathrm{e}^{\mathrm{i}\phi\chi_{\{x_2\geq 0\}}} P \mathrm{e}^{-\mathrm{i}\phi\chi_{\{x_2\geq 0\}}}$
2. The flow is also generated by $P^\uparrow(\phi) = U(\phi)PU^*(\phi)$ with

$$\begin{cases} \partial_\phi U(\phi) = \mathrm{i}K(\phi)U(\phi) \\ U(0) = 1 \end{cases}, \qquad K(\phi) := -\mathrm{i}[\partial_\phi P(\phi), P(\phi)]$$

3. Truncation to the left $D(\phi) = \chi_{\{x_1\leq 0\}} K(\phi)\chi_{\{x_1\leq 0\}}$, and consider the associated non-autonomous evolution $U^D(\phi_1, \phi_2)$.

The defect state is $Q^D_{2\pi} = U^D(2\pi, 0)PU^D(2\pi, 0)^*$.

Proposition

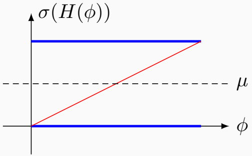
$$\mathcal{I}(\omega_P) = \mathrm{Tr}(Q^D_{2\pi} - P) = \mathrm{Ind}(Q^D_{2\pi}, P)$$

Spectral flow interpretation

Let H be the (physical) parent Hamiltonian for P, and consider the ϕ-flux insertion procedure on it. It reads

$$H(\phi)_{x,y} = \begin{cases} \mathrm{e}^{\mathrm{i}\phi \operatorname{sgn}(x_2 - y_2)} H_{x,y}, & x_1, y_1 \leq 0 \\ H_{x,y}, & \text{otherwise.} \end{cases}$$

$H(0)$ is gaped, and $H(2\pi) = H(0)$. However, $Q^D_\phi \neq \chi_{(-\infty,\mu)}(H(\phi))$.



Proposition

$$\mathcal{I}(\omega_P) = \mathrm{SF}_\mu([0, 2\pi] \ni \phi \mapsto H(\phi))$$

See De Nittis and Schulz-Baldes '15 for a detailed spectral flow interpretation of QHE in the single particle picture.

Conclusion

Conclusion

- A many-body index in the infinite dimensional setting which classifies $U(1)$-symmetric invertible states and covers the non-interacting models for Quantum Hall Effect.
- Physical interpretation: many-body spectral flow associated to a magnetic flux insertion at the origin
- We classify ground states, parent Hamiltonians are tools.
- Invertible states appear as tailored for integer QHE (SRE is not enough and $\mathcal{I} \in \mathbb{Z}$ so no FQHE)

A many-body index of a pair of projection

Recall that if P and Q are two projections on a Hilbert space with $P-Q$ compact then

$$\mathrm{Index}(P,Q) = \dim(\mathrm{im}P \cap \ker Q) - \dim(\mathrm{im}Q \cap \ker P) \in \mathbb{Z}$$

is a topological index — Avron Seiler Simon '94

Theorem — **Bachmann, Shapiro, T. '25**

Let Q be the generator of a $U(1)$ symmetry on a C^*-algebra $\mathcal{A}$. Let ω_1 and ω_2 be two symmetric pure states which are locally comparable:

$$\exists U \in \mathcal{U}(\mathcal{A}) \cap \mathcal{D}(\delta^Q), \qquad \omega_2 = \omega_1 \circ \mathrm{Ad}(U)$$

Then the quantity

$$\mathcal{N}(\omega_1, \omega_2) = \omega_1(U^* \delta^Q(U)) \in \mathbb{Z}$$

is a topological index, called index of a pair of pure states.

Prop: If P and Q are two projections on $\mathcal{H}$ with $P-Q \in \mathcal{J}_1(\mathcal{H})$, then $\mathcal{N}(\omega_P, \omega_Q) = \mathrm{Index}(P,Q)$.

(no need for SRE property in the abstract framework)

Bachmann, Shapiro, T.
The index of a pair of pure states and the interacting integer quantum Hall effect.
arXiv:2507.10807

Thank you!