

**The Brownian net,  
the branching-coalescing point set,  
and the one-dimensional 1-facilitated  
Frederickson-Anderson model**

J.M. Swart

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# Chapter 1

## Some interacting particle systems

### 1.1 A branching-killing process

The motivation for these lectures comes from an interacting particle system that has been called the the biased annihilating branching process by Claudia Neuhauser and Aidan Sudbury, who introduced the model in [NS93]. We will focus on the one-dimensional case, for which the best results to date were proved by Sudbury in [Sud99]. Unknown to these authors, the model belongs to the class of kinetically constrained models that has been intensively studied by both physicists and mathematicians in recent years [RS03]. In fact, it is very similar to the 1-facilitated Fredrickson-Anderson model studied, for example, in [BCMRT, BDT19, MV19].

The one-dimensional biased annihilating branching process, or BAB, to be short, is a Markov process  $(X_t)_{t \geq 0}$  taking values in the space  $\{0, 1\}^{\mathbb{Z}}$  of all configurations  $x = (x(i))_{i \in \mathbb{Z}}$  of zeros and ones on the one-dimensional integer lattice. We call  $X_t(i)$  the *local state* at the *site*  $i \in \mathbb{Z}$  at time  $t \geq 0$ . If  $X_t(i) = 1$ , then we say that the site  $i$  is *occupied*, and if  $X_t(i) = 0$ , then we say that  $i$  is *empty*. At each site, there is located an independent “exponential clock”, that rings at i.i.d. time intervals with a mean one exponential distribution. When the clock rings, the site chooses one of its two neighbours at random, checks its local state, and if it is occupied, updates its own state by making it occupied with probability  $p$  and empty with probability  $1 - p$ .

More formally, we can construct the BAB as follows. For each  $i, j \in \mathbb{Z}$ , let us define maps  $\mathbf{bra}_{ij}$  and  $\mathbf{kill}_{ij}$  from the state space  $\{0, 1\}^{\mathbb{Z}}$  into itself by

$$\mathbf{bra}_{ij}(x)(k) := \begin{cases} 1 & \text{if } k = j, x(i) = 1 \\ x(k) & \text{otherwise,} \end{cases} \quad (1.1)$$

and

$$\mathbf{kill}_{ij}(x)(k) := \begin{cases} 0 & \text{if } k = j, x(i) = 1, \\ x(k) & \text{otherwise.} \end{cases} \quad (1.2)$$

We let  $\vec{E} := \{(i, j) \in \mathbb{Z}^2 : |i - j| = 1\}$ . We let  $\omega^b$  and  $\omega^k$  be independent Poisson point subsets of  $\vec{E} \times \mathbb{R}$  with intensities  $\frac{1}{2}p$  and  $\frac{1}{2}(1-p)$ , respectively. More precisely, if  $\mu$  denotes the counting measure on  $\vec{E}$ , that gives each element weight one, and  $\ell$  denotes the Lebesgue measure on  $\mathbb{R}$ , then the intensity of the Poisson point set  $\omega^b$  is the measure  $\frac{1}{2}p(\mu \otimes \ell)$  and the intensity of  $\omega^k$  is  $\frac{1}{2}(1-p)(\mu \otimes \ell)$ . Note that elements of  $\omega^b$  or  $\omega^k$  are pairs  $((i, j), t)$  where  $(i, j) \in \vec{E}$  and  $t \in \mathbb{R}$ .

Let  $X_0$  be a  $\{0, 1\}$ -valued random variable, independent of  $\omega^b$  and  $\omega^k$ . Then it can be shown [Swa22, Thm 4.19] that almost surely, there exists a unique function  $(X_t)_{t \geq 0}$  with initial state  $X_0$ , such that  $t \mapsto X_t(i)$  is piecewise constant and right-continuous for each  $i \in \mathbb{Z}$ , and satisfies

$$X_t = \begin{cases} \mathbf{bra}_{ij}(X_{t-}) & \text{for all } t \text{ such that } ((i, j), t) \in \omega^b, \\ \mathbf{kill}_{ij}(X_{t-}) & \text{for all } t \text{ such that } ((i, j), t) \in \omega^k, \\ X_{t-} & \text{for all other } t, \end{cases} \quad (1.3)$$

where  $X_{t-}(i) := \lim_{s \uparrow t} X_s(i)$  ( $i \in \mathbb{Z}$ ) denotes the left-continuous modification of  $t \mapsto X_t(i)$ . The process  $(X_t)_{t \geq 0}$  is a Markov process [Swa22, Thm 4.20]. Its transition probabilities form a Feller semigroup with generator

$$G_{\text{BAB}}f(x) = \frac{1}{2}p \sum_{(i,j) \in \vec{E}} \{f(\mathbf{bra}_{ij}(x)) - f(x)\} + \frac{1}{2}(1-p) \sum_{(i,j) \in \vec{E}} \{f(\mathbf{kill}_{ij}(x)) - f(x)\}, \quad (1.4)$$

which is a closed linear operator that is first defined for functions  $f$  that depend on finitely many coordinates and then for more general functions by taking the closure [Swa22, Thm 4.30].

We can visualise the construction in (1.3) as follows. We draw space  $\mathbb{Z}$  horizontally and time vertically. For each  $((i, j), t) \in \omega^b$ , we draw a straight arrow from  $(i, t)$  to  $(j, t)$ , and for each  $((i, j), t) \in \omega^k$ , we draw a squiggly arrow from  $(i, t)$  to  $(j, t)$ . These arrows indicate that the maps  $\mathbf{bra}_{ij}$  or  $\mathbf{kill}_{ij}$  should be applied. The effect of a straight arrow is that if the site at the tail of the arrow is occupied, then the site at its tip becomes occupied as well. The effect of a squiggly arrow is that if the site at the tail of the arrow is occupied, then the site at its tip becomes empty, i.e., the particle at the tail of the arrow kills the particle at the tip, if there is one. See Figure 1.1.

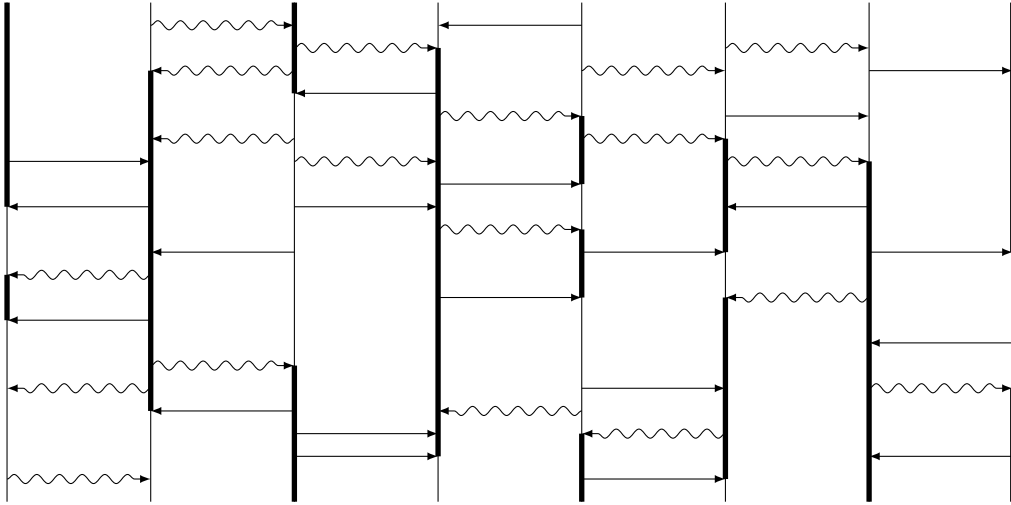


Figure 1.1: Graphical representation of the one-dimensional biased annihilating branching process. Thick lines indicate occupied sites. Straight and squiggly arrows indicate the application of a branching or killing map, respectively.

This sort of construction is called a *graphical representation* of an interacting particle system.

Our conventions differ from those of Neuhauser and Sudbury [NS93, Sud99], who choose for  $\omega^b$  and  $\omega^k$  Poisson point sets with intensities  $\lambda$  and 1, respectively, instead of  $\frac{1}{2}p$  and  $\frac{1}{2}(1-p)$  as we do. The two processes can directly be translated into each other by a change of parameter and a rescaling of time: if  $(X_t)_{t \geq 0}$  is our process and  $(Y_t)_{t \geq 0}$  is the process in [NS93, Sud99], then

$$Y_t = X_{2(1+\lambda)t} \quad \text{with} \quad p = \frac{\lambda}{1+\lambda}. \quad (1.5)$$

Our conventions differ from the literature on kinetically constrained models in the sense that sites that we call occupied (value 1) are called empty (value 0) in [BCMRT, BDT19, MV19]. The model studied in these latter papers differs moreover in the fact that in the BAB, the rate at which a site is updated is  $\frac{1}{2}$  times the number of occupied neighbours, while for the *Fredrickson-Andersen one spin facilitated model* studied in these papers the rate of updating is the indicator function of the event that at least one neighbour is occupied (or, by their conventions, empty). This model cannot easily be translated into the BAB, but its behaviour and the methods available to study it seem to be similar. We stick to the BAB because its graphical representation is simpler. For the Fredrickson-Andersen model, one would have to define maps that

look at three sites instead of just two as for our branching and killing maps.

By checking detailed balance, it is easy to see that for any  $p > 0$ , product measure with intensity  $p$  is a reversible law for the BAB. Trivially, the delta measure on the empty configuration is another reversible law. It has been proved in [NS93, Thm 3] that on quite general lattices, these are the only two extremal invariant laws. In one dimension, it has moreover been proved in [NS93, Thm 4] that for any  $p \geq 1/4$ , the law of the process started in any nonzero initial state converges weakly to product measure with intensity  $p$ . This sort of behaviour is called *complete convergence* (compare the definition in [Lig99, Sect. I.1]). Sudbury later improved this result [Sud99]. With a computer-assisted proof, he was able to prove complete convergence for all  $p \geq 0.0335$ . It has been conjectured in [NS93] that complete convergence holds for all  $p > 0$ . A key step in the proofs of complete convergence is showing that the process has a positive *edge speed*, i.e., if  $R_t := \sup\{i \in \mathbb{Z} : X_t(i) = 1\}$  is the position of the right-most particle, then

$$\liminf_{t \rightarrow \infty} R_t/t > 0 \quad \text{a.s.} \quad (1.6)$$

A difficulty in showing this is that one has to control how often the right-most particle is killed by a particle on the left of it. Since the process is non-monotone, the usual comparison arguments do not work and there is no known explicit formula for the equilibrium distribution of the process as seen from the right-most particle.

It is easy to simulate the BAB on a computer. Some simulation results are shown in Figure 1.2. When  $p$  is small, we observe that the BAB effectively functions as a system of branching and coalescing particles. It is useful to rescale time by a factor  $p^{-1}$ , so that branching (in each direction) takes place with rate  $\frac{1}{2}$  and killing (in each direction) with rate  $\frac{1}{2}(p^{-1} - 1)$ . A single particle now creates a new particle on one of its neighbouring sites with rate 1. When  $p$  is small, with high probability, one of the two neighbouring particles will kill the other one before anything else can happen. Each of the particles has an equal chance of being killed, so with equal probabilities we end up either with one particle on the site of the original particle, or on one of the neighbouring sites. Therefore, in the limit  $p \rightarrow 0$ , on time scales of order  $p^{-1}$ , the BAB started with a single particle effectively behaves like a random walk that jumps with rate  $\frac{1}{2}$  to a random neighbouring site.

The behaviour on longer time scales is harder to analyse. Occasionally, after a particle branches into two, one of the two will create a third particle on a free site before any of the two have time to kill each other. Once three neighbouring sites are occupied, there is a probability of approximately  $\frac{1}{2}$  that the middle particle gets killed before the other two. The result of this



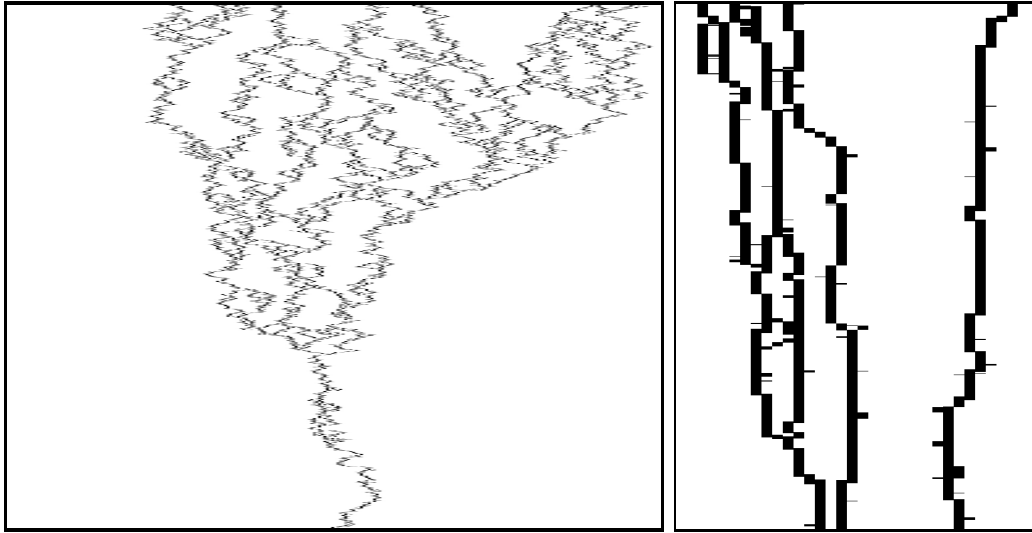


Figure 1.2: Simulations of one-dimensional biased annihilating branching processes. Space is plotted horizontally, time vertically, and black indicates the presence of a particle. On the left a process with  $p = 0.074$ . Total elapsed time in this picture is 54,000 and the shown part of space contains 250 sites. On the right detail of a process with  $p = 0.13$ .

is that the original particle has effectively split into two particles. These two particles can now each start to move through space as independent random walks, until they get close enough to each other again so that one kills the other. The result of all this is that on sufficiently long time scales, when  $p$  is small, the BAB should effectively behave like a system of branching and coalescing random walks with a small branching rate.

In the coming section, we will study simpler systems of branching and coalescing particles. For these systems, it is known that in the small branching rate limit, after a suitable rescaling of space and time, they converge to a continuum process known as the *branching-coalescing point set*. It is believed that the branching-coalescing point set is a *universal limit object*, that occurs as the scaling limit of a variety of models. In particular, we will formulate a precise conjecture that says that the branching-coalescing point set is the scaling limit of the BAB as  $p \rightarrow 0$ .

## 1.2 Branching-coalescing random walks

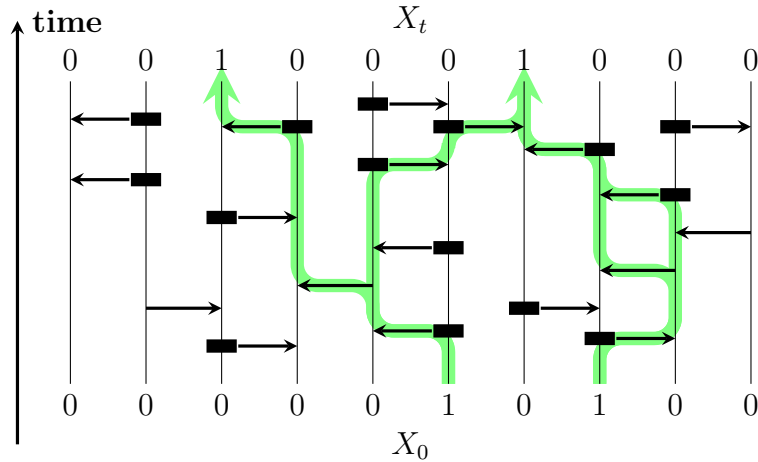
For each  $i, j \in \mathbb{Z}$ , let us define a *coalescing random walk map*  $\mathbf{rw}_{ij}$  and  $\mathbf{kill}_{ij}$  from the state space  $\{0, 1\}^{\mathbb{Z}}$  into itself by

$$\mathbf{rw}_{ij}(x)(k) := \begin{cases} 0 & \text{if } k = i \\ 1 & \text{if } k = j, x(i) = 1 \\ x(k) & \text{otherwise.} \end{cases} \quad (1.7)$$

In words, this has the effect that if there is a particle at  $i$ , then this particle moves to  $j$ , coalescing with any particle that may already be present on that site. In graphical representations, we represent the map  $\mathbf{rw}_{ij}$  by an arrow with a black rectangle at its tail. This black rectangle is a “blocking symbol” that indicates that any particle present there at previous times cannot continue on that site. We will be interested in the interacting particle system with generator

$$\begin{aligned} G_{\text{braco}}f(x) = & \frac{1}{2}p \sum_{(i,j) \in \vec{E}} \{f(\mathbf{bra}_{ij}(x)) - f(x)\} \\ & + \frac{1}{2}(1-p) \sum_{(i,j) \in \vec{E}} \{f(\mathbf{rw}_{ij}(x)) - f(x)\}, \end{aligned} \quad (1.8)$$

i.e., compared to (1.4) we have replaced the killing map by the coalescing random walk map. We can again construct the process from a graphical representation, which now looks like this:



Trivially, the delta measure on the empty configuration is a reversible law for the branching coalescing random walks with generator (1.8). We claim that moreover, just as for the BAB, product measure with intensity  $p$  is a reversible law.

**Lemma 1.1 (Reversible product law)** *Product measure with intensity  $p$  is a reversible law for the system of branching and coalescing random walks with generator as in (1.8).*

**Proof** We check detailed balance. For a product measure with intensity  $r$ , the frequency of jumps  $10 \mapsto 11$  is  $\frac{1}{2}pr(1-r)$  and the frequency of jumps  $11 \mapsto 10$  is  $\frac{1}{2}(1-p)r^2$ . Detailed balance then gives  $\frac{1}{2}pr(1-r) = \frac{1}{2}(1-p)r^2$  which is solved for  $r = 0, p$ . Also, the configurations  $10$  and  $01$  have the same probability and due to the coalescing random walk dynamics, the process jumps between these configurations with rate  $\frac{1}{2}(1-p)$  in each direction.

An alternative way to obtain the result, that was suggested to me by Cristina Toninelli, is to observe that the dynamics in (1.8) can be reformulated by saying that each pair of neighbouring sites is activated with rate  $\frac{1}{2}(1-p)^2p^{-1}$ , and then the configuration on these sites is changed according to the following rules. If both sites are empty, then they remain empty. If at least one of the sites is occupied, then a new configuration is chosen according to the product measure with intensity  $p$ , conditioned on the event that at least one of the two sites is occupied. ■

It is easy to see that the branching coalescing random walks have a positive edge speed for all  $p > 0$ . If the process is started in a finite initial configuration, then the position of the right-most particle  $R_t := \sup\{i \in \mathbb{Z} : X_t(i) = 1\}$  is a random walk that jumps one step to the right with rate  $\frac{1}{2}$  and one step to the left with rate  $\frac{1}{2}(1-p)$ . The strong law of large numbers then gives

$$\lim_{t \rightarrow \infty} R_t/t = \frac{1}{2}p \quad \text{a.s.} \quad (1.9)$$

The main reason why branching coalescing random walks are easier to study than the BAB is that they are *monotone*, in the sense that if we construct two processes  $(X_t)_{t \geq 0}$  and  $(X'_t)_{t \geq 0}$  with initial states satisfying  $X_0 \leq X'_0$  a.s., using the same graphical representation, then  $X_t \leq X'_t$  for all  $t \geq 0$  a.s. Even better, the process is *additive* in the sense that for three processes that are constructed using the same graphical representation,

$$X_0 = X'_0 \vee X''_0 \quad \text{implies} \quad X_t = X'_t \vee X''_t \quad (t \geq 0). \quad (1.10)$$

There exists a very useful duality theory for general additive interacting particle systems [Gri79]. In the next section, we will see that the additive dual of branching coalescing random walks is the biased voter model.

Because of all these tools that are available, a lot is known about branching coalescing random walks. In particular, it is known that they have a scaling limit as  $p \rightarrow 0$ , which is called the *branching-coalescing point set* and which we will introduce in Section 1.4 below.

### 1.3 The biased voter model

The graphical representation of branching coalescing random walks is reminiscent of oriented percolation. Elaborating on this similarity, let us define an *open path* in the graphical representation defined by the Poisson point sets  $\omega^b$  and  $\omega^k$  to be a piecewise constant, right-continuous function  $\gamma : [s, u] \rightarrow \mathbb{Z}$ , defined on some compact interval  $[s, u] \subset \mathbb{R}$ , such that:

- (i)  $\gamma(t) \neq \gamma(t-)$  if there is a blocking symbol at  $(\gamma(t-), t)$ ,
- (ii) if  $\gamma(t) \neq \gamma(t-)$ , then there is an arrow from  $(\gamma(t-), t)$  to  $(\gamma(t), t)$ .

Then for each  $t \geq 0$ ,

$$\begin{aligned} X_t(i) = 1 &\Leftrightarrow \\ &\exists \text{ open path } \gamma : [0, t] \rightarrow \mathbb{Z} \text{ s.t. } X_0(\gamma(0)) = 1 \text{ and } \gamma(t) = i. \end{aligned} \quad (1.11)$$

This leads to the following idea. Let  $Y_0$  be a random variable with values in  $\{0, 1\}^{\mathbb{Z}}$ , independent of the Poisson point sets  $\omega^b$  and  $\omega^k$  and of  $X_0$ . Fix  $T > 0$  and for  $t \geq 0$ , define  $Y_t$  by

$$\begin{aligned} Y_t(i) = 1 &\Leftrightarrow \\ &\exists \text{ open path } \gamma : [T-t, T] \rightarrow \mathbb{Z} \text{ s.t. } Y_0(\gamma(T)) = 1 \text{ and } \gamma(T-t) = i. \end{aligned} \quad (1.12)$$

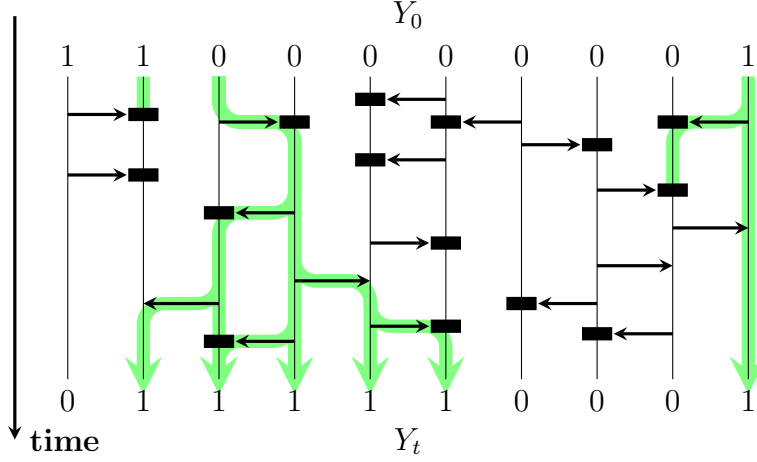
Then, letting  $\underline{0}$  denote the configuration in  $\{0, 1\}^{\mathbb{Z}}$  that is identically zero, for each  $t \in [0, T]$ , it will be true that

$$\begin{aligned} X_t \wedge Y_{T-t} \neq \underline{0} &\Leftrightarrow \\ &\exists \text{ open path } \gamma : [0, T] \rightarrow \mathbb{Z} \text{ s.t. } X_0(\gamma(0)) = 1 \text{ and } X_0(\gamma(T)) = 1. \end{aligned} \quad (1.13)$$

In particular, this event a.s. does not depend on  $t \in [0, T]$ , so setting  $t = 0, T$ , we see that

$$X_0 \wedge Y_T \neq \underline{0} \Leftrightarrow X_T \wedge Y_0 \neq \underline{0}. \quad (1.14)$$

The interesting thing about this construction is that  $(Y_t)_{t \geq 0}$  is itself a Markov process. Indeed, we can turn the graphical presentation for the process  $(X_t)_{t \geq 0}$  into a graphical presentation for the process  $(Y_t)_{t \geq 0}$  by reversing the direction of time, reversing the direction of all arrows, and keeping the blocking symbols:



The new symbol that we see here, an arrow with a blocking symbol at its tip (rather than at its tail as for the map  $\text{rw}_{ij}$ ) corresponds to the *voter model map*  $\text{vot}_{ij}$  which has the following simple description:

$$\text{vot}_{ij}(y)(k) := \begin{cases} y(i) & \text{if } k = j \\ y(k) & \text{otherwise,} \end{cases} \quad (1.15)$$

where  $i$  represents the tail of the arrow and  $j$  the tip. Since there is a blocking symbol at  $j$ , the previous local state at  $j$  does not matter and the new local state at  $j$  will be 0 or 1 depending on the local state at  $i$  (which remains unchanged). The generator of the biased voter model is

$$\begin{aligned} G_{\text{vot}}f(y) = & \frac{1}{2}p \sum_{(i,j) \in \vec{E}} \{f(\text{bra}_{ij}(y)) - f(y)\} \\ & + \frac{1}{2}(1-p) \sum_{(i,j) \in \vec{E}} \{f(\text{vot}_{ij}(y)) - f(y)\}. \end{aligned} \quad (1.16)$$

The special case  $p = 0$  corresponds to the standard voter model. Because of the branching arrows, for  $p > 0$ , the 1's have an advantage over the 0's, which is why this model is called the biased voter model.

**Lemma 1.2 (Biased voter model duality)** *Let  $(X_t)_{t \geq 0}$  be a system of branching coalescing random walks with generator as in (1.8) and let  $(Y_t)_{t \geq 0}$  be an independent biased voter model with generator as in (1.16) Then*

$$\mathbb{P}[X_t \wedge Y_0 \neq \emptyset] = \mathbb{P}[X_0 \wedge Y_t \neq \emptyset] \quad (t \geq 0). \quad (1.17)$$

**Proof** This follows by taking expectations in (1.14), using the fact that in (1.14),  $X_0$  is independent of  $Y_T$  and  $X_T$  is independent of  $Y_0$ .<sup>1</sup> ■

Lemma 1.2 relates a system of branching coalescing random walks that runs forwards in time to a biased voter model that runs backwards in time. There is also an interesting relation between the two processes both running forward in time. To formulate this, we need some notation. Let  $(P_t)_{t \geq 0}$  denote the transition probabilities of the branching coalescing random walks, i.e.,  $P_t$  is the probability kernel on  $\{0, 1\}^{\mathbb{Z}}$  defined as

$$P_t(x, \cdot) := \mathbb{P}^x[X_t \in \cdot] \quad (t \geq 0, x \in \{0, 1\}^{\mathbb{Z}}), \quad (1.18)$$

where  $\mathbb{P}^x$  denotes the law of the process started in the deterministic initial state  $X_0 = x$ . Similarly, let  $(Q_t)_{t \geq 0}$  denote the transition probabilities of the biased voter model, i.e.,

$$Q_t(y, \cdot) := \mathbb{P}^y[Y_t \in \cdot] \quad (t \geq 0, y \in \{0, 1\}^{\mathbb{Z}}). \quad (1.19)$$

Finally, for each  $r \in [0, 1]$ , let  $(\chi_r(i))_{i \in \mathbb{Z}}$  be i.i.d. Bernoulli random variables with  $\mathbb{P}[\chi_r(i) = 1] = r$  and  $\mathbb{P}[\chi_r(i) = 0] = 1 - r$ , and let  $K_r$  denote the probability kernel on  $\{0, 1\}^{\mathbb{Z}}$  defined as

$$K_r(y, \cdot) := \mathbb{P}[(\chi_r(i)y(i))_{i \in \mathbb{Z}} \in \cdot] \quad (y \in \{0, 1\}^{\mathbb{Z}}). \quad (1.20)$$

In words,  $K_r(y, \cdot)$  is the law of a random variable that is obtained by *thinning* the configuration  $y$ , where independently each occupied site remains occupied with probability  $r$  and is emptied with probability  $1 - r$ . We define the concatenation of two probability kernels in the usual way, i.e.,

$$(KL)(x, dz) = \int K(x, dy)L(y, dz). \quad (1.21)$$

For a proof of the following proposition, we refer to [Swa22, Prop. 6.26].

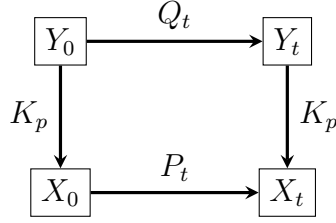
**Proposition 1.3 (Thinning relation)** *The probability kernels defined in (1.18), (1.19), and (1.20) satisfy*

$$K_p P_t = Q_t K_p \quad (t \geq 0). \quad (1.22)$$

Proposition 1.3 says that the following diagram commutes:

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<sup>1</sup>More generally, in the set-up of (1.14),  $X_t$  is independent of  $Y_{T-t}$  for each  $t \in [0, T]$ , since these random variables depend on disjoint parts of space-time, and the restrictions of a Poisson point set to disjoint sets are independent.



In other words, for any initial state  $y \in \{0, 1\}^{\mathbb{Z}}$ , the following two procedures are equivalent: 1. first run biased voter model dynamics for time  $t$ , then thin the resulting configuration with  $p$ , and: 2. first thin the configuration with  $p$ , then run branching coalescing random walk dynamics for time  $t$ .

Proposition 1.3 has a natural interpretation. In our original graphical representation for branching coalescing random walks, let us set

$$\bar{X}_t(i) := 1 \quad \Leftrightarrow \quad \exists \text{ open path } \gamma : (-\infty, t] \rightarrow \mathbb{Z} \text{ s.t. } \gamma(t) = i. \quad (1.23)$$

Then  $(\bar{X}_t)_{t \in \mathbb{R}}$  is a stationary process that at each time is distributed according to the product law with intensity  $p$ . Since this law is reversible, the processes  $(\bar{X}_t)_{t \in \mathbb{R}}$  and  $(\bar{X}_{-t})_{t \in \mathbb{R}}$  are equal in law, so we can also view  $(\bar{X}_t)_{t \in \mathbb{R}}$  as a system of branching-coalescing random walks running backwards in time. Now if we fix  $T \in \mathbb{R}$  and define  $(Y_t)_{t \geq 0}$  as in (1.12), then setting  $X_t := \bar{X}_{T-t} \cap Y_t$  defines a system of branching coalescing random walks  $(X_t)_{t \geq 0}$  that is a  $p$ -thinning of the biased voter model  $(Y_t)_{t \geq 0}$ .

If  $(Y_t)_{t \geq 0}$  is a biased voter model, then we can define a process  $(Z_t)_{t \geq 0}$  with values in  $\{-1, 0, 1\}^{\mathbb{Z}}$  as follows:

$$Z_t(i) := Y_t(i) - Y_t(i+1) \quad (t \geq 0, i \in \mathbb{Z}). \quad (1.24)$$

It is easy to see that  $(Z_t)_{t \geq 0}$  is itself a Markov process. We call  $(Z_t)_{t \geq 0}$  the *interface process* associated with  $(Y_t)_{t \geq 0}$ . Because of the nearest-neighbour property of our processes, the interface process has the following simple description:

- (i)  $-1$ 's behave as random walks that jump with rate  $\frac{1}{2}(1-p)$  to the right and with rate  $\frac{1}{2}$  to the left.
- (ii)  $1$ 's behave as random walks that jump with rate  $\frac{1}{2}$  to the right and with rate  $\frac{1}{2}(1-p)$  to the left.
- (iii) Once a  $-1$  and a  $1$  land on the same position, the result is a  $0$ .

Note that since the  $-1$ 's and  $1$ 's alternate, it cannot happen that two  $-1$ 's, or two  $1$ 's land on the same position.

## 1.4 The branching-coalescing point set

In this section, we describe the diffusive scaling limits of the biased voter model and of systems of branching coalescing random walks in the limit  $p \rightarrow 0$ , and we formulate a conjecture for the diffusive scaling limit of the BAB in the limit  $p \rightarrow 0$ . The limiting processes will be Markov processes taking values in the space of closed subsets of the real line.

For any  $A \subset \mathbb{R}$ , we let  $\text{int}(A)$  and  $\bar{A}$  denote the interior and closure of  $A$ , respectively, and we denote its boundary by  $\partial A := \bar{A} \setminus \text{int}(A)$ . We also set

$$\begin{aligned}\partial_- A &:= \{x \in \bar{A} : A \cap (x - \varepsilon, x) = \emptyset \text{ for some } \varepsilon > 0\}, \\ \partial_+ A &:= \{x \in \bar{A} : A \cap (x, x + \varepsilon) = \emptyset \text{ for some } \varepsilon > 0\}.\end{aligned}\tag{1.25}$$

We let  $\text{Cl}(\mathbb{R})$  denote the set of all closed subsets of  $\mathbb{R}$  and set

$$\text{Cl}_I(\mathbb{R}) := \{A \in \text{Cl}(\mathbb{R}) : \partial A \text{ is finite and } \partial_- A \cap \partial_+ A = \emptyset\}.\tag{1.26}$$

Fix  $A \in \text{Cl}_I(\mathbb{R})$ . We wish to define a Markov process  $(\mathcal{Y}_t)_{t \geq 0}$  with values in  $\text{Cl}_I(\mathbb{R})$ , started in the initial state  $\mathcal{Y}_0 = A$ , that is the scaling limit of the biased voter model in the limit  $p \rightarrow 0$ . Motivated by the structure of the interface process defined in (1.24), we proceed as follows. For each  $y \in \partial_- A$ , we let  $(B_t^y)_{t \geq 0}$  be a Brownian motion with drift  $-1$ , started in  $B_0^y = y$ . Similarly, for each  $y \in \partial_+ A$ , we let  $(B_t^y)_{t \geq 0}$  be a Brownian motion with drift  $1$ , started in  $B_0^y = y$ . We assume that the Brownian motions  $(B_t^y)_{t \geq 0}$  with  $y \in \partial A$  are independent. We order the elements of  $\partial A$  as  $y_0 < \dots < y_n$ . For  $0 < k \leq n$ , we let

$$\tau_{k-1,k} := \inf\{t \geq 0 : B_t^{y_{k-1}} = B_t^{y_k}\}\tag{1.27}$$

denote the first meeting time of  $(B_t^{y_{k-1}})_{t \geq 0}$  and  $(B_t^{y_k})_{t \geq 0}$ . We set

$$\tau_{y_0} := \tau_{0,1}, \quad \tau_{y_k} := \tau_{k-1,k} \wedge \tau_{k,k+1} \quad (0 < k < n), \quad \text{and} \quad \tau_{y_n} := \tau_{n-1,n}.\tag{1.28}$$

Then there exists a unique process  $(\mathcal{Y}_t)_{t \geq 0}$  with values in  $\text{Cl}_I(\mathbb{R})$  such that

$$\partial_{\pm} \mathcal{Y}_t = \{B_t^y : t < \tau_y, y \in \partial_{\pm} A\} \quad (t \geq 0).\tag{1.29}$$

We call  $(\mathcal{Y}_t)_{t \geq 0}$  the *expanding interval process*. We formulate the following theorem a bit sloppily, since we do not want to go into the topological details, which will follow later, but the basic idea is quite simple. In (1.30) below, we rescale space by a factor  $\frac{1}{2}p$  and time by a factor  $(\frac{1}{2}p)^2$ . Since  $\pm \frac{1}{2}p$  is the drift of the boundaries of the discrete process, after diffusive rescaling with  $\frac{1}{2}p$ , they should converge to Brownian motions with drift  $\pm 1$ .



**Theorem 1.4 (Scaling limit of biased voter models)** *For each  $p$ , let  $(Y_t^p)_{t \geq 0}$  be a biased voter model with generator as in (1.16). Let  $(\mathcal{Y}_t^p)_{t \geq 0}$  be the process with values in  $\text{Cl}(\mathbb{R})$  defined as*

$$\mathcal{Y}_{(p/2)^{2t}}^p := \left\{ \frac{p}{2}i : i \in \mathbb{Z}, Y_t^p(i) = 1 \right\} \quad (t \geq 0). \quad (1.30)$$

*Assume that the initial states  $\mathcal{Y}_0^p$  converge as  $p \rightarrow 0$  in an appropriate sense to a set  $A \in \text{Cl}_1(\mathbb{R})$ . Then, in an appropriate sense,*

$$\mathbb{P}[(\mathcal{Y}_t^p)_{t \geq 0} \in \cdot] \xrightarrow[p \rightarrow 0]{} \mathbb{P}[(\mathcal{Y}_t)_{t \geq 0} \in \cdot], \quad (1.31)$$

*where  $(\mathcal{Y}_t)_{t \geq 0}$  is the expanding interval process started in the initial state  $\mathcal{Y}_0 = A$ .*

We now turn our attention to the scaling limit of branching coalescing random walks. We call the Markov process  $(\mathcal{X}_t)_{t \geq 0}$  from the following theorem the *branching-coalescing point set*.

**Theorem 1.5 (Branching-coalescing point set)** *There exists a Markov process  $(\mathcal{X}_t)_{t \geq 0}$  with state space  $\text{Cl}(\mathbb{R})$ , whose transition probabilities are uniquely characterised by the duality relation*

$$\mathbb{P}[\mathcal{X}_t \cap \mathcal{Y}_0 \neq \emptyset] = \mathbb{P}[\mathcal{X}_0 \cap \mathcal{Y}_t \neq \emptyset] \quad (t \geq 0), \quad (1.32)$$

*which holds for each deterministic  $\mathcal{X}_0 \in \text{Cl}(\mathbb{R})$  and for each expanding interval process  $(\mathcal{Y}_t)_{t \geq 0}$  started in a deterministic initial state  $\mathcal{Y}_0 \in \text{Cl}_1(\mathbb{R})$ .*

**Remark** In order to determine the transition probabilities of  $(\mathcal{X}_t)_{t \geq 0}$ , in (1.32) it suffices to consider deterministic initial states. By integrating over the initial laws, it is then easy to see that (1.32) holds more generally when  $\mathcal{X}_t$  is independent of  $\mathcal{Y}_0$  and  $\mathcal{X}_0$  is independent of  $\mathcal{Y}_t$ .

**Theorem 1.6 (Scaling limit of branching coalescing random walks)** *For each  $p$ , let  $(X_t^p)_{t \geq 0}$  be a system of branching coalescing random walks with generator as in (1.8). Let  $(\mathcal{X}_t^p)_{t \geq 0}$  be the process with values in  $\text{Cl}(\mathbb{R})$  defined as*

$$\mathcal{X}_{(p/2)^{2t}}^p := \left\{ \frac{p}{2}i : i \in \mathbb{Z}, X_t^p(i) = 1 \right\} \quad (t \geq 0).$$

*Assume that the initial states  $\mathcal{X}_0^p$  converge as  $p \rightarrow 0$  in an appropriate sense to a set  $A \in \text{Cl}(\mathbb{R})$ . Then, in an appropriate sense,*

$$\mathbb{P}[(\mathcal{X}_t^p)_{t \geq 0} \in \cdot] \xrightarrow[p \rightarrow 0]{} \mathbb{P}[(\mathcal{X}_t)_{t \geq 0} \in \cdot],$$

*where  $(\mathcal{X}_t)_{t \geq 0}$  is the branching-coalescing point set started in the initial state  $\mathcal{X}_0 = A$ .*

The branching-coalescing point set has first been introduced in [SS08] (although with a different definition than in Theorem 1.5) and Theorem 1.6 can be proved using the methods of that paper. The larger part of these lectures will be devoted to a sketch of the proof of Theorem 1.6. Our motivation is the following conjecture. Note that here, the time scaling is different, since we rescale by  $(p/2)^3$  instead of  $(p/2)^2$ .

**Conjecture 1.7 (Scaling limit of the BAB)** *For each  $p$ , let  $(X_t^p)_{t \geq 0}$  be a system of biased annihilating branching process (BAB) with generator as in (1.4). Let  $(\mathcal{X}_t^p)_{t \geq 0}$  be the process with values in  $\text{Cl}(\mathbb{R})$  defined as*

$$\mathcal{X}_{(p/2)^3 t}^p := \left\{ \frac{p}{2}i : i \in \mathbb{Z}, X_t^p(i) = 1 \right\} \quad (t \geq 0).$$

*Assume that the initial states  $\mathcal{X}_0^p$  converge as  $p \rightarrow 0$  in an appropriate sense to a set  $A \in \text{Cl}(\mathbb{R})$ . Then, in an appropriate sense,*

$$\mathbb{P}[(\mathcal{X}_t^p)_{t \geq 0} \in \cdot] \xrightarrow[p \rightarrow 0]{} \mathbb{P}[(\mathcal{X}_t)_{t \geq 0} \in \cdot], \quad (1.33)$$

*where  $(\mathcal{X}_t)_{t \geq 0}$  is the branching-coalescing point set started in the initial state  $\mathcal{X}_0 = A$ .*

Related to Conjecture 1.7 is the conjecture that the BAB has a positive edge speed for all  $p > 0$ . In particular, we believe that for small  $p$ , this edge speed is approximately  $p^2/4$ . This is based on Conjecture 1.7 and the fact that the limiting process has edge speed one. In Section 5.3, we will give arguments for Conjecture 1.7. Before we can do this, we need to better understand the branching-coalescing point set as well as the precise form of convergence we will need.

# Chapter 2

## Topological prerequisites

### 2.1 Topological spaces

We are interested in diffusive scaling limits of systems of branching and coalescing particles with small branching rate. In order to be able to formulate the convergence, in the present chapter, we introduce the right spaces. In particular, we will need a space of paths, introduced in Section 2.7, and the space of all compact sets of paths, equipped with the Hausdorff metric, introduced in Section 2.5.

A *topological space* is a set  $\mathcal{X}$  equipped with a collection  $\mathcal{O}$  of subsets of  $\mathcal{X}$  that are called *open sets*, such that

- (i) If  $(O_\gamma)_{\gamma \in \Gamma}$  is any collection of (possibly uncountably many) sets  $O_\gamma \in \mathcal{O}$ , then  $\bigcup_{\gamma \in \Gamma} O_\gamma \in \mathcal{O}$ .
- (ii) If  $O_1, O_2 \in \mathcal{O}$ , then  $O_1 \cap O_2 \in \mathcal{O}$ .
- (iii)  $\emptyset, \mathcal{X} \in \mathcal{O}$ .

Any such collection of sets is called a *topology*. It is fairly standard to also assume the *Hausdorff* property

- (iv) For each  $x_1, x_2 \in \mathcal{X}$ ,  $x_1 \neq x_2$   $\exists O_1, O_2 \in \mathcal{O}$  s.t.  $O_1 \cap O_2 = \emptyset$ ,  $x_1 \in O_1$ ,  $x_2 \in O_2$ .

A set  $V \subset \mathcal{X}$  is a *neighbourhood* of a point  $x \in \mathcal{X}$  if  $x \in O \subset V$  for some  $O \in \mathcal{O}$ . We let  $\mathcal{V}_x$  denote the set of all neighbourhoods of  $x$ . A *fundamental system* of neighbourhoods of  $x$  is a set  $\mathcal{V}'_x \subset \mathcal{V}_x$  such that

$$\forall V \in \mathcal{V}_x \exists V' \in \mathcal{V}'_x \text{ s.t. } V' \subset V.$$

For example, the set of all  $O \in \mathcal{O}$  such that  $x \in O$  is a fundamental system of neighbourhoods of  $x$ . A sequence of points  $x_n \in \mathcal{X}$  converges to a limit  $x$  in a given topology  $\mathcal{O}$  if for each  $V \in \mathcal{V}_x$  there is an  $n$  such that  $x_m \in V$  for all  $m \geq n$ . It suffices to check this condition for a fundamental system of neighbourhoods  $\mathcal{V}'_x$ . If the topology is Hausdorff, then limits are unique, i.e.,  $x_n \rightarrow x$  and  $x_n \rightarrow x'$  implies  $x = x'$ .

If  $(\mathcal{X}, \mathcal{O})$  is a topological space (with  $\mathcal{O}$  the collection of open subsets of  $\mathcal{X}$ ) and  $\mathcal{X}' \subset \mathcal{X}$  is any subset of  $\mathcal{X}$ , then  $\mathcal{X}'$  is also naturally equipped with a topology given by the collection of open subsets  $\mathcal{O}' := \{O \cap \mathcal{X}' : O \in \mathcal{O}\}$ . This topology is called the *induced* topology from  $\mathcal{X}$ . If  $x_n, x \in \mathcal{X}'$ , then  $x_n \rightarrow x$  in the induced topology on  $\mathcal{X}'$  if and only if  $x_n \rightarrow x$  in  $\mathcal{X}$ .

A *basis* of a topology is a subset  $\mathcal{O}' \subset \mathcal{O}$  such that each element of  $\mathcal{O}$  can be written as the union of (possibly uncountably many) elements of  $\mathcal{O}'$ . Equivalently, this says that

$$\mathcal{O} = \{O \subset \mathcal{X} : \forall x \in O \exists \mathcal{O}' \in \mathcal{O}' \text{ s.t. } x \in \mathcal{O}' \subset O\}.$$

If  $\mathcal{O}'$  is a basis for  $\mathcal{O}$ , then  $\mathcal{V}'_x := \{O \in \mathcal{O}' : x \in O\}$  is a fundamental system of neighbourhoods of  $x$ . A topology is *first countable* if every  $x \in \mathcal{X}$  has a countable fundamental system of neighbourhoods. A topology is *second countable* if there exists a countable basis of the topology.

A set  $C \subset \mathcal{X}$  is called *closed* if its complement is open. Because of property (i) in the definition of a topology, for each  $A \subset \mathcal{X}$ , the union of all open sets contained in  $A$  is itself an open set. We call this the *interior* of  $A$ , denoted as  $\text{int}(A) := \bigcup\{O : O \subset A, O \text{ open}\}$ . Then clearly  $\text{int}(A)$  is the largest open set contained in  $A$ . Similarly, by taking complements, for each set  $A \subset \mathcal{X}$  there exists a smallest closed set containing  $A$ . We call this the *closure* of  $A$ , denoted as  $\bar{A} := \bigcap\{C : C \supset A, C \text{ closed}\}$ . If the topology is first countable, then

$$\bar{A} = \{x \in \mathcal{X} : \exists x_n \in A \text{ s.t. } x_n \rightarrow x\}, \quad (2.1)$$

i.e.,  $\bar{A}$  is the set of all limits of sequences in  $A$ . A similar statement holds for general topological spaces if we replace sequences by the more general concept of a *net*, that we will not discuss here. Since a set is closed if and only if it coincides with its closure, it follows from (2.1) that in a first countable topological space, knowing all convergent sequences and their limits uniquely determines the closed sets and their complements, the open sets, and hence the whole topology.

A topological space is called *separable* if there exists a countable set  $D \subset \mathcal{X}$  such that  $D$  is dense in  $\mathcal{X}$ , where we say that a set  $D \subset \mathcal{X}$  is *dense* if

its closure is  $\mathcal{X}$ , or equivalently, if every nonempty open subset of  $\mathcal{X}$  has a nonempty intersection with  $D$ .

A *metric* on a set  $\mathcal{X}$  is a function  $d : \mathcal{X} \times \mathcal{X} \rightarrow [0, \infty)$  such that for all  $x, y, z \in \mathcal{X}$ ,

- (i)  $d(x, y) = d(y, x)$ ,
- (ii)  $d(x, z) \leq d(x, y) + d(y, z)$ ,
- (iii)  $d(x, y) = 0$  implies  $x = y$ .

A *metric space* is a space with a metric defined on it. If  $d$  is a metric on  $\mathcal{X}$ , and  $B_\varepsilon(x) := \{y \in \mathcal{X} : d(x, y) < \varepsilon\}$  denotes the open ball around  $x$  of radius  $\varepsilon$ , then

$$\mathcal{O} := \{O \subset \mathcal{X} : \forall x \in O \exists \varepsilon > 0 \text{ s.t. } B_\varepsilon(x) \subset O\}$$

defines a Hausdorff topology on  $\mathcal{X}$  such that convergence  $x_n \rightarrow x$  in this topology is equivalent to  $d(x_n, x) \rightarrow 0$ . Note that the open balls form a basis for this topology. Since open balls of radius  $1/n$  around a point  $x$  form a fundamental system of neighbourhoods, metric spaces are first countable. We say that the metric  $d$  *generates* the topology  $\mathcal{O}$ . If for a given topology  $\mathcal{O}$  there exists a metric  $d$  that generates  $\mathcal{O}$ , then we say that the topological space  $(\mathcal{X}, \mathcal{O})$  is *metrisable*. Such a metric, if it exist, can always be chosen such that it is bounded. For example, if  $d$  is any metric on  $\mathcal{X}$ , then  $d'(x, y) := d(x, y) \wedge 1$  is a bounded metric that generates the same topology. A metrisable space is always first countable. It is second countable if and only if it is separable.

A sequence  $x_n$  in a metric space  $(\mathcal{X}, d)$  is a *Cauchy sequence* if for all  $\varepsilon > 0$  there is an  $n$  such that  $d(x_k, x_l) \leq \varepsilon$  for all  $k, l \geq n$ . A metric space is *complete* if every Cauchy sequence converges. Every metric space  $(\mathcal{X}, d)$  has a *completion*, i.e., there exists a complete metric space  $(\overline{\mathcal{X}}, \overline{d})$  such that  $\mathcal{X} \subset \overline{\mathcal{X}}$  is dense and the metric on  $\mathcal{X}$  is the *induced metric* from  $\overline{\mathcal{X}}$ , i.e.,  $d(x, y) = \overline{d}(x, y)$  for all  $x, y \in \mathcal{X}$ . Such a completion is unique up to isometries.

A *Polish space* is a separable topological space  $(\mathcal{X}, \mathcal{O})$  such that there exists a metric  $d$  on  $\mathcal{X}$  with the property that  $(\mathcal{X}, d)$  is complete and  $d$  generates  $\mathcal{O}$ . *Warning:* there may be many different metrics on  $\mathcal{X}$  that generate the same topology. It may even happen that  $\mathcal{X}$  is not complete in some of these metrics, and complete in others (in which case  $\mathcal{X}$  is still Polish). Example:  $\mathbb{R}$  is separable and complete in the usual metric  $d(x, y) = |x - y|$ , and therefore  $\mathbb{R}$  is a Polish space. But  $d'(x, y) := |\arctan(x) - \arctan(y)|$  is another metric that generates the same topology, while  $(\mathbb{R}, d')$  is not complete. Indeed, the completion of  $\mathbb{R}$  w.r.t. the metric  $d'$  is  $[-\infty, \infty]$ .

## 2.2 Compactness

A subset  $K$  of a general topological space  $\mathcal{X}$  (with collection of open sets  $\mathcal{O}$ ) is called *compact* if every open cover has a finite subcover, i.e., if for any collection  $(O_\gamma)_{\gamma \in \Gamma}$  of open subsets of  $\mathcal{X}$  such that  $\bigcup_{\gamma \in \Gamma} O_\gamma \supset K$ , there exists a finite  $\Delta \subset \Gamma$  such that  $\bigcup_{\gamma \in \Delta} O_\gamma \supset K$ . Using this definition, it is easy to see that the image of a compact set under a continuous function is again compact. Compact subsets of Hausdorff topological spaces are closed. A subset  $K$  of a metric space  $\mathcal{X}$  is compact if and only if it is closed and *totally bounded*, which means that for every  $\varepsilon > 0$  there exists a finite collection  $\{B_\varepsilon(x_1), \dots, B_\varepsilon(x_n)\}$  of open balls such that

$$B_\varepsilon(x_1) \cup \dots \cup B_\varepsilon(x_n) \supset K.$$

From this, it is not hard to see that compact metrisable spaces are always separable. If  $(x_n)_{n \in \mathbb{N}}$  is a sequence and  $m : \mathbb{N} \rightarrow \mathbb{N}$  is a function such that  $m(n) \rightarrow \infty$  as  $n \rightarrow \infty$ , then setting  $x'_n := x_{m(n)}$  ( $n \in \mathbb{N}$ ) defines a new sequence. Such a sequence is called a *subsequence* of the original sequence. A *cluster point* of a sequence is a limit of a subsequence.

**Theorem 2.1 (Bolzano-Weierstrass)** *Let  $\mathcal{X}$  be a metrisable space and let  $K \subset \mathcal{X}$ . Then  $K$  is compact if and only if every sequence in  $K$  has a subsequence that converges to a limit in  $K$ .*

The Bolzano-Weierstrass theorem also holds for second countable spaces. (Note that metrisable spaces need in general not be second countable, and conversely, not every second countable space is metrisable.) There is also a version of the Bolzano-Weierstrass theorem that holds in general topological spaces but in this case one has to replace sequences by the more general nets. A set  $A$  is *precompact* if its closure is compact. In metrisable spaces, this is equivalent to the statement that each sequence of points  $x_n \in A$  has a convergent subsequence. Note that in this case we do not require that the limit is an element of  $A$ . The following simple lemma is often useful.

**Lemma 2.2 (Convergence and compactness)** *Let  $\mathcal{X}$  be a metrisable space and let  $x, x_n \in \mathcal{X}$ . Then  $x_n \rightarrow x$  if and only if the following two conditions are satisfied.*

- (i) *The set  $\{x_n : n \in \mathbb{N}\}$  is precompact.*
- (ii) *For every subsequence  $x_{n(m)}$  such that  $x_{n(m)} \xrightarrow{m \rightarrow \infty} x'$  for some  $x' \in \mathcal{X}$ , one has  $x' = x$ .*

If  $(\mathcal{X}, \mathcal{O})$  is a topological space, then a *compactification* of  $\mathcal{X}$  is a compact topological space  $\overline{\mathcal{X}}$  such that  $\mathcal{X}$  is a dense subset of  $\overline{\mathcal{X}}$  and the topology on  $\mathcal{X}$  is the induced topology from  $\overline{\mathcal{X}}$ . If  $\overline{\mathcal{X}}$  is metrisable, then we say that  $\overline{\mathcal{X}}$  is a *metrisable compactification* of  $\mathcal{X}$ . It turns out that each separable metrisable space  $\mathcal{X}$  has a metrisable compactification [Cho69, Theorem 6.3].

A topological space  $\mathcal{X}$  is called *locally compact* if for every  $x \in \mathcal{X}$  there exists a compact neighbourhood of  $x$ . We cite the following proposition from [Eng89, Thms 3.3.8 and 3.3.9].

**Proposition 2.3 (Compactification of locally compact spaces)** *Let  $\mathcal{X}$  be a metrisable topological space. Then the following statements are equivalent.*

- (i)  $\mathcal{X}$  is locally compact and separable.
- (ii) There exists a metrisable compactification  $\overline{\mathcal{X}}$  of  $\mathcal{X}$  such that  $\mathcal{X}$  is an open subset of  $\overline{\mathcal{X}}$ .
- (iii) For each metrisable compactification  $\overline{\mathcal{X}}$  of  $\mathcal{X}$ ,  $\mathcal{X}$  is an open subset of  $\overline{\mathcal{X}}$ .

We note that if  $\mathcal{X}$  satisfies the equivalent conditions of Proposition 2.3, then it is possible to find a metrisable compactification  $\overline{\mathcal{X}}$  of  $\mathcal{X}$  such that  $\overline{\mathcal{X}} \setminus \mathcal{X}$  consists of just one point, usually denoted by  $\infty$ . In this case,  $\overline{\mathcal{X}} = \mathcal{X} \cup \{\infty\}$  is called the *one-point compactification* of  $\mathcal{X}$ . The open sets of  $\mathcal{X} \cup \{\infty\}$  are all open sets of  $\mathcal{X}$  plus all sets of the form  $\{\infty\} \cup O$  where  $\mathcal{X} \setminus O$  is a compact subset of  $\mathcal{X}$ .

A subset  $A \subset \mathcal{X}$  of a topological space  $\mathcal{X}$  is called a  $G_\delta$ -set if  $A$  is a countable intersection of open sets (i.e., there exist  $O_i \in \mathcal{O}$  such that  $A = \bigcap_{i=1}^{\infty} O_i$ ). If  $\mathcal{X}$  is metrisable, then every closed set  $A \subset \mathcal{X}$  is a  $G_\delta$ -set, since it is the intersection of the open sets  $\{x \in \mathcal{X} : d(x, A) < 1/n\}$ . The following result can be found in [Bou58, §6 No. 1, Theorem. 1]. See also [Oxt80, Thms 12.1 and 12.3].

**Proposition 2.4 (Compactification of Polish spaces)** *Let  $\mathcal{X}$  be a metrisable topological space. Then the following statements are equivalent.*

- (i)  $\mathcal{X}$  is Polish.
- (ii) There exists a metrisable compactification  $\overline{\mathcal{X}}$  of  $\mathcal{X}$  such that  $\mathcal{X}$  is a  $G_\delta$ -subset of  $\overline{\mathcal{X}}$ .
- (iii) For each metrisable compactification  $\overline{\mathcal{X}}$  of  $\mathcal{X}$ ,  $\mathcal{X}$  is a  $G_\delta$ -subset of  $\overline{\mathcal{X}}$ .

Moreover, a subset  $\mathcal{Y} \subset \mathcal{X}$  of a Polish space  $\mathcal{X}$  is Polish in the induced topology if and only if  $\mathcal{Y}$  is a  $G_\delta$ -subset of  $\mathcal{X}$ .

We note that if  $\overline{\mathcal{X}}$  is a compactification of a Polish space  $\mathcal{X}$ , equipped with a concrete metric, then  $\overline{\mathcal{X}}$  is also the completion of  $\mathcal{X}$  in this metric. Thus, unless  $\mathcal{X}$  is itself compact, it will never be complete in such a metric (even though, by the definition of a Polish space, there exists metrics generating the same topology with respect to which  $\mathcal{X}$  is complete).

## 2.3 Weak convergence

Let  $\mathcal{X}$  be a metrisable space. We let  $\mathcal{B}(\mathcal{X})$  denote *Borel- $\sigma$ -field* on  $\mathcal{X}$ , i.e., the  $\sigma$ -field generated by the open sets. We let  $\mathcal{C}(\mathcal{X})$  denote the space of all continuous functions  $f : \mathcal{X} \rightarrow \mathbb{R}$ . We let  $B_b(\mathcal{X})$  denote the space of all bounded Borel-measurable real functions on  $\mathcal{X}$  and we let  $\mathcal{C}_b(\mathcal{X}) := \mathcal{C}(\mathcal{X}) \cap B_b(\mathcal{X})$  denote the space of all bounded continuous real functions on  $\mathcal{X}$ . We equip with  $\mathcal{C}_b(\mathcal{X})$  with the *supremum norm*

$$\|f\|_\infty := \sup_{x \in \mathcal{X}} |f(x)|.$$

With this norm,  $\mathcal{C}_b(\mathcal{X})$  is a Banach space [Dud02, Theorem 2.4.9]. We let  $\mathcal{M}(\mathcal{X})$  denote the space of all finite measures on  $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$  and write  $\mathcal{M}_1(\mathcal{X})$  for the subspace of all probability measures. We cite the following well-known fact from [EK86, Theorems 3.1.7 and 3.3.1].

**Proposition 2.5 (Weak convergence)** *Let  $\mathcal{X}$  be a separable metrisable space. Then it is possible to equip  $\mathcal{M}_1(\mathcal{X})$  with a metric  $d_P$  such that*

- (i)  $(\mathcal{M}_1(\mathcal{X}), d_P)$  is a separable metric space,
- (ii)  $d_P(\mu_n, \mu) \rightarrow 0$  if and only if  $\int f d\mu_n \rightarrow \int f d\mu$  for all  $f \in \mathcal{C}_b(\mathcal{X})$ .

*If  $\mathcal{X}$  is a Polish space, then  $d_P$  can be chosen such that  $(\mathcal{M}_1(\mathcal{X}), d_P)$  is moreover complete.*

In many applications, we are not interested in the precise choice of  $d_P$  (there are several canonical ways to define such a metric). Since a metrisable topology is uniquely characterized by its convergent sequences, property (ii) uniquely characterizes the topology generated by  $d_P$  in terms of the topology on  $\mathcal{X}$ . We call this topology the *topology of weak convergence* and denote convergence in this topology as

$$\mu_n \Rightarrow \mu.$$

Proposition 2.5 shows in particular that if  $\mathcal{X}$  is a Polish space, then so is  $\mathcal{M}_1(\mathcal{X})$ , equipped with the topology of weak convergence.



One possible choice for a metric  $d_{\mathbb{P}}$  as in Proposition 2.5 is the Prohorov metric. For each subset  $A \subset \mathcal{X}$  and  $\varepsilon > 0$ , we set

$$A^\varepsilon := \{x \in \mathcal{X} : d(x, A) < \varepsilon\} \quad \text{with} \quad d(x, A) := \inf_{y \in A} d(x, y).$$

If  $(\mathcal{X}, d)$  is a metric space, then the *Prohorov metric* is the metric  $d_{\mathbb{P}}$  on  $\mathcal{M}_1(\mathcal{X})$  defined as

$$d_{\mathbb{P}}(\mu, \nu) := \inf\{\varepsilon > 0 : \mu(A) \leq \nu(A^\varepsilon) + \varepsilon \forall A \in \mathcal{B}(\mathcal{X})\}.$$

It follows from [EK86, Lemma 3.1.1] that  $d_{\mathbb{P}}$  is a metric. It is possible to give an alternative characterisation of  $d_{\mathbb{P}}$  in terms of coupling. Let  $C(\mu, \nu)$  denote the space of all probability measures  $\eta$  on  $X \times X$  whose first and second marginals are given by  $\mu$  and  $\nu$ , respectively. We cite the following lemma from [EK86, Thm 3.1.2].

**Lemma 2.6 (Prohorov metric and coupling)** *Let  $(\mathcal{X}, d)$  be a separable metric space and let  $\mu, \nu \in \mathcal{M}_1(\mathcal{X})$ . Then*

$$d_{\mathbb{P}}(\mu, \nu) = \inf\{\varepsilon > 0 : \exists \eta \in C(\mu, \nu) \text{ s.t. } \eta(\{(x, y) \in \mathcal{X}^2 : d(x, y) \geq \varepsilon\}) \leq \varepsilon\}. \quad (2.2)$$

In words, (2.2) says that  $d_{\mathbb{P}}(\mu, \nu)$  is the infimum of all  $\varepsilon > 0$  for which it is possible to couple random variables  $X, Y$  with laws  $\mu, \nu$  such that  $\mathbb{P}[d(X, Y) \geq \varepsilon] \leq \varepsilon$ . We cite the following lemmas from [EK86, Thms 3.1.7 and 3.3.1].

**Lemma 2.7 (Properties of Prohorov metric)** *Let  $(\mathcal{X}, d)$  be a separable metric space and let  $d_{\mathbb{P}}$  be the Prohorov metric. Then  $(\mathcal{M}_1(\mathcal{X}), d_{\mathbb{P}})$  is a separable metric space. If  $(\mathcal{X}, d)$  is complete, then so is  $(\mathcal{M}_1(\mathcal{X}), d_{\mathbb{P}})$ .*

**Lemma 2.8 (Prohorov metric and weak convergence)** *Let  $(\mathcal{X}, d)$  be a separable metric space and let  $d_{\mathbb{P}}$  be the Prohorov metric. Then  $\mu_n, \mu \in \mathcal{M}_1(\mathcal{X})$  satisfy  $d_{\mathbb{P}}(\mu_n, \mu) \rightarrow 0$  if and only if  $\int f d\mu_n \rightarrow \int f d\mu$  for all  $f \in \mathcal{C}_b(\mathcal{X})$ .*

In particular, Lemmas 2.7 and 2.8 imply Proposition 2.5. The following well-known alternative characterisation of weak convergence [EK86, Theorem 3.3.1] is sometimes useful.

**Lemma 2.9 (Characterization with open and closed sets)** *Let  $\mu_n$  and  $\mu$  be probability measures on a metrisable space  $\mathcal{X}$ . Then the following statements are equivalent.*

- (i)  $\mu_n \Rightarrow \mu$ .
- (ii)  $\limsup_{n \rightarrow \infty} \mu_n(C) \leq \mu(C)$  for all closed  $C \subset \mathcal{X}$ .
- (iii)  $\liminf_{n \rightarrow \infty} \mu_n(O) \geq \mu(O)$  for all open  $O \subset \mathcal{X}$ .

**Exercise 2.10 (Measures concentrated on a subset)** Let  $\mathcal{X}$  be a Polish space and let  $\mathcal{X}' \subset \mathcal{X}$  be a  $G_\delta$ -set, equipped with the induced topology. We naturally identify  $\mathcal{M}_1(\mathcal{X}')$  with the subset of  $\mathcal{M}_1(\mathcal{X})$  consisting of all  $\mu \in \mathcal{M}_1(\mathcal{X})$  such that  $\mu(\mathcal{X}') = 1$ . Show that the topology on  $\mathcal{M}_1(\mathcal{X}')$  coincides with the induced topology from its embedding in  $\mathcal{M}_1(\mathcal{X})$ . (Hint: Lemma 2.9.) Use this to conclude that  $\mathcal{M}_1(\mathcal{X}')$  is a  $G_\delta$ -subset of  $\mathcal{M}_1(\mathcal{X})$ . (Hint: Proposition 2.4).

A very useful characterization of weak convergence in terms of coupling is given by the next theorem [EK86, Cor 3.1.6 and Thm 3.1.8].

**Theorem 2.11 (Skorohod representation)** Let  $\mu_n$  and  $\mu$  be probability measures on a Polish space  $\mathcal{X}$ . Then  $\mu_n \Rightarrow \mu$  if and only if it is possible to couple random variables  $X_n, X$  with laws  $\mu_n, \mu$ , respectively, in such a way that  $X_n \rightarrow X$  a.s.

The next result is known as Prohorov's theorem (see, e.g., [EK86, Theorem 3.2.2] or [Bil99, Theorems 5.1 and 5.2]).

**Theorem 2.12 (Prohorov)** Let  $\mathcal{X}$  be a Polish space. Let  $\mathcal{M}_1(\mathcal{X})$  be equipped with the topology of weak convergence. Then a subset  $\mathcal{C} \subset \mathcal{M}_1(\mathcal{X})$  is precompact if and only if  $\mathcal{C}$  is tight, i.e.,

$$\forall \varepsilon > 0 \exists K \subset \mathcal{X} \text{ compact, s.t. } \sup_{\mu \in \mathcal{C}} \mu(\mathcal{X} \setminus K) \leq \varepsilon.$$

## 2.4 Locally uniform convergence

Let  $E$  be a metric space and let  $I \subset \mathbb{R}$  be a closed interval. We let  $\mathcal{C}_I(E)$  denote the space of all continuous functions  $w : I \rightarrow \mathbb{R}$ .

**Lemma 2.13 (Locally uniform convergence)** For  $w_n, w \in \mathcal{C}_I(E)$ , the following conditions are equivalent:

- (i)  $\sup_{t \in C} d(w_n(t), w(t)) \xrightarrow{n \rightarrow \infty} 0$  for all compact  $C \subset I$ ,
- (ii)  $w_n(t_n) \xrightarrow{n \rightarrow \infty} w(t)$  for all  $t_n, t \in I$  such that  $t_n \xrightarrow{n \rightarrow \infty} t$ .

**Proof** Assume (i) and let  $t_n, t \in I$  satisfy  $t_n \xrightarrow{n \rightarrow \infty} t$ . By Lemma 2.2 (i), there exists a compact set  $C \subset I$  such that  $t_n \in C$  for all  $n$  (and hence also  $t \in C$ ). Then for each  $\varepsilon > 0$ , there exists an  $N < \infty$  such that  $d(w_n(t), w(t)) \leq \varepsilon$  for all  $n \geq N$ . Now

$$d(w_n(t_n), w(t)) \leq d(w_n(t_n), w(t_n)) + d(w(t_n), w(t)) \leq \varepsilon + d(w(t_n), w(t))$$

for all  $n \geq N$ , and hence

$$\limsup_{n \rightarrow \infty} d(w_n(t_n), w(t)) \leq \varepsilon$$

by the continuity of  $w$ . Since  $\varepsilon > 0$  is arbitrary, this shows that (i) implies (ii). On the other hand, if (i) fails for some compact  $C \subset I$ , then we can choose  $t_n \in C$  and  $\varepsilon > 0$  such that

$$d(w_n(t_n), w(t_n)) \geq \varepsilon \quad \forall n.$$

Since  $C$  is compact, by going to a subsequence, we can without loss of generality assume that  $t_n \rightarrow t$  for some  $t \in C$ . Since

$$d(w_n(t_n), w(t)) \geq d(w_n(t_n), w(t_n)) - d(w(t_n), w(t)) \geq \varepsilon + d(w(t_n), w(t)),$$

using the continuity of  $w$ , we see that

$$\liminf_{n \rightarrow \infty} d(w_n(t_n), w(t)) \geq \varepsilon,$$

which contradicts (ii). ■

There exists a metrisable topology on  $\mathcal{C}_I(E)$  such that a  $w_n \in \mathcal{C}_I(E)$  converges to a limit  $w$  if and only if the equivalent conditions of Lemma 2.13 are satisfied. Note that by (2.1) and the remarks below it, these conditions uniquely determine the topology. Note also that by condition (ii) of Lemma 2.13, the topology on  $\mathcal{C}_I(E)$  depends only on the topology on  $E$  and not on the precise choice of the metric on  $E$ . A possible choice of a metric on  $\mathcal{C}_I(E)$  is

$$\rho(v, w) := \sum_{n=1}^{\infty} 2^{-n} \sup_{t \in [0, n]} d(v(t), w(t)),$$

where  $d$  is a bounded metric that generates the topology on  $E$ . Such a metric can always be found: if  $d$  is any metric generating the topology on  $E$ , then  $d'(x, y) := d(x, y) \wedge 1$  is a bounded metric that generates the same topology. Usually, we do not care about the precise choice of the metric on  $\mathcal{C}_I(E)$ ; apart from  $\rho$ , there are many other possible choices. We call this the topology on  $\mathcal{C}_I(E)$  the *topology of locally uniform convergence*.

## 2.5 The Hausdorff metric

Let  $(E, d)$  be a metric space, let  $\mathcal{K}(E)$  be the space of all compact subsets of  $E$  and set  $\mathcal{K}_+(E) := \{K \in \mathcal{K}(E) : K \neq \emptyset\}$ . Then the *Hausdorff metric*  $d_H$  on  $\mathcal{K}_+(E)$  is defined as

$$\begin{aligned} d_H(K_1, K_2) &:= \sup_{x_1 \in K_1} d(x_1, K_2) \vee \sup_{x_2 \in K_2} d(x_2, K_1) \\ &= \inf \left\{ \varepsilon > 0 : K_1 \subset K_2^\varepsilon \text{ and } K_2 \subset K_1^\varepsilon \right\}, \end{aligned} \quad (2.3)$$

where as before  $d(x, A) := \inf_{y \in A} d(x, y)$  denotes the distance between a point  $x \in E$  and a set  $A \subset E$  and  $A^\varepsilon := \{x \in \mathcal{X} : d(x, A) < \varepsilon\}$ . The corresponding topology is naturally called the *Hausdorff topology*. Note the subtle difference between “the Hausdorff topology” (the topology generated by the Hausdorff metric) and “a Hausdorff topology” (any topology satisfying condition (iv) of Section 2.1). We extend this topology to  $\mathcal{K}(E)$  by adding  $\emptyset$  as an isolated point.

A good source for the Hausdorff topology is [SSS14, Appendix B], where one can find the proofs of all the lemmas in this section. Some more information can be found in [BBI01, Chapter 7]. The first lemma of this section shows that the Hausdorff topology depends only on the topology on  $E$ , and not on the choice of the metric.

**Lemma 2.14 (Convergence criterion)** *Let  $K_n, K \in \mathcal{K}_+(E)$  ( $n \geq 1$ ). Then  $K_n \rightarrow K$  in the Hausdorff topology if and only if there exists a  $C \in \mathcal{K}_+(E)$  such that  $K_n \subset C$  for all  $n \geq 1$  and*

$$\begin{aligned} K &= \{x \in E : \exists x_n \in K_n \text{ s.t. } x_n \rightarrow x\} \\ &= \{x \in E : \exists x_n \in K_n \text{ s.t. } x \text{ is a cluster point of } (x_n)_{n \in \mathbb{N}}\}. \end{aligned} \quad (2.4)$$

The following lemma shows that  $\mathcal{K}(E)$  is Polish if  $E$  is.

**Lemma 2.15 (Properties of the Hausdorff metric)**

- (a) *If  $(E, d)$  is separable, then so is  $(\mathcal{K}_+(E), d_H)$ .*
- (b) *If  $(E, d)$  is complete, then so is  $(\mathcal{K}_+(E), d_H)$ .*

The following lemma shows in particular that  $\mathcal{K}(E)$  is compact if  $E$  is compact.

**Lemma 2.16 (Compactness in the Hausdorff topology)** *A set  $\mathcal{A} \subset \mathcal{K}_+(E)$  is precompact if and only if there exists a compact  $C \subset E$  such that  $K \subset C$  for each  $K \in \mathcal{A}$ .*

The following lemma is useful when proving convergence of  $\mathcal{K}_+(E)$ -valued random variables.

**Lemma 2.17 (Tightness criterion)** *Assume that  $E$  is a Polish space and let  $K_n$  ( $n \geq 1$ ) be  $\mathcal{K}_+(E)$ -valued random variables. Then the collection of laws  $\{\mathbb{P}[K_n \in \cdot] : n \geq 1\}$  is tight if and only if for each  $\varepsilon > 0$  there exists a compact  $C \subset E$  such that  $\mathbb{P}[K_n \subset C] \geq 1 - \varepsilon$  for all  $n \geq 1$ .*

## 2.6 Squeezed space

Let  $(E, d)$  be a metric space, let  $\{*\}$  be a set containing a single element called  $*$ , which we assume is not an element of  $E$ , and let

$$\mathcal{R}(E) := (E \times \mathbb{R}) \cup \{(*, -\infty), (*, +\infty)\}. \quad (2.5)$$

We extend  $d$  to  $E \cup \{*\}$  by setting  $d(x, *) = d(*, x) := \infty$  if  $x \neq *$  and  $:= 0$  otherwise. Let  $\overline{\mathbb{R}} := [-\infty, \infty]$  denote the usual two-point compactification of the real line. We fix a continuous function  $\phi : \overline{\mathbb{R}} \rightarrow [0, \infty)$  such that  $\phi(t) > 0$  for all  $t \in \mathbb{R}$  and  $\phi(\pm\infty) = 0$ , we choose a metric  $d_{\overline{\mathbb{R}}}$  that generates the topology on  $\overline{\mathbb{R}}$ , and we define  $\rho : \mathcal{R}(E)^2 \rightarrow [0, \infty)$  by

$$\rho((x, s), (y, t)) := (\phi(s) \wedge \phi(t))(d(x, y) \wedge 1) + |\phi(s) - \phi(t)| + d_{\overline{\mathbb{R}}}(s, t) \quad (2.6)$$

**Lemma 2.18 (Metric on squeezed space)** *The function  $\rho$  is a metric on  $\mathcal{R}(E)$ .*

**Proof** For brevity, we write  $d'(x, y) := d(x, y) \wedge 1$ . Then  $d'$  is a metric on  $E$ . The only nontrivial statement that we have to prove is the triangle inequality, and it suffices to prove this for the function

$$\rho'((x, s), (y, t)) := (\phi(s) \wedge \phi(t))d'(x, y) + |\phi(s) - \phi(t)|.$$

We estimate

$$\rho'((x, s), (z, u)) \leq (\phi(s) \wedge \phi(u))(d'(x, y) + d'(y, z)) + |\phi(s) - \phi(u)|. \quad (2.7)$$

If  $\phi(t) \geq \phi(s) \wedge \phi(u)$ , then  $\phi(s) \wedge \phi(u)$  is less than  $\phi(s) \wedge \phi(t)$  and also less than  $\phi(t) \wedge \phi(u)$ , so we can simply estimate the expression in (2.7) from above by

$$(\phi(s) \wedge \phi(t))d'(x, y) + (\phi(t) \wedge \phi(u))d'(y, z) + |\phi(s) - \phi(t)| + |\phi(t) - \phi(u)|$$

and we are done. On the other hand, if  $\phi(t) < \phi(s) \wedge \phi(u)$ , then

$$|\phi(s) - \phi(t)| + |\phi(t) - \phi(u)| = |\phi(s) - \phi(u)| + 2(\phi(s) \wedge \phi(u) - \phi(t)).$$

Using the fact that  $d' \leq 1$ , we can now estimate the right-hand side of (2.7) from above by

$$\begin{aligned} & \phi(t)(d'(x, y) + d'(y, z)) + 2(\phi(s) \wedge \phi(u) - \phi(t)) + |\phi(s) - \phi(u)| \\ &= (\phi(s) \wedge \phi(t))d'(x, y) + (\phi(t) \wedge \phi(u))d'(y, z) \\ & \quad + |\phi(s) - \phi(t)| + |\phi(t) - \phi(u)|, \end{aligned}$$

and again we are done.  $\blacksquare$

The following lemma shows that the topology generated by the metric  $\rho$  depends only on the topology on  $E$  and not on the choice of the metric on  $E$ . Recall that by (2.1), a metrisable topology is uniquely characterised by its convergent sequences, so the topology on  $\mathcal{R}(E)$  is uniquely characterised by the conditions (i) and (ii) below.

**Lemma 2.19 (Topology on squeezed space)** *A sequence  $(x_n, t_n) \in \mathcal{R}(E)$  converges to a limit  $(x, t)$  in the metric  $\rho$  defined in (2.6) if and only if the following two conditions are satisfied:*

- (i)  $t_n \rightarrow t$  in the topology on  $\overline{\mathbb{R}}$ ,
- (ii) if  $t \in \mathbb{R}$ , then  $x_n \rightarrow x$  in the topology on  $E$ .

**Proof** This is immediate from the definition of  $\rho$ .  $\blacksquare$

We can think of the space  $\mathcal{R}(E)$  as being obtained from  $E \times \overline{\mathbb{R}}$  by squeezing the sets  $E \times \{\pm\infty\}$  into the single points  $(*, \pm\infty)$ . For this reason, we call  $\mathcal{R}(E)$  the *squeezed space*. In the special case that  $E = \overline{\mathbb{R}}$ , we can make a picture of  $\mathcal{R}(\overline{\mathbb{R}})$  by mapping  $\overline{\mathbb{R}} \times \mathbb{R}$  into the closed unit disc using the function

$$(x, t) \mapsto (\sqrt{1 - \psi(t)^2}\psi(x), \psi(t)) \quad \text{with} \quad \psi(z) := \frac{z}{1 + |z|}$$

(with  $\psi(\pm\infty) := \pm 1$ ), and mapping the points  $(*, \pm\infty)$  to  $(0, \pm 1)$ . The following lemma shows that  $\mathcal{R}(E)$  is a Polish space if  $E$  is Polish.

**Lemma 2.20 (Properties of squeezed space)**

- (a) *If  $(E, d)$  is separable, then so is  $(\mathcal{R}(E), \rho)$ .*
- (b) *If  $(E, d)$  is complete, then so is  $(\mathcal{R}(E), \rho)$ .*

**Proof** If  $D$  is a countable dense subset of  $(E, d)$ , then  $D \times \mathbb{Q}$  is a countable dense subset of  $(\mathcal{R}(E), \rho)$ , proving (a).

To prove (b), let  $(x_n, t_n)$  be a Cauchy sequence in  $(\mathcal{R}(E), \rho)$ . Then by (2.6)  $t_n$  is a Cauchy sequence in  $\overline{\mathbb{R}}$  and hence  $t_n \rightarrow t$  for some  $t \in \overline{\mathbb{R}}$ . If  $t \in \mathbb{R}$ ,

then by (2.6)  $x_n$  is a Cauchy sequence in  $(E, d)$  so by the completeness of the latter,  $x_n \rightarrow x$  for some  $x \in E$ . By Lemma 2.19, it follows that  $(x_n, t_n)$  converges, proving the completeness of  $(\mathcal{R}(E), \rho)$ . ■

The following lemma identifies the compact subsets of  $\mathcal{R}(E)$ . In particular, the lemma shows that  $\mathcal{R}(E)$  is compact if  $E$  is compact.

**Lemma 2.21 (Compactness criterion)** *A set  $A \subset \mathcal{R}(E)$  is precompact if and only if for each  $T < \infty$ , there exists a compact set  $K \subset E$  such that  $\{x \in E : (x, t) \in A, t \in [-T, T]\} \subset K$ .*

**Proof** Assume that  $A \subset \mathcal{R}(E)$  has the property that for each  $T < \infty$ , there exists a compact set  $K \subset E$  such that  $\{x \in E : (x, t) \in A, t \in [-T, T]\} \subset K$ . To show that  $A$  is precompact, we will show that each sequence  $(x_n, t_n) \in A$  has a convergent subsequence. By the compactness of  $\overline{\mathbb{R}}$ , we can select a subsequence  $(x'_n, t'_n)$  such that  $t'_n \rightarrow t$  for some  $t \in \overline{\mathbb{R}}$ . If  $t = \pm\infty$ , then by Lemma 2.19  $(x'_n, t'_n) \rightarrow (*, \pm\infty)$  and we are done. Otherwise, there exists a  $T < \infty$  such that  $t'_n \in [-T, T]$  for all  $n$  large enough. By assumption, there then exists a compact set  $K \subset E$  such that  $x'_n \in K$  for all  $n$  large enough, so we can select a further subsequence such that  $(x''_n, t''_n)$  converges to a limit  $(x, t) \in E \times \mathbb{R}$ .

Assume, on the other hand, that  $A \subset \mathcal{R}(E)$  has the property that for some  $T < \infty$ , there does not exist a compact set  $K \subset E$  such that  $\{x \in E : (x, t) \in A, t \in [-T, T]\} \subset K$ . Set

$$B := \{x \in E : (x, t) \in A \text{ for some } t \in [-T, T]\}$$

The closure of  $B$  cannot be compact, since this would contradict our assumption. It follows that there exists a sequence  $x_n \in B$  that does not contain a convergent subsequence, and there exist  $t_n \in [-T, T]$  such that  $(x_n, t_n) \in A$ . But then, in view of Lemma 2.19, the sequence  $(x_n, t_n)$  cannot contain a convergent subsequence either, proving that  $A$  is not precompact. ■

## 2.7 Path space

Let  $E$  be a metrisable space and let  $\mathcal{R}(E)$  be the squeezed space defined in Section 2.6. By definition, a *path* in  $E$  is a nonempty compact subset  $\pi \subset \mathcal{R}(E)$  such that  $\{x \in E : (x, t) \in \pi\}$  has at most one element for each given  $t \in \mathbb{R}$  and the set

$$\bar{I}_\pi := \{t \in \overline{\mathbb{R}} : \exists x \in E \cup \{*\} \text{ s.t. } (x, t) \in \pi\} \quad (2.8)$$

is a closed subinterval of  $\overline{\mathbb{R}}$ . We call  $I_\pi := \overline{I}_\pi \cap \mathbb{R}$  the *domain* of  $\pi$  and we call

$$\sigma_\pi := \inf \overline{I}_\pi \quad \text{and} \quad \tau_\pi := \sup \overline{I}_\pi \quad (2.9)$$

the *starting time* and *final time* of the path  $\pi$ . For each  $t \in \overline{I}_\pi$ , we define  $\pi(t) \in E \cup \{*\}$  by  $\{\pi(t)\} := \{x \in E : (x, t) \in \pi\}$ . Then  $I_\pi \ni t \mapsto \pi(t)$  is a function from  $I_\pi$  to  $E$ . We let  $\Pi(E)$  denote the set of all paths in  $E$ .

**Lemma 2.22 (Path viewed as a function)** *The domain  $I_\pi$  of a path  $\pi \in \Pi(E)$  is a closed subinterval of  $\mathbb{R}$ , and  $t \mapsto \pi(t)$  is a continuous function from  $I_\pi$  to  $E$ . Conversely, if  $I \subset \mathbb{R}$  is a closed interval and  $t \mapsto f(t)$  is a continuous function from  $I$  to  $E$ , then there exists a path  $\pi \in \Pi(E)$  such that  $I_\pi = I$  and  $\pi(t) = f(t)$  ( $t \in I$ ). The path  $\pi$  is uniquely determined by the interval  $I$  and function  $f$ , except in the trivial case when  $I = \emptyset$ , in which case there are two possible choices for  $\pi$ .*

**Proof** We first show that for each  $\pi \in \Pi(E)$ , the function  $I_\pi \ni t \mapsto \pi(t)$  is continuous. Assume that  $t_n, t \in I_\pi$  and  $t_n \rightarrow t$ . Since  $\pi$  is compact, the sequence  $(\pi(t_n), t_n)$  is precompact. Since  $\pi(t)$  is the only element of  $\{x \in E : (x, t) \in \pi\}$ , each subsequence of the  $(\pi(t_n), t_n)$  must converge to  $(\pi(t), t)$ . By Lemma 2.2, we conclude that  $(\pi(t_n), t_n) \rightarrow (\pi(t), t)$ . Since  $t \in \mathbb{R}$ , by Lemma 2.19, we conclude that  $\pi(t_n) \rightarrow \pi(t)$ , which shows that  $I_\pi \ni t \mapsto \pi(t)$  is continuous on  $I$  as claimed.

Let  $I \subset \mathbb{R}$  be a closed interval and let  $f : I \rightarrow E$  be continuous. Assume that  $I$  is nonempty. Let  $\overline{I}$  be the closure of  $I$  in  $\overline{\mathbb{R}}$ . Extend  $f$  to  $\overline{I}$  by setting  $f(t) := *$  if  $t = \pm\infty$ . Let  $\pi := \{(f(t), t) : t \in \overline{I}\}$ . It follows from Lemma 2.19 and the continuity of  $f$  that the map

$$\overline{I} \ni t \mapsto (f(t), t) \in \mathcal{R}(E) \quad (2.10)$$

is continuous. Since  $\overline{I}$  is compact and since  $\pi$  is the image of  $\overline{I}$  under the continuous map (2.10), we conclude that  $\pi$  is compact. Clearly,  $\{x \in E : (x, t) \in \pi\}$  has precisely one element for  $t \in \overline{I}$ , and is empty for  $t \notin \overline{I}$ . This shows that  $\pi \in \Pi(E)$ . Since  $\overline{I}$  is the only closed subinterval of  $\overline{\mathbb{R}}$  such that  $\overline{I} \cap \mathbb{R} = I$ , we see that  $\pi$  is uniquely determined by the interval  $I$  and function  $f$ .

In the special case that  $I = \emptyset$ , it is easy to see that there exist precisely two paths  $\pi$  such that  $I_\pi = I$  (the condition  $\pi(t) = f(t)$  ( $t \in I$ ) is void in this case). These are the trivial paths with  $\overline{I}_\pi = \{-\infty\}$  or  $\{\infty\}$ , respectively. ■

In view of Lemma 2.22, we often view a path  $\pi \in \Pi(E)$  as a continuous function defined on a closed interval  $I_\pi \subset \mathbb{R}$ . If  $I \subset \mathbb{R}$  is a closed nonempty



interval, then we identify the space  $\mathcal{C}_I(E)$  defined in Section 2.4 with the subset of  $\Pi(E)$  defined as  $\{\pi \in \Pi(E) : I_\pi = I\}$ .

Let  $\mathcal{K}_+(\mathcal{R}(E))$  be the set of nonempty compact subsets of the squeezed space  $\mathcal{R}(E)$ . We equip  $\mathcal{K}_+(\mathcal{R}(E))$  with the Hausdorff topology. We observe that  $\Pi(E)$  is a subset of  $\mathcal{K}_+(\mathcal{R}(E))$ . We naturally equip  $\Pi(E)$  with the induced topology from its embedding in  $\mathcal{K}_+(\mathcal{R}(E))$ .

**Lemma 2.23 (Paths with a fixed domain)** *Let  $I \subset \mathbb{R}$  be a closed nonempty interval. The induced topology on  $\mathcal{C}_I(E)$  from its embedding in  $\Pi(E)$  is the topology of locally uniform convergence.*

**Proof** Assume that  $\pi_n, \pi \in \mathcal{C}_I(E)$ , viewed as functions, satisfy  $\pi_n \rightarrow \pi$  locally uniformly. We need to show that viewed as compact subsets of  $\mathcal{R}(E)$ , the sets  $\pi_n, \pi$  satisfy  $\pi_n \rightarrow \pi$  in the Hausdorff topology on  $\mathcal{K}(\mathcal{R}(E))$ . Let  $\bar{I}$  denote the closure of  $I$  in  $\mathbb{R}$ . By Lemma 2.14, we need to show that  $\bigcup_n \pi_n$  is precompact and

$$\begin{aligned} \pi \subset \{ & (x, t) \in \mathcal{R}(E) : \exists t_n \in \bar{I} \text{ s.t. } (\pi_n(t_n), t_n) \rightarrow (x, t)\}, \\ \{ & (x, t) \in \mathcal{R}(E) : (x, t) \text{ is a cluster} \\ & \text{point of } (\pi_n(t_n), t_n) \text{ for some } t_n \in \bar{I}\} \subset \pi. \end{aligned} \quad (2.11)$$

To see that  $\bigcup_n \pi_n$  is precompact, we need to show that each sequence of the form  $(\pi_{n(m)}(t_m), t_m)_{m \geq 1}$  has a convergent subsequence. If  $n(m)$  infinitely often takes the same value  $n$ , then the claim is obvious from the compactness of  $\pi_n$ , so without loss of generality we may assume that  $n(m) \rightarrow \infty$ . Going to a subsequence if necessary, we may assume that  $t_m \rightarrow t$  for some  $t \in \bar{I}$ . If  $t = \pm\infty$ , then the claim is again obvious so we may assume that  $t \in I$ . In this case Lemma 2.13 (ii) tells us that  $\pi_{n(m)}(t_m) \rightarrow \pi(t)$  so we have found a convergent subsequence as required.

To prove the first inclusion in (2.11), let  $(\pi(t), t) \in \pi$  and set  $t_n := t$  for all  $n$ . If  $t \in I$ , then  $\pi_n(t) \rightarrow \pi(t)$  since locally uniform convergence implies pointwise convergence, and if  $t = \pm\infty$  then trivially  $(*, t) \rightarrow (*, t)$  as  $n \rightarrow \infty$ . To prove the second inclusion, assume that  $(\pi_{n(m)}(t_{n(m)}), t_{n(m)}) \rightarrow (x, t)$  as  $m \rightarrow \infty$  for some  $(x, t) \in \mathcal{R}(E)$ ,  $t_n \in \bar{I}$ , and  $n(m) \rightarrow \infty$ . If  $t \in I$ , then we can use Lemma 2.13 (ii) which tells us that  $\pi_{n(m)}(t_{n(m)}) \rightarrow \pi(t)$  and hence  $(x, t) = (\pi(t), t) \in \pi$ . If  $t = \pm\infty$ , then trivially  $x = *$  and  $(*, t) \in \pi$ .

Assume, conversely, that  $\pi_n \rightarrow \pi$  in the Hausdorff topology on  $\mathcal{K}(\mathcal{R}(E))$ . We need to show that  $\pi_n, \pi \in \mathcal{C}_I(E)$  and that  $\pi_n \rightarrow \pi$  locally uniformly. Assume that  $t_n, t \in I$  such that  $t_n \rightarrow t$ . By Lemma 2.13 (ii), it suffices to show that  $\pi_n(t_n) \rightarrow \pi(t)$  for all such  $t_n, t$ . Equivalently, we may show that  $(\pi_n(t_n), t_n) \rightarrow (\pi(t), t)$ . By Lemma 2.2, it suffices to show that the set  $\{(\pi_n(t_n), t_n) : n \in \mathbb{N}\}$  is precompact and  $(\pi(t), t)$  is the only cluster

point of the sequence  $(\pi_n(t_n), t_n)$ . By Lemma 2.14, there exists a compact set  $C \subset \mathcal{R}(E)$  such that  $\pi_n \subset C$  for all  $n$ , so  $\{(\pi_n(t_n), t_n) : n \in \mathbb{N}\}$  is precompact as required. Let  $(x, t)$  be any cluster point. By Lemma 2.14 (ii),  $(x, t) \in \pi$  and hence  $x = \pi(t)$ , which shows that  $\pi_n(t_n) \rightarrow \pi(t)$  as required. ■

Our next proposition says that the space of paths in  $E$  is Polish provided  $E$  has this property.

**Proposition 2.24 (Polish space)** *If  $E$  is a Polish space, then so is  $\Pi(E)$ .*

The proof of Proposition 2.24 needs some preparations. Let  $d$  be a metric generating the topology on  $E$  and let  $\pi \in \Pi(E)$ . For each  $\pi \in \Pi(E)$ ,  $\delta > 0$  and  $T < \infty$ , we define

$$m_{T,\delta}(\pi) := \sup \{d(\pi(s), \pi(t)) : s, t \in I_\pi, -T \leq s \leq t \leq T, t - s \leq \delta\}. \quad (2.12)$$

The quantity  $m_{T,\delta}(\pi)$  is called the *modulus of continuity* of the path  $\pi$ . More generally, for any compact subset  $K \subset \mathcal{R}(E)$ , we can define

$$m_{T,\delta}(K) := \sup \{d(x, y) : (x, s), (y, t) \in K, -T \leq s \leq t \leq T, t - s \leq \delta\},$$

which coincides with our previous definition if  $\pi$  is a path. In analogy with (2.8), we also define

$$\bar{I}_K := \{t \in \bar{\mathbb{R}} : \exists x \in E \cup \{*\} \text{ s.t. } (x, t) \in K\}.$$

**Lemma 2.25 (Characterisation of paths)** *A compact subset  $\pi \subset \mathcal{R}(E)$  is an element of the path space  $\Pi(E)$  if and only if  $\bar{I}_K$  is a closed subinterval of  $\bar{\mathbb{R}}$  and  $\lim_{\delta \rightarrow 0} m_{T,\delta}(\pi) = 0$  for all  $T < \infty$ .*

**Proof** Assume that  $\pi \in \mathcal{K}(\mathcal{R}(E))$  and  $\limsup_{\delta \rightarrow 0} m_{T,\delta}(\pi) > 0$  for some  $T < \infty$ . Then we can find  $(x_n, s_n), (y_n, t_n) \in \pi$  and  $\delta > 0$  with  $d(x_n, y_n) \geq \delta$ ,  $-T \leq s_n \leq t_n \leq T$ , and  $t_n - s_n \leq 1/n$ . Since  $\pi$  is compact, by going to a subsequence, we can assume that  $(x_n, s_n) \rightarrow (x, s)$  and  $(y_n, t_n) \rightarrow (y, t)$  for some  $(x, s), (y, t) \in \pi$  with  $d(x, y) \geq \delta > 0$ ,  $-T \leq s \leq t \leq T$ , and  $t - s = 0$ . This shows that  $\pi \notin \Pi(E)$ .

Conversely, if  $\pi \notin \Pi(E)$ , then either  $\bar{I}_\pi$  is not a closed subinterval of  $\bar{\mathbb{R}}$  or there exist  $(x, t), (y, t) \in \pi$  with  $x \neq y$ . In the latter case, since  $(*, \pm\infty)$  are the only points in  $\mathcal{R}(E)$  with time coordinate  $\pm\infty$  we must have  $t \in \mathbb{R}$ . But then  $m_{T,\delta}(\pi) \geq d(x, y) > 0$  for all  $T \geq |t|$ , which shows that  $\limsup_{\delta \rightarrow 0} m_{T,\delta}(\pi) > 0$  for some  $T < \infty$ . ■

**Proof of Proposition 2.24** If  $E$  is a Polish space, then by Lemma 2.20 so is  $\mathcal{R}(E)$  and hence by Lemma 2.15 so is  $\mathcal{K}(\mathcal{R}(E))$ . Let us set

$$\mathcal{K}' := \{K \in \mathcal{K}(\mathcal{R}(E)) : \bar{I}_K \text{ is a closed subinterval of } \bar{\mathbb{R}}\}. \quad (2.13)$$

Then  $\mathcal{K}'$  is a closed subset of  $\mathcal{K}(\mathcal{R}(E))$  and hence Polish in the induced topology by Proposition 2.4. For each  $\varepsilon, \delta > 0$  and  $T < \infty$ , the set

$$A_{T,\varepsilon,\delta} := \{K \in \mathcal{K}' : m_{T,\delta}(K) \geq \varepsilon\}$$

is a closed subset of  $\mathcal{K}'$  and hence its complement  $A_{T,\varepsilon,\delta}^c$  is open. By Lemma 2.25,

$$\Pi(E) = \bigcap_{n,m} \bigcup_k A_{n,1/m,1/k}^c,$$

which is a countable intersection of open sets, i.e., a  $G_\delta$ -set. By Proposition 2.4, it follows that  $\Pi(E)$  is a Polish space.  $\blacksquare$

A set  $\mathcal{A} \subset \Pi(E)$  is called *equicontinuous* if

$$\limsup_{\delta \rightarrow 0} \sup_{\pi \in \mathcal{A}} m_{T,\delta}(\pi) = 0 \quad (T < \infty).$$

The following theorem identifies the compact subsets of  $\Pi(E)$ . Condition (ii) is called the *compact containment* condition. If  $I \subset \mathbb{R}$  is a closed nonempty interval, then  $\mathcal{C}_I(E)$  is a closed subset of  $\Pi$  and hence the following theorem can also be used to identify the precompact subsets of  $\mathcal{C}_I(E)$ . In this context, the result is known as the *Arzela-Ascoli theorem*. Note that while the definition of equicontinuity depends (at least a priori) on the choice of the metric  $d$  on  $E$ , whether a set  $\mathcal{A} \subset \Pi(E)$  is precompact only depends on the topology on  $E$ , so when verifying conditions (i) and (ii) below, we are free to choose any metric  $d$  that generates the topology on  $E$ .

**Theorem 2.26 (Arzela-Ascoli)** *A set  $\mathcal{A} \subset \Pi(E)$  is precompact if and only if*

- (i)  $\mathcal{A}$  is equicontinuous,
- (ii) for each  $T < \infty$ , there exists a compact set  $C \subset E$  such that  $\pi(t) \in C$  for all  $\pi \in \mathcal{A}$ ,  $t \in [-T, T]$ .

**Proof** Let  $\mathcal{K}'$  be the space defined in (2.13), equipped with the Hausdorff topology. Let  $\overline{\mathcal{A}}$  denote the closure of  $\mathcal{A}$ , viewed as a subset of the space  $\mathcal{K}'$ . Then  $\mathcal{A}$  is a precompact subset of  $\Pi(E)$  if and only if  $\overline{\mathcal{A}}$  is a compact subset of  $\mathcal{K}'$  and  $\overline{\mathcal{A}} \subset \Pi(E)$ . By Lemmas 2.16 and 2.21,  $\overline{\mathcal{A}}$  is a compact subset of  $\mathcal{K}'$  if and only if condition (ii) holds. To complete the proof, it suffices to show that assuming that (ii) holds, one has  $\overline{\mathcal{A}} \subset \Pi(E)$  if and only if (i) holds.

We first show that (i) implies  $\overline{\mathcal{A}} \subset \Pi(E)$ . Assume that  $\pi_n \in \mathcal{A}$  converge in the Hausdorff topology to a compact subset  $\pi \subset \mathcal{R}(E)$ . To show that  $\pi \in \Pi(E)$ , will apply Lemma 2.25. If  $(x, s), (y, t) \in \pi$ , then by Lemma 2.14,

there exist  $(x_n, s_n), (y_n, t_n) \in \pi_n$  such that  $(x_n, s_n) \rightarrow (x, s)$  and  $(y_n, t_n) \rightarrow (y, t)$ . If  $s, t \in [-T, T]$  and  $|t - s| \leq \delta$ , then for  $n$  large enough we have  $s_n, t_n \in [-T - 1, T + 1]$  and  $|t_n - s_n| \leq 2\delta$ . Since  $d(x_n, y_n) \rightarrow d(x, y)$ , it follows that

$$\limsup_{\delta \rightarrow 0} m_{T, \delta}(\pi) \leq \limsup_{\delta \rightarrow 0} \sup_n m_{T+1, 2\delta}(\pi_n) = 0 \quad (\delta > 0, T < \infty),$$

which by Lemma 2.25 implies that  $\pi \in \Pi(E)$ .

Assume now that (ii) holds but (i) fails. Then there exist  $T < \infty$  and  $\varepsilon > 0$  such that for each  $n \geq 1$ , we can find  $\pi_n \in \mathcal{A}$  with  $m_{T, 1/n}(\pi_n) \geq \varepsilon$ . This means that there exist  $-T \leq s_n \leq t_n \leq T$  such that  $d(\pi_n(s_n), \pi_n(t_n)) \geq \varepsilon$  and  $t_n - s_n \leq 1/n$ . By (ii),  $\overline{\mathcal{A}}$  is a compact subset of  $\mathcal{K}'$ , so by going a subsequence we may assume that  $\pi_n \rightarrow \pi \in \mathcal{K}'$ . By going to a further subsequence, we may assume that  $s_n \rightarrow s$  and  $t_n \rightarrow t$  for some  $s, t \in [-T, T]$ . But then  $s = t$  since  $t_n - s_n \leq 1/n$ . Let  $x_n := \pi_n(s_n)$  and  $y_n := \pi_n(t_n)$ . By (ii), we can select a further subsequence such that  $x_n \rightarrow x$  and  $y_n \rightarrow y$  for some  $x, y$  with  $d(x, y) \geq \varepsilon$ . By Lemma 2.14, we have  $(x, t), (y, t) \in \pi$  which shows that  $\pi \notin \Pi(E)$  and hence  $\overline{\mathcal{A}}$  is not a subset of  $\Pi(E)$ . ■

## 2.8 Tightness

In this section, we use the general results from the previous section to derive a tightness criterion for sequences of random variables with values in the space  $\Pi(\overline{\mathbb{R}})$ .

**Lemma 2.27 (Precompactness)** *Let  $\mathcal{A}$  be a subset of  $\Pi(\overline{\mathbb{R}})$ . Then  $\mathcal{A}$  is precompact if and only if for all  $T < \infty$  and  $\varepsilon > 0$ , there exists a  $\delta > 0$  such that*

$$\begin{aligned} |\pi(u) - \pi(t)| &\leq \varepsilon \text{ for all } \pi \in \mathcal{A} \text{ and } \sigma_\pi \leq t \leq u \\ \text{s.t. } &(\pi(t), t), (\pi(u), u) \in [-T, T]^2, \quad u - t \leq \delta. \end{aligned}$$

**Proof** Let  $\phi : \overline{\mathbb{R}} \rightarrow [-1, 1]$  be strictly increasing and continuous with  $\phi(\pm\infty) = \pm 1$ . Then

$$d(x, y) := |\phi(x) - \phi(y)| \quad (x, y \in \overline{\mathbb{R}}).$$

is a metric generating the topology on  $\overline{\mathbb{R}}$ . Since  $\overline{\mathbb{R}}$  is compact, by the Arzela-Ascoli theorem (Theorem 2.26),  $\mathcal{A}$  is precompact if and only if it is equicontinuous, i.e.,

$$\sup \left\{ d(\pi(t), \pi(u)) : \pi \in \mathcal{A}, \sigma_\pi \leq t \leq u \leq \tau_\pi, \right. \\ \left. t, u \in [-T, T], u - t \leq \delta \right\} \xrightarrow{\delta \rightarrow 0} 0 \quad \forall T < \infty.$$

In other words,  $\mathcal{A}$  is *not* precompact if and only if

$$\begin{aligned} \exists T < \infty \text{ and } \varepsilon > 0 \text{ s.t. } \forall \delta > 0 \exists \pi \in \mathcal{A} \text{ and } \sigma_\pi \leq t \leq u \leq \tau_\pi \\ \text{s.t. } t, u \in [-T, T], u - t \leq \delta \text{ and } d(\pi(t), \pi(u)) > \varepsilon. \end{aligned} \quad (2.14)$$

We claim that this is equivalent to

$$\begin{aligned} \exists S, T < \infty \text{ and } \varepsilon > 0 \text{ s.t. } \forall \delta > 0 \exists \pi \in \mathcal{A} \text{ and } \sigma_\pi \leq t \leq u \leq \tau_\pi \text{ s.t.} \\ t, u \in [-T, T], \pi(t), \pi(u) \in [-S, S], u - t \leq \delta \text{ and } d(\pi(t), \pi(u)) > \varepsilon/3. \end{aligned} \quad (2.15)$$

The implication (2.15) $\Rightarrow$ (2.14) is trivial. To prove the converse, assume that (2.14) holds for some  $T < \infty$  and  $\varepsilon > 0$ . Making  $\varepsilon$  smaller if necessary, we can without loss of generality assume that  $0 < \varepsilon < 1$ . We can choose the function  $\phi$  in the definition of our metric  $d$  on  $\overline{\mathbb{R}}$  to be symmetric and then define  $S > 0$  by  $d(\pm S, \pm\infty) = \varepsilon/3$ . Now fix  $\delta > 0$  and let  $\pi$  be as in (2.14). If  $\pi(t), \pi(u) \in [-S, S]$  already holds we are done. If  $\pi(t) \notin [-S, S]$ , then either 1.  $\pi(t) \in [-\infty, -S)$  or 2.  $\pi(t) \in (S, \infty]$ . Assume that we are in case 1. Since  $d(\pi(t), \pi(u)) > \varepsilon$ , we must have  $\pi(u) \in (-S, \infty]$ . Therefore, by continuity, there must be some  $t' \in [t, u]$  such that  $\pi(t') = -S$ . Then  $d(\pi(t'), \pi(u)) > (2/3)\varepsilon$ . If  $\pi(u) \leq S$  we are done. Otherwise, by continuity, there must be some  $u' \in [t', u]$  such that  $\pi(u') = S$  and now  $d(\pi(t'), \pi(u')) = d(-S, S) > \varepsilon/3$ . Case 2 is similar, by symmetry, and the case that  $\pi(t) \in [-S, S]$  but  $\pi(u) \notin [-S, S]$  can also be treated in the same way.

Replacing  $S$  and  $T$  by  $S \vee T$  if necessary, we see (2.15) is equivalent to

$$\begin{aligned} \exists T < \infty \text{ and } \varepsilon > 0 \text{ s.t. } \forall \delta > 0 \exists \pi \in \mathcal{A} \text{ and } \sigma_\pi \leq t \leq u \leq \tau_\pi \text{ s.t.} \\ (\pi(t), t), (\pi(u), u) \in [-T, T]^2, u - t \leq \delta \text{ and } d(\pi(t), \pi(u)) > \varepsilon/3. \end{aligned}$$

Using the fact that for each  $T < \infty$ , there exist constants  $0 < c < C < \infty$  such that

$$c|x - y| \leq d(x, y) \leq C|x - y| \quad (x, y \in [-T, T]),$$

the claim of the lemma now follows.  $\blacksquare$

**Proposition 2.28 (Almost sure precompactness)** *Let  $\mathcal{A}$  be a random subset of  $\Pi(\overline{\mathbb{R}})$ . Then  $\mathcal{A}$  is almost surely a precompact subset of  $\Pi(\overline{\mathbb{R}})$  if and only if*

$$\begin{aligned} \mathbb{P} \left[ \left| \pi(u) - \pi(t) \right| \geq \varepsilon \text{ for some } \pi \in \mathcal{A} \text{ and } \sigma_\pi \leq t \leq u \leq \tau_\pi \right. \\ \left. \text{s.t. } (\pi(t), t), (\pi(u), u) \in [-T, T]^2, u - t \leq \delta \right] \xrightarrow[\delta \rightarrow 0]{} 0 \quad \forall T < \infty, \varepsilon > 0. \end{aligned}$$

**Proof** Let  $A_{T,\varepsilon}^\delta$  denote the event that

$$\begin{aligned} & |\pi(u) - \pi(t)| \geq \varepsilon \text{ for some } \pi \in \mathcal{A} \text{ and } \sigma_\pi \leq t \leq u \leq \tau_\pi \\ & \text{s.t. } (\pi(t), t), (\pi(u), u) \in [-T, T]^2, \quad u - t \leq \delta. \end{aligned}$$

Then  $\delta \leq \delta'$  implies  $A_{T,\varepsilon}^\delta \subset A_{T,\varepsilon}^{\delta'}$  and  $A_{T,\varepsilon} := \bigcap_{\delta > 0} A_{T,\varepsilon}^\delta$  is the event that

$$\begin{aligned} & \forall \delta > 0 \exists \pi \in \mathcal{A} \text{ and } \sigma_\pi \leq t \leq u \leq \tau_\pi \\ & \text{s.t. } (\pi(t), t), (\pi(u), u) \in [-T, T]^2, \quad u - t \leq \delta, \text{ and } |\pi(u) - \pi(t)| \geq \varepsilon. \end{aligned}$$

The assumption of the proposition implies that  $P(A_{T,\varepsilon}) = 0$  for all  $T < \infty$  and  $\varepsilon > 0$ . In particular, if  $(T_n)_{n \geq 1}$  and  $(\varepsilon_m)_{m \geq 1}$  are sequences of positive constants such that  $T_n \rightarrow \infty$  and  $\varepsilon_n \rightarrow 0$ , then

$$\mathbb{P}\left(\bigcup_{n \geq 1} \bigcup_{m \geq 1} A_{T_n, \varepsilon_m}\right) = 0,$$

which shows that almost surely, for all  $n \geq 1$  and  $m \geq 1$ , there exists a  $\delta > 0$  such that

$$\begin{aligned} & |\pi(u) - \pi(t)| < \varepsilon_m \text{ for all } \pi \in \mathcal{A} \text{ and } \sigma_\pi \leq t \leq u \leq \tau_\pi \\ & \text{s.t. } (\pi(t), t), (\pi(u), u) \in [-T_n, T_n]^2, \quad u - t \leq \delta. \end{aligned}$$

By Lemma 2.27, it follows that  $\mathcal{A}$  is almost surely precompact.

On the other hand, if the assumption of the proposition does not hold, then the event  $A_{T,\varepsilon}$  has positive probability for some  $T < \infty$  and  $\varepsilon > 0$ , which by Lemma 2.27 implies that  $\mathcal{A}$  is with positive probability not precompact.  $\blacksquare$

**Proposition 2.29 (Tightness of random compact sets of paths)** *Let  $\mathcal{K}(\Pi(\overline{\mathbb{R}}))$  be the set of compact subsets of  $\Pi(\overline{\mathbb{R}})$ , equipped with the Hausdorff topology. Let  $(\mathcal{A}_n)_{n \geq 1}$  be a sequence of random variables with values in  $\mathcal{K}(\Pi(\overline{\mathbb{R}}))$ . Then the probability laws  $(\mathbb{P}[\mathcal{A}_n \in \cdot])_{n \geq 1}$  are tight if and only if*

$$\begin{aligned} & \sup_{n \geq 1} \mathbb{P}\left[|\pi(u) - \pi(t)| \geq \varepsilon \text{ for some } \pi \in \mathcal{A}_n \text{ and } \sigma_\pi \leq t \leq u \leq \tau_\pi \right. \\ & \left. \text{s.t. } (\pi(t), t), (\pi(u), u) \in [-T, T]^2, \quad u - t \leq \delta\right] \xrightarrow{\delta \rightarrow 0} 0 \end{aligned}$$

for all  $T < \infty$  and  $\varepsilon > 0$ .

**Proof (partial)** By Theorem 2.12, the probability laws  $(\mathbb{P}[\mathcal{A}_n \in \cdot])_{n \geq 1}$  are tight if and only if for each  $\eta > 0$ , there exists a compact set  $C \subset \mathcal{K}(\Pi(\overline{\mathbb{R}}))$  such that

$$\inf_{n \geq 1} \mathbb{P}[\mathcal{A}_n \in C] \geq 1 - \eta.$$

Equivalently, we may show that there exists a precompact set  $C \subset \mathcal{K}(\Pi(\overline{\mathbb{R}}))$  with this property, because its closure  $\overline{C}$  is then compact with  $\mathbb{P}[\mathcal{A}_n \in \overline{C}] \geq \mathbb{P}[\mathcal{A}_n \in C]$ . By Lemma 2.16, a subset  $C \subset \mathcal{K}(\Pi(\overline{\mathbb{R}}))$  is precompact if and only if there exists a compact  $\mathcal{C} \subset \Pi(\overline{\mathbb{R}})$  such that  $\mathcal{A} \subset \mathcal{C}$  for all  $\mathcal{A} \in C$ . It follows that the probability laws  $(\mathbb{P}[\mathcal{A}_n \in \cdot])_{n \geq 1}$  are tight if and only if for each  $\eta > 0$ , there exists a compact set  $\mathcal{C} \subset \Pi(\overline{\mathbb{R}})$  such that<sup>1</sup>

$$\inf_{n \geq 1} \mathbb{P}[\mathcal{A}_n \subset \mathcal{C}] \geq 1 - \eta.$$

Again, this is equivalent to the existence, for each  $\eta > 0$ , of a precompact set  $\mathcal{C}$  with this property.

The rest of the argument is similar to the proof of Proposition 2.28. For brevity, we skip the details. ■

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<sup>1</sup>Indeed, the existence of such a  $\mathcal{C}$  is necessary by our previous condition and Lemma 2.16, and conversely, if such a  $\mathcal{C}$  exists, then by Lemma 2.16  $C := \{\mathcal{A} : \mathcal{A} \subset \mathcal{C}\}$  is compact so we can apply our previous condition.





# Chapter 3

## The Brownian web

### 3.1 Arrow configurations

We are interested in diffusive scaling limits of systems of branching and coalescing particles with small branching rate. To prepare for this, in the present chapter, we will study the case without branching. In this case, the limit is coalescing Brownian motions. Although this may seem relatively easy (and indeed it is significantly easier than the case with branching), we will see that nonetheless there are already nontrivial phenomena to be studied in this case, and that a detailed understanding of these phenomena will help us hugely when we take on the case with branching.

We will be interested in the diffusive scaling limit of coalescing random walks. Instead of working with coalescing random walks in continuous time, as we did in Section 1.2, it will often be convenient to work with coalescing random walks in discrete time, which we introduce now. By definition, we call

$$\mathbb{Z}_{\text{even}}^2 := \{(x, t) \in \mathbb{Z}^2 : x + t \text{ is even}\}$$

the *even sublattice* of  $\mathbb{Z}^2$ . Let  $\omega = (\omega_z)_{z \in \mathbb{Z}_{\text{even}}^2}$  be an i.i.d. collection of random variables that are uniformly distributed on  $\{-1, +1\}$ . We can use  $\omega$  to define a random directed graph with vertex set  $\mathbb{Z}_{\text{even}}^2$  and set of oriented edges

$$\vec{E} := \{(x, t), (x + \omega_{(x,t)}, t + 1) : (x, t) \in \mathbb{Z}_{\text{even}}^2\}.$$

We call the random directed graph  $(\mathbb{Z}_{\text{even}}^2, \vec{E})$  an *arrow configuration*. See Figure 3.1 for a picture.

In Section 2.7, for any metrisable space  $\mathcal{X}$ , we gave a definition of the path space  $\Pi(\mathcal{X})$ . Recall that  $I_\pi$  denotes the domain of a path  $\pi \in \Pi(\mathcal{X})$  and that  $\sigma_\pi, \tau_\pi$  denote its starting time and final time, respectively. We will especially

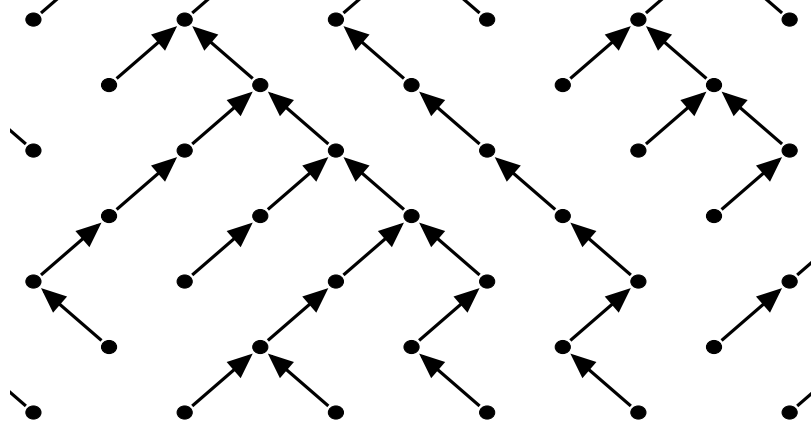


Figure 3.1: An arrow configuration.

be interested in the case that the metrisable space  $\mathcal{X}$  is  $\overline{\mathbb{R}} := [-\infty, \infty]$ , the extended real line. We let

$$\Pi^\uparrow := \{\pi \in \Pi(\overline{\mathbb{R}}) : \tau_\pi = \infty\}.$$

We call  $\Pi^\uparrow$  the space of all *upward paths*. In view of Lemma 2.22, elements of  $\Pi^\uparrow$  correspond to continuous functions  $\pi : I_\pi \rightarrow \overline{\mathbb{R}}$ , where  $I_\pi$  is an interval of the form  $[\sigma_\pi, \infty)$  if the starting time  $\sigma_\pi$  is finite, and

$$I_\pi = \mathbb{R} \text{ if } \sigma_\pi = -\infty \quad \text{and} \quad I_\pi = \emptyset \text{ if } \sigma_\pi = +\infty.$$

We will call the point

$$z_\pi := (\pi(\sigma_\pi), \sigma_\pi)$$

the *starting point* of the path  $\pi$ . Note that in general  $z_\pi$  is an element of  $\mathcal{R}(\overline{\mathbb{R}})$ , the squeezed space defined in Section 2.6. By definition, a *path in the arrow configuration*  $(\mathbb{Z}_{\text{even}}^2, \vec{E})$ , or simply a *path in  $\omega$* , is a path  $\pi \in \Pi^\uparrow$  with the following properties:

- (i)  $(\pi(t), t) \in \mathbb{Z}_{\text{even}}^2$  ( $t \in \mathbb{Z}$ ,  $t \geq \sigma_\pi$ ),
- (ii)  $\pi(t+1) = \pi(t) + \omega_{(\pi(t), t)}$  ( $t \in \mathbb{Z}$ ,  $t \geq \sigma_\pi$ ),
- (iii)  $\pi(t+s) = (1-s)\pi(t) + s\pi(t+1)$  ( $0 \leq s \leq 1$ ,  $t \in \mathbb{Z}$ ,  $t \geq \sigma_\pi$ ).

In words, these are upward paths that visit points in the even sublattice at integer times and follow the arrows, with linear interpolation between integer times. We let

$$\mathcal{U} = \mathcal{U}(\omega) := \{\pi \in \Pi^\uparrow : \pi \text{ is a path in } \omega\}. \quad (3.1)$$

We let  $\overline{\mathcal{U}}$  denote the closure of  $\mathcal{U}$  in the topology on  $\Pi^\uparrow$ . The following proposition says that  $\overline{\mathcal{U}}$  is a.s. compact and compared to  $\mathcal{U}$  only contains a few extra trivial paths. Below, we use the notation  $\overline{\mathbb{Z}} := \mathbb{Z} \cup \{-\infty, \infty\}$ , i.e., this is the closure of  $\mathbb{Z}$  in  $\overline{\mathbb{R}}$ .

**Proposition 3.1 (Compact set of paths)** *The closure  $\overline{\mathcal{U}}$  of the random set of upward paths  $\mathcal{U}$  defined in (3.1) is almost surely a compact subset of  $\Pi^\uparrow$ . Moreover, almost surely, the set  $\overline{\mathcal{U}} \setminus \mathcal{U}$  consists of all paths  $\pi \in \Pi^\uparrow$  with  $\sigma_\pi \in \overline{\mathbb{Z}}$  and either  $\pi(t) = -\infty$  for all  $t \in I_\pi$  or  $\pi(t) = +\infty$  for all  $t \in I_\pi$ .*

**Proof** Since paths in  $\mathcal{U}$  are Lipschitz continuous with Lipschitz constant one, equicontinuity is obvious so  $\mathcal{U}$  is precompact by Proposition 2.28 and hence  $\overline{\mathcal{U}}$  is compact.

Let  $s \in \mathbb{Z}$  and let  $\pi \in \Pi^\uparrow$  be defined by  $\sigma_\pi := s$  and  $\pi(t) := -\infty$  for all  $\sigma_\pi \leq t < \infty$ . To see that  $\pi \in \overline{\mathcal{U}}$ , choose  $x_n \in \mathbb{Z}$  such that  $(x_n, s) \in \mathbb{Z}_{\text{even}}^2$  and  $x_n \rightarrow -\infty$ . Let  $\pi_n \in \mathcal{U}$  be the unique path started at  $(x_n, t)$ . Since  $\overline{\mathcal{U}}$  is compact, by going to a subsequence if necessary, we can assume that  $\pi_n \rightarrow \pi'$  for some  $\pi' \in \Pi^\uparrow$ . Since  $\pi_n$  is a random walk starting from  $(x_n, t)$  and  $x_n \rightarrow -\infty$ , the law of  $\pi_n(t)$  converges weakly to the delta measure on  $-\infty$  for each  $t \geq s$ , from which we conclude that  $\pi' = \pi$  and hence  $\pi \in \overline{\mathcal{U}}$ . In the same way, we see that  $\overline{\mathcal{U}}$  contains all trivial paths  $\pi$  with  $\sigma_\pi \in \mathbb{Z}$  and  $\pi(t) = \infty$  for all  $\sigma_\pi \leq t < \infty$ . Since  $\overline{\mathcal{U}}$  is closed, it also contains all limits of such paths, so letting  $\sigma_\pi \rightarrow \infty$  or  $\sigma_\pi \rightarrow -\infty$  we see that  $\overline{\mathcal{U}}$  also contains all trivial paths with  $\sigma_\pi = -\infty$  and either  $\pi(t) = -\infty$  for all  $t \in \mathbb{R}$  or  $\pi(t) = +\infty$  for all  $t \in \mathbb{R}$ , as well as the trivial path with  $\sigma_\pi = +\infty$ .

To complete the proof, we must show that if  $\pi \in \overline{\mathcal{U}}$  satisfies  $\pi(t) \in \mathbb{R}$  for some  $t \geq \sigma_\pi$ , then  $\pi(t) \in \mathbb{R}$  for all  $t \geq \sigma_\pi$ . We first note that paths in  $\mathcal{U}$  are *noncrossing* in the sense that there do not exist  $\pi, \pi' \in \mathcal{U}$  and  $\sigma_\pi \vee \sigma_{\pi'} \leq s < t < \infty$  such that  $\pi(s) < \pi'(s)$  while  $\pi'(t) < \pi(t)$ . It is easy to see that this property is preserved in the limit so paths in  $\overline{\mathcal{U}}$  are noncrossing too. Now assume that  $\pi \in \overline{\mathcal{U}}$  satisfies  $\pi(t) \in \mathbb{R}$  for some  $t \geq \sigma_\pi$ . Choose  $z_n = (x_n, s_n) \in \mathbb{Z}_{\text{even}}^2$  with  $s_n < t$  such that  $z_n \rightarrow (\infty, s)$  for some  $s \in \mathbb{R}$ , and let  $\pi_n \in \mathcal{U}$  denote the path started from  $z_n$ . Then  $\pi_n$  is a random walk started from  $z_n$ . By our previous arguments,  $\pi_n(t) \rightarrow \infty$  a.s. so  $\pi(t) < \pi_n(t)$  for all  $n$  large enough. Since paths in  $\overline{\mathcal{U}}$  are noncrossing it follows that there exists an  $n$  such that  $\pi(t) \leq \pi_n(t) < \infty$  for all  $t \geq \sigma_\pi$ . In the same way, by symmetry, we see that  $-\infty < \pi(t)$  for all  $t \geq \sigma_\pi$ .  $\blacksquare$

We now turn to what we are mainly interested in, which is the diffusive scaling limit of arrow configurations. For each  $\varepsilon > 0$ , we define a diffusive scaling map  $\theta_\varepsilon : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  by

$$\theta_\varepsilon(x, t) := (\varepsilon x, \varepsilon^2 t) \quad ((x, t) \in \mathbb{R}^2). \quad (3.2)$$

Let  $\mathcal{R}(\overline{\mathbb{R}})$  be the squeezed space defined in Section 2.6. We extend  $\theta_\varepsilon$  continuously to  $\mathcal{R}(\overline{\mathbb{R}})$  in the obvious way, by setting

$$\theta_\varepsilon(\pm\infty, t) := (\pm\infty, \varepsilon^2 t) \quad (t \in \mathbb{R}) \quad \text{and} \quad \theta_\varepsilon(*, \pm\infty) := (*, \pm\infty).$$

For any subset  $A \subset \mathcal{R}(\overline{\mathbb{R}})$ , we let

$$\theta_\varepsilon(A) := \{\theta_\varepsilon(z) : z \in A\}$$

denote the image of  $A$  under  $\theta_\varepsilon$ . In particular, this notation applies to paths  $\pi \in \Pi(\overline{\mathbb{R}})$ , which according to their definition in Section 2.7 correspond to compact subsets of  $\mathcal{R}(\overline{\mathbb{R}})$ . It is easy to see that  $\theta_\varepsilon(\pi) \in \Pi^\uparrow$  for all  $\pi \in \Pi^\uparrow$ , so the diffusive scaling map  $\theta_\varepsilon : \mathcal{R}(\overline{\mathbb{R}}) \rightarrow \mathcal{R}(\overline{\mathbb{R}})$  naturally gives rise to a diffusive scaling map from  $\Pi^\uparrow$  to  $\Pi^\uparrow$  which by a slight abuse of notation we also denote by  $\theta_\varepsilon$ . Going one step further, for any subset  $\mathcal{A} \subset \Pi^\uparrow$ , we let

$$\theta_\varepsilon(\mathcal{A}) := \{\theta_\varepsilon(\pi) : \pi \in \mathcal{A}\}$$

denote the image of  $\mathcal{A}$  under this map.

In Section 2.5, we equipped the space  $\mathcal{K}(\mathcal{X})$  of all compact subsets of a metrisable topological space  $\mathcal{X}$  with the Hausdorff topology. We make a simple observation.

**Lemma 3.2 (Map acting on compact sets)** *Let  $\mathcal{X}$  be a metrisable topological space and let  $\mathcal{K}(\mathcal{X})$  be the set of all compact subsets of  $\mathcal{X}$ . Let  $\psi : \mathcal{X} \rightarrow \mathcal{X}$  be a continuous map and let*

$$\hat{\psi}(K) := \{\psi(x) : x \in K\} \quad (K \in \mathcal{K}(\mathcal{X})).$$

*Then  $\hat{\psi}(K) \in \mathcal{K}(\mathcal{X})$  for all  $K \in \mathcal{K}(\mathcal{X})$ , and the map  $\hat{\psi} : \mathcal{K}(\mathcal{X}) \rightarrow \mathcal{K}(\mathcal{X})$  is continuous with respect to the Hausdorff topology.*

**Proof** The well-known fact that the continuous image of a compact set is itself a compact set has already been mentioned at the beginning of Section 2.2. To see that  $\hat{\psi} : \mathcal{K}(\mathcal{X}) \rightarrow \mathcal{K}(\mathcal{X})$  is continuous, assume that  $K_n \rightarrow K$ . Then by Lemma 2.14,

$$\exists C \in \mathcal{K}(\mathcal{X}) \text{ s.t. } K_n \subset C \quad \forall n \geq 1 \tag{3.3}$$

and

$$\begin{aligned} K &= \{x \in \mathcal{X} : \exists x_n \in K_n \text{ s.t. } x_n \rightarrow x\} \\ &= \{x \in \mathcal{X} : \exists x_n \in K_n \text{ s.t. } x \text{ is a cluster point of } (x_n)_{n \in \mathbb{N}}\}. \end{aligned} \tag{3.4}$$

Since  $\hat{\psi}(C)$  is compact and  $\hat{\psi}(K_n) \subset \hat{\psi}(C)$  for all  $n \geq 1$ , by Lemma 2.14, to prove that  $\hat{\psi}(K_n) \rightarrow \hat{\psi}(K)$ , it suffices to show that

$$\begin{aligned} \hat{\psi}(K) &= \{y \in \mathcal{X} : \exists y_n \in \hat{\psi}(K_n) \text{ s.t. } y_n \rightarrow y\} \\ &= \{y \in \mathcal{X} : \exists y_n \in \hat{\psi}(K_n) \text{ s.t. } y \text{ is a cluster point of } (y_n)_{n \in \mathbb{N}}\}. \end{aligned}$$

The latter condition can be rewritten as

$$\begin{aligned} \{\psi(x) : x \in K\} &= \{y \in \mathcal{X} : \exists x_n \in K_n \text{ s.t. } \psi(x_n) \rightarrow y\} \\ &= \{y \in \mathcal{X} : \exists x_n \in K_n \text{ s.t. } y \text{ is a cluster point of } (\psi(x_n))_{n \in \mathbb{N}}\}. \end{aligned}$$

It therefore suffices to prove that

- (i)  $\{\psi(x) : x \in K\} \subset \{y \in \mathcal{X} : \exists x_n \in K_n \text{ s.t. } \psi(x_n) \rightarrow y\}$ ,
- (ii)  $\{y \in \mathcal{X} : \exists x_n \in K_n \text{ s.t. } y \text{ is a cluster point of } (\psi(x_n))_{n \in \mathbb{N}}\} \subset \{\psi(x) : x \in K\}$ .

To prove (i), we use that by (3.4), for each  $x \in K$  there exist  $x_n \in K_n$  such that  $x_n \rightarrow x$ , and hence  $\psi(x_n) \rightarrow \psi(x)$  by the continuity of  $\psi$ . To prove (ii), assume that  $x_n \in K_n$  ( $n \in \mathbb{N}$ ) and there exists a sequence  $(n(m))_{m \geq 1}$  with  $\lim_{m \rightarrow \infty} n(m) = \infty$  such that  $y = \lim_{m \rightarrow \infty} \psi(x_{n(m)})$ . By (3.3) and the compactness of  $C$ , by going to a further subsequence if necessary, we can assume without loss of generality that  $\lim_{m \rightarrow \infty} x_{n(m)} = x$  for some  $x \in C$ . Then  $x \in K$  by (3.4) and  $\lim_{m \rightarrow \infty} \psi(x_{n(m)}) = \psi(x)$  by the continuity of  $\psi$  which shows that  $y = \psi(x)$ .  $\blacksquare$

As an immediate consequence of Lemma 3.2, we obtain:

**Lemma 3.3 (Scaling of paths)** *For each  $\varepsilon > 0$ , the map  $\theta_\varepsilon : \Pi^\uparrow \rightarrow \Pi^\uparrow$  is continuous.*

**Proof** Immediate from Lemma 3.2, the continuity of the map  $\theta_\varepsilon : \mathcal{R}(\overline{\mathbb{R}}) \rightarrow \mathcal{R}(\overline{\mathbb{R}})$ , and the fact that in Section 2.7 we viewed the path space  $\Pi(\overline{\mathbb{R}})$  as a subset of  $\mathcal{K}(\mathcal{R}(\overline{\mathbb{R}}))$  and equipped it with the induced topology from this embedding.  $\blacksquare$

Let  $\mathcal{U}$  be the set of all paths in an arrow configuration and let  $\overline{\mathcal{U}}$  be its closure, which by Proposition 3.1 is a random compact subset of  $\Pi^\uparrow$ . Then, since the continuous image of a compact set is compact, by Lemma 3.3, for each  $\varepsilon > 0$ , the diffusively rescaled set of paths  $\theta_\varepsilon(\overline{\mathcal{U}})$  is a random compact subset of  $\Pi^\uparrow$ . Our aim is to prove that

$$\mathbb{P}[\theta_\varepsilon(\overline{\mathcal{U}}) \in \cdot] \xrightarrow[\varepsilon \rightarrow 0]{\Rightarrow} \mathbb{P}[\mathcal{W} \in \cdot] \quad (3.5)$$

where  $\Rightarrow$  denotes weak convergence of probability laws on the space  $\mathcal{K}(\Pi^\uparrow)$ , equipped with the Hausdorff topology, and  $\mathcal{W}$  is a random compact subset of  $\Pi^\uparrow$  that will be called the *Brownian web*.

### 3.2 Coalescing Brownian motions

As a first step towards proving (3.5), we start by proving something like convergence of finite dimensional distributions. More precisely, for each  $\varepsilon > 0$ , we choose finitely many points  $z_1^\varepsilon, \dots, z_n^\varepsilon$  in the diffusively rescaled lattice  $\theta_\varepsilon(\mathbb{Z}_{\text{even}}^2)$ , in such a way that

$$(z_1^\varepsilon, \dots, z_n^\varepsilon) \xrightarrow{\varepsilon \rightarrow 0} (z_1, \dots, z_n)$$

for some  $z_1, \dots, z_n \in \mathbb{R}^2$ . Letting  $\pi_1^\varepsilon, \dots, \pi_n^\varepsilon$  denote the paths in  $\mathcal{U}$  with starting points  $z_1^\varepsilon, \dots, z_n^\varepsilon$ , we will argue that  $(\pi_1^\varepsilon, \dots, \pi_n^\varepsilon)$  converges in distribution to a collection of coalescing Brownian motions.

Let  $B^1 = (B_t^1)_{t \geq 0}$  and  $B^2 = (B_t^2)_{t \geq 0}$  be two independent standard one-dimensional Brownian motions started from initial states  $B_0^i = x_i$  ( $i = 1, 2$ ), and let

$$\tau := \inf\{t \geq 0 : B_t^1 = B_t^2\},$$

which is a.s. finite since  $(B_t^1 - B_t^2)_{t \geq 0}$  is a Brownian motion (with double the quadratic variation of a standard Brownian motion), and one-dimensional Brownian motion is point recurrent. Let  $\tilde{B}^2 = (\tilde{B}_t^2)_{t \geq 0}$  be defined by

$$\tilde{B}_t^2 := \begin{cases} B_t^2 & \text{if } t \leq \tau, \\ B_t^1 & \text{if } \tau \leq t. \end{cases}$$

Then it is easy to check that  $\tilde{B}^2$  is a standard Brownian motion. However,  $B^1$  and  $\tilde{B}^2$  are of course not independent. The process  $(B_t^1, B_t^2)_{t \geq 0}$  is a Markov process that is known as *coalescing Brownian motions*. Although this is not completely immediate from our definition (at least if one wants to give a formal proof), our definition is symmetric in the sense that  $(B_t^2, B_t^1)_{t \geq 0}$  is a Markov process with the same transition probabilities as  $(B_t^1, B_t^2)_{t \geq 0}$ .

We can carry out the same construction for any finite number of Brownian motions, that can moreover start at different times. Let  $z_1, \dots, z_n \in \mathbb{R}^2$  with  $z_i = (x_i, s_i)$  ( $i = 1, \dots, n$ ), and let  $B^1, \dots, B^n$  be independent Brownian motions such that  $B^i = (B_t^i)_{t \geq s_i}$  starts at time  $s_i$  in  $B_{s_i}^i = x_i$ . We set  $\tau_1 := \infty$ ,  $A_1 := \{(B_t^1, t) : s_1 \leq t < \infty\}$  and define inductively for  $j = 2, \dots, n$

$$\begin{aligned} \tau_j &:= \inf\{t \geq s_j : (B_t^j, t) \in A_1 \cup \dots \cup A_{j-1}\}, \\ A_j &:= \{(B_t^j, t) : s_j \leq t < \tau_j\}. \end{aligned}$$

By the recurrence of one-dimensional Brownian motion, almost surely  $\tau_j < \infty$  for all  $2 \leq j \leq n$ . Note that the sets  $A_1, \dots, A_n$  are disjoint. In view of this, we can uniquely define  $\iota(j) \in \{1, \dots, j-1\}$  by the requirement that

$$(B_{\tau_j}^j, \tau_j) \in A_{\iota(j)}.$$

Using this, we define inductively  $\tilde{B}^1 := B^1$  and

$$\tilde{B}_t^j := \begin{cases} B_t^j & \text{if } s_i \leq t \leq \tau_j, \\ \tilde{B}_t^{(j)} & \text{if } \tau_j \leq t. \end{cases}$$

We call  $\tilde{B}^1, \dots, \tilde{B}^n$  *coalescing Brownian motions* starting from the space-time points  $z_1, \dots, z_n \in \mathbb{R}^2$ .

We are now ready to formulate a result about the convergence in law of finitely many paths in an arrow configuration. We have already become used (hopefully!) to the slight abuse of notation by which  $\theta_\varepsilon$  can denote both a diffusive scaling map acting on space-time points, or on sets of space-time points such as paths, or even sets of paths. Taking this one step further, we also denote

$$\theta_\varepsilon(z_1, \dots, z_n) := (\theta_\varepsilon(z_1), \dots, \theta_\varepsilon(z_n)), \quad \theta_\varepsilon(\pi_1, \dots, \pi_n) := (\theta_\varepsilon(\pi_1), \dots, \theta_\varepsilon(\pi_n))$$

when  $z_1, \dots, z_n$  are space-time points and  $\pi_1, \dots, \pi_n$  are paths.

**Proposition 3.4 (Convergence of finite dimensional distributions)**

Let  $\varepsilon_k > 0$  satisfy  $\varepsilon_k \rightarrow 0$ . Fix  $n \geq 1$  and for each  $k$ , let  $z_1^k, \dots, z_n^k \in \mathbb{Z}_{\text{even}}^2$ . Assume that

$$\theta_{\varepsilon_k}(z_1^k, \dots, z_n^k) \xrightarrow[k \rightarrow \infty]{} (z_1, \dots, z_n) \in (\mathbb{R}^2)^n.$$

Fix an arrow configuration and for each  $k$ , let  $\pi_1^k, \dots, \pi_n^k$  be the unique paths in the arrow configuration with starting points  $z_1^k, \dots, z_n^k$ . Then

$$\mathbb{P}[\theta_{\varepsilon_k}(\pi_1^k, \dots, \pi_n^k) \in \cdot] \xRightarrow[k \rightarrow \infty]{} \mathbb{P}[(\pi_1, \dots, \pi_n) \in \cdot],$$

where  $\Rightarrow$  denotes weak convergence of probability measures on  $(\Pi^\uparrow)^n$  and  $\pi_1, \dots, \pi_n$  are coalescing Brownian motions starting from  $z_1, \dots, z_n$ .

**Proof** Our definition of coalescing Brownian motions involved a procedure that started with  $n$  independent Brownian motions  $(B^1, \dots, B^n)$  and used them to construct  $n$  coalescing Brownian motions  $(\tilde{B}^1, \dots, \tilde{B}^n)$ . More formally, we can view  $(\tilde{B}^1, \dots, \tilde{B}^n)$  as the image of  $(B^1, \dots, B^n)$  under a map

$$(\pi_1, \dots, \pi_n) \mapsto (\tilde{\pi}_1, \dots, \tilde{\pi}_n) \tag{3.6}$$

that takes  $n$  paths  $\pi_1, \dots, \pi_n$  in  $\Pi^\uparrow$  with starting points in  $\mathbb{R}^2$  and maps them into  $n$  new paths  $\tilde{\pi}_1, \dots, \tilde{\pi}_n$  with the same starting points.

For each  $k$ , let  $(R^{k,1}, \dots, R^{k,n})$  be a collection of independent random walks started from  $z_1^k, \dots, z_n^k$ , and let  $(\tilde{R}^{k,1}, \dots, \tilde{R}^{k,n})$  be its image under the map from (3.6). Then  $(\tilde{R}^{k,1}, \dots, \tilde{R}^{k,n})$  are coalescing random walks. It is

easy to see that they are equal in law with  $(\pi_1^k, \dots, \pi_n^k)$ . We want to show that

$$\mathbb{P}[\theta_{\varepsilon_k}(\tilde{R}^{k,1}, \dots, \tilde{R}^{k,n}) \in \cdot] \xrightarrow[k \rightarrow \infty]{} \mathbb{P}[(\tilde{B}^1, \dots, \tilde{B}^n) \in \cdot].$$

It is easy to see that the diffusive scaling map commutes with the map in (3.6), i.e., the random variable in the left-hand side of our equation is the same as what we would obtain if we first diffusively rescale the independent random walk paths and then apply the map from (3.6).

Weak convergence in law of diffusively rescaled independent random walks to independent Brownian motions follows from Donsker's invariance principle. Using Skorohod's representation theorem (Theorem 2.11), we can couple our random variables such that

$$\theta_{\varepsilon_k}(R^{k,1}, \dots, R^{k,n}) \xrightarrow[k \rightarrow \infty]{} (B^1, \dots, B^n) \quad \text{a.s.}$$

in the topology on  $(\Pi^\uparrow)^n$ . If the map in (3.6) would be continuous with respect to the topology on  $(\Pi^\uparrow)^n$ , then the rest of the proof would now be easy, since we would just apply this map to both sides of our last equation and we would be done.

Things are not quite so simple, however, since it is easy to check (even for  $n = 2$ ) that the map in (3.6) is not continuous with respect to the topology on  $(\Pi^\uparrow)^n$ . It turns out, however, that  $(B^1, \dots, B^n)$  is almost surely a point of continuity of this map, which is just as good. Here, with a point of continuity of the map in (3.6) we mean, of course, a collection of paths  $(\pi_1, \dots, \pi_n)$  with the property that for each  $(\pi_1^k, \dots, \pi_n^k)$  such that

$$(\pi_1^k, \dots, \pi_n^k) \xrightarrow[k \rightarrow \infty]{} (\pi_1, \dots, \pi_n),$$

one also has

$$(\tilde{\pi}_1^k, \dots, \tilde{\pi}_n^k) \xrightarrow[k \rightarrow \infty]{} (\tilde{\pi}_1, \dots, \tilde{\pi}_n).$$

That  $(B^1, \dots, B^n)$  is almost surely a point of continuity follows quite easily from our definitions and from Lemma 3.5 and Exercise 3.6 below. We leave the details to the reader.  $\blacksquare$

**Lemma 3.5 (Brownian paths cross when they meet)** *Let  $B^i = (B_t^i)_{t \geq s_i}$  ( $i = 1, 2$ ) be independent Brownian motions started from deterministic space-time points  $z_i = (x_i, s_i)$  ( $i = 1, 2$ ), respectively, and let*

$$\tau := \inf\{t \geq s_1 \vee s_2 : B_t^1 = B_t^2\}.$$

*Then almost surely, for each  $\varepsilon > 0$ , there exist times  $t_-, t_+ \in [\tau, \tau + \varepsilon]$  such that*

$$B_{t_-}^1 < B_{t_-}^2 \quad \text{and} \quad B_{t_+}^1 > B_{t_+}^2.$$



**Proof** By the strong Markov property,  $(B_{\tau+t}^1 - B_{\tau+t}^2)_{t \geq 0}$  is a Brownian motion, so it suffices to prove that a Brownian motion  $B$  started in zero takes both positive and negative values in each time interval of the form  $[0, \varepsilon]$  with  $\varepsilon > 0$ . By symmetry, it suffices to prove that  $\tau := \inf\{t \geq 0 : B_t < 0\} = 0$  a.s. By Blumenthal's zero-one law,  $\mathbb{P}[\tau = 0] \in \{0, 1\}$ , so it suffices to prove  $\mathbb{P}[\tau = 0] \neq 0$ . Assume that, conversely,  $\mathbb{P}[\tau = 0] = 0$ . Then a.s., there exists some random  $\varepsilon > 0$  such that  $B_t \geq 0$  for all  $t \in [0, \varepsilon)$ . By symmetry, there then also a.s. exists some random  $\varepsilon' > 0$  such that  $B_t \leq 0$  for all  $t \in [0, \varepsilon')$ . Setting  $\varepsilon'' := \varepsilon \wedge \varepsilon'$  we see that there then also a.s. exists some random  $\varepsilon'' > 0$  such that  $B_t = 0$  for all  $t \in [0, \varepsilon'')$ . But then  $P[B_t = 0] \geq \mathbb{P}[t < \varepsilon''] > 0$  for  $t > 0$  small enough, which contradicts the fact that  $\mathbb{P}[B_t = 0] = 0$  for all  $t > 0$ . ■

**Exercise 3.6 (Convergence of meeting times)** Let  $\pi_1, \pi_2 \in \Pi^\uparrow$  have starting points  $z_i = (x_i, s_i)$  ( $i = 1, 2$ ), respectively, and assume that their first meeting time

$$\tau := \inf\{t \geq s_1 \vee s_2 : \pi_1(t) = \pi_2(t)\}$$

satisfies  $\tau < \infty$ . Assume moreover that for each  $\varepsilon > 0$ , there exist times  $t_-, t_+ \in [\tau, \tau + \varepsilon]$  such that

$$\pi_1(t_-) < \pi_2(t_-) \quad \text{and} \quad \pi_1(t_+) > \pi_2(t_+).$$

Let  $\pi_1^k, \pi_2^k \in \Pi^\uparrow$  satisfy  $\pi_i^k \rightarrow \pi_i$  ( $i = 1, 2$ ). Then the first meeting times  $\tau_k$  of  $\pi_1^k$  and  $\pi_2^k$  satisfy  $\tau_k \rightarrow \tau$ . Hint: First show that generally  $\tau \leq \liminf_{k \rightarrow \infty} \tau_k$ . Then use the assumption about crossing to prove that  $\limsup_{k \rightarrow \infty} \tau_k \leq \tau_k$ .

### 3.3 The Brownian web

Let  $\mathcal{D} \subset \mathbb{R}^2$  be countable. Since  $\mathcal{D}$  is countable, we can enumerate it as  $\mathcal{D} := \{z_i : i \geq 1\}$  where  $(z_i)_{i \geq 1}$  be a sequence of space-time points  $z_i \in \mathbb{R}^2$ . Then for each  $n \geq 1$ , we can construct a collection of random paths  $(\pi_1, \dots, \pi_n)$  that are distributed as coalescing Brownian motions starting from  $(z_1, \dots, z_n)$ . Since these laws are consistent, by Kolmogorov's extension theorem, we can construct a random collection of paths  $(\pi_z)_{z \in \mathcal{D}}$  such that for each finite set  $\Delta \subset \mathcal{D}$ , the paths  $(\pi_z)_{z \in \Delta}$  that are distributed as coalescing Brownian motions starting from the points in  $\Delta$ . We call  $(\pi_z)_{z \in \mathcal{D}}$  a *collection of coalescing Brownian motions* started from the countable set  $\mathcal{D}$ .

**Proposition 3.7 (Precompactness)** Let  $(\pi_z)_{z \in \mathcal{D}}$  be a collection of coalescing Brownian motions started from a countable set  $\mathcal{D} \subset \mathbb{R}^2$ . Then  $\{\pi_z : z \in \mathcal{D}\}$  is almost surely a precompact subset of  $\Pi^\uparrow$ .

**Proof (sketch)** We apply Proposition 2.28 to  $\mathcal{A} := \{\pi_z : z \in \mathcal{D}\}$ . Fix  $T < \infty$  and  $\varepsilon, \delta > 0$  and consider the grid

$$\mathcal{G}_{\varepsilon, \delta} := \left\{ \left( \frac{1}{3}k\varepsilon, l\delta \right) : k, l \in \mathbb{Z} \right\}.$$

Let  $\mathcal{A}' = \{\pi'_z : z \in \mathcal{D} \cup \mathcal{G}_{\varepsilon, \delta}\}$  be a collection of coalescing Brownian motions started from the countable set  $\mathcal{D} \cup \mathcal{G}_{\varepsilon, \delta}$ . We can couple  $\mathcal{A}'$  to  $\mathcal{A}$  such that  $\pi'_z = \pi_z$  for each  $z \in \mathcal{D}$ . Since paths in  $\mathcal{A}$  cannot cross paths in  $\{\pi'_z : z \in \mathcal{G}_{\varepsilon, \delta}\}$ , it is not hard to see (see Figure 3.2) that almost surely on the event

$$\begin{aligned} |\pi(u) - \pi(t)| &\geq \varepsilon \text{ for some } \pi \in \mathcal{A} \text{ and } \sigma_\pi \leq t \leq u \\ \text{s.t. } (\pi(t), t) &\in [-T, T]^2, \quad u - t \leq \delta \end{aligned}$$

one has that

$$\begin{aligned} |\pi'_{(x,s)}(s+r) - x| &\geq \frac{1}{3}\varepsilon \\ \text{for some } (x, s) &\in \mathcal{G}_{\varepsilon, \delta} \cap [-T - \varepsilon, T + \varepsilon]^2 \text{ and } r \in [0, 2\delta]. \end{aligned} \tag{3.7}$$

By Lemma 3.12 below, if  $B$  is a standard Brownian motion, then

$$\mathbb{P} \left[ \sup_{r \in [0, 2\delta]} |B_r| \geq \frac{1}{3}\varepsilon \right] \leq C e^{-c\varepsilon^2/\delta},$$

for some  $C < \infty$  and  $c > 0$ . A simple union bound then tells us that the probability of the event in (3.7) can be estimated from above by

$$C_T \varepsilon^{-1} \delta^{-1} e^{-c\varepsilon^2/\delta}$$

for some  $C_T < \infty$  and  $c > 0$ . This quantity goes to zero as  $\delta \rightarrow 0$  for fixed  $T < \infty$  and  $\varepsilon > 0$ , so by Proposition 2.28 we conclude that  $\{\pi_z : z \in \mathcal{D}\}$  is almost surely precompact.  $\blacksquare$

We adopt the following notation. If  $\mathcal{A} \subset \mathcal{K}(\Pi^\uparrow)$  is a collection of paths and  $D \subset \mathcal{R}(\overline{\mathbb{R}})$  is a set, then we let

$$\mathcal{A}(D) := \{\pi \in \mathcal{A} : z_\pi \in D\} \tag{3.8}$$

denote the subset of  $\mathcal{A}$  consisting of all paths that have their starting points in  $D$ . In particular, for  $z \in \mathcal{R}(\overline{\mathbb{R}})$ , we write  $\mathcal{A}(z) := \mathcal{A}(\{z\})$ . As before, we let  $\overline{\mathcal{A}}$  denote the closure of a set  $\mathcal{A} \subset \Pi^\uparrow$ .

**Theorem 3.8 (The Brownian web)** *There exists a random compact set  $\mathcal{W} \subset \mathcal{K}(\Pi^\uparrow)$  whose distribution is uniquely determined by the following properties.*

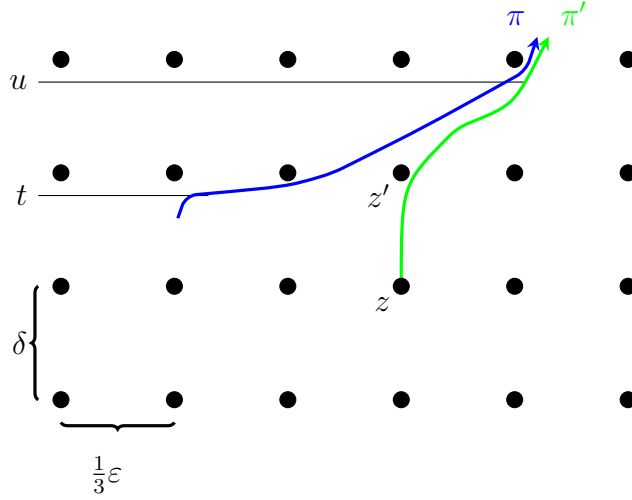


Figure 3.2: The tightness argument. The blue path  $\pi$  moves a distance  $\geq \varepsilon$  during a time interval  $[t, u]$  of length  $\leq \delta$ , forcing the green path  $\pi'$  starting from the point  $z \in \mathcal{G}_{\varepsilon, \delta}$  to move a distance  $\geq \varepsilon/3$  from its starting position during a time interval of length  $2d$ . Note that the blue path could have passed below the point  $z' \in \mathcal{G}_{\varepsilon, \delta}$  that lies just above  $z$ .

- (i) For each  $z \in \mathbb{R}^2$ , almost surely there exists a unique  $\pi_z \in \Pi^\uparrow$  such that  $\mathcal{W}(z) = \{\pi_z\}$ .
- (ii) For each  $z_1, \dots, z_n \in \mathbb{R}^2$ , the paths  $(\pi_{z_1}, \dots, \pi_{z_n})$  are distributed as coalescing Brownian motions starting from  $z_1, \dots, z_n$ .
- (iii) For each countable dense set  $\mathcal{D} \subset \mathbb{R}^2$ , almost surely  $\mathcal{W} = \overline{\mathcal{W}(\mathcal{D})}$ .

**Remark** In Section 3.8, we will see that in point (i) of Theorem 3.8, the order of the “for all” and “almost surely” statements cannot be interchanged. Although for a fixed, deterministic  $z \in \mathbb{R}^2$ , it is true that almost surely,  $\mathcal{W}(z)$  consists of a single path, there exist random points  $z \in \mathbb{R}^2$  in which  $\mathcal{W}(z)$  has two, or even three elements.

**Proof of Theorem 3.8** Let  $\mathcal{D} \subset \mathbb{R}^2$  be countable and dense and let  $(\pi_z)_{z \in \mathcal{D}}$  be a collection of coalescing Brownian motions started from  $\mathcal{D}$ . Then  $\{\pi_z : z \in \mathcal{D}\}$  is precompact by Proposition 3.7 and hence

$$\mathcal{W} := \overline{\{\pi_z : z \in \mathcal{D}\}} \tag{3.9}$$

is a random compact subset of  $\Pi^\uparrow$ . We claim that paths in  $\mathcal{W}$  do not cross, in the sense that there do not exist  $\pi, \pi' \in \mathcal{W}$  and  $\sigma_\pi \vee \sigma_{\pi'} \leq s < t$  such that

$\pi(s) < \pi'(s)$  but  $\pi'(t) < \pi(t)$ . Indeed, if such paths would exist, then they would be limits of paths  $\pi_n, \pi'_n$  in  $\{\pi_z : z \in \mathcal{D}\}$  that would also have to cross for  $n$  large enough, which is not possible.

We will now show that  $\mathcal{W}$  has the properties (i)–(iii) from the theorem. Fix  $z = (x, s) \in \mathbb{R}^2$ . Let  $\varepsilon_n$  be positive constants converging to zero, let  $z_n^\pm := (x, s \pm \varepsilon_n)$ , and let  $\mathcal{D}' := \mathcal{D} \cup \{z_n^\pm : n \geq 1\}$ . We can couple  $(\pi_z)_{z \in \mathcal{D}}$  to a collection of coalescing Brownian motions  $(\pi'_z)_{z \in \mathcal{D}'}$  started from  $\mathcal{D}'$  such that  $\pi_z = \pi'_z$  for all  $z \in \mathcal{D}$ . Let

$$\tau_n := \inf\{t \geq 0 : \pi'_{z_n^-}(t) = \pi'_{z_n^+}(t)\}. \quad (3.10)$$

Since paths cannot cross, we see that  $\tau_1 \geq \tau_2 \geq \dots$  and hence  $\tau_n \rightarrow \tau_\infty$  a.s. for some random variable  $\tau_\infty$ . Using Lemma 3.5, it is easy to see that if we start two independent Brownian motions from  $z_n^-$  and  $z_n^+$ , then their first meeting time converges to zero in probability as  $n \rightarrow \infty$ . Together with our earlier observation, this implies that  $\tau_\infty = s$  a.s. Since paths in  $\mathcal{W}$  do not cross the paths  $\pi'_{z_n^\pm}$ , any path  $\pi \in \mathcal{W}$  that starts in  $(\pi(\sigma_\pi), \sigma_\pi) = z$  must satisfy

$$\pi'_{z_n^-}(t) \leq \pi(t) \leq \pi'_{z_n^+}(t) \quad (t \geq s). \quad (3.11)$$

Since  $\tau_\infty = s$  a.s., there can be at most one such path, proving property (i).

Property (ii) now follows from the fact that we can couple  $(\pi_z)_{z \in \mathcal{D}}$  to a collection of coalescing Brownian motions  $(\pi'_z)_{z \in \mathcal{D} \cup \{z_1, \dots, z_n\}}$  such that  $\pi_z = \pi'_z$  for all  $z \in \mathcal{D}$ . To prove property (iii), we must show that our construction does not depend on the choice of the countable dense set  $\mathcal{D}$ . Let  $\mathcal{D}$  and  $\mathcal{D}'$  be countable dense subsets of  $\mathbb{R}^2$ , let  $(\pi_z)_{z \in \mathcal{D} \cup \mathcal{D}'}$  be coalescing Brownian motions started from  $\mathcal{D} \cup \mathcal{D}'$ , and let

$$\begin{aligned} \mathcal{W} &:= \overline{\{\pi_z : z \in \mathcal{D}\}}, & \mathcal{W}' &:= \overline{\{\pi_z : z \in \mathcal{D}'\}}, \\ \text{and } \mathcal{W}'' &:= \overline{\{\pi_z : z \in \mathcal{D} \cup \mathcal{D}'\}}. \end{aligned} \quad (3.12)$$

To prove (iii), it suffices to show that  $\mathcal{W} = \mathcal{W}'$ . By symmetry, it suffices to show that  $\mathcal{W} \subset \mathcal{W}'$ . Since both  $\mathcal{W}$  and  $\mathcal{W}'$  are closed, it suffices to show that for each  $z \in \mathcal{D}$ , the path  $\pi_z$  satisfies  $\pi_z \in \mathcal{W}'$ . By what we have already proved, there exists unique paths  $\pi' \in \mathcal{W}'$  and  $\pi'' \in \mathcal{W}''$  with starting points  $z_{\pi'} = z_{\pi''} = z$ . Since  $\pi' \in \mathcal{W}''$  we must have  $\pi' = \pi''$  and since  $\pi_z \in \mathcal{W}''$  we must have  $\pi'' = \pi_z$ , so we conclude that  $\pi_z = \pi'' = \pi' \in \mathcal{W}'$ .  $\blacksquare$

For the next lemma, we let

$$\Pi_{\text{triv}}^\uparrow := \{\pi \in \Pi^\uparrow : \pi(t) = -\infty \forall t \geq \sigma_\pi\} \cup \{\pi \in \Pi^\uparrow : \pi(t) = +\infty \forall t \geq \sigma_\pi\}$$

denote the set of trivial paths that are constantly  $-\infty$  or  $\infty$ .

**Lemma 3.9 (Trivial paths)** *Let  $\mathcal{W}$  be a Brownian web. Then  $\Pi_{\text{triv}}^\uparrow \subset \mathcal{W}$  a.s. and each  $\pi \in \mathcal{W} \setminus \Pi_{\text{triv}}^\uparrow$  satisfies  $\pi(t) \in \mathbb{R}$  for all  $\sigma_\pi \leq t < \infty$ .*

**Proof** This follows from the same argument as in the proof of Proposition 3.1.  $\blacksquare$

We still need to provide an estimate that we have used in the proof of Proposition 3.7.

**Lemma 3.10 (Reflection principle)** *Let  $(B_t)_{t \geq 0}$  be Brownian motion. Then*

$$\mathbb{P}\left[\sup_{s \in [0, t]} B_s < a\right] = \mathbb{P}[|B_t| \leq a] \quad (t, a > 0). \quad (3.13)$$

**Proof** Let  $\tau := \inf\{t > 0 : B_t = a\}$ . By the strong Markov property and the symmetry of Brownian motion, conditional on the event  $\{\tau < t\}$ , the events  $\{B_t > a\}$  and  $\{B_t < a\}$  have equal probabilities. Since  $\mathbb{P}[B_t = a] = 0$  and the event  $\{B_t > a\}$  almost surely implies  $\{\tau < t\}$ , it follows that

$$\mathbb{P}\left[\sup_{s \in [0, t]} B_s < a\right] = 1 - 2\mathbb{P}[B_t > a] = \mathbb{P}[|B_t| \leq a]. \quad (3.14)$$

$\blacksquare$

**Lemma 3.11 (Tail estimate)** *Let  $N$  be a standard normal random variable. Then*

$$\mathbb{P}[N \geq a] \leq \frac{1}{2}e^{-a^2/2}. \quad (3.15)$$

**Proof** This follows by writing

$$\begin{aligned} \mathbb{P}[N \geq a] &= \frac{1}{\sqrt{2\pi}} \int_a^\infty e^{-x^2/2} dx = \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-(x+a)^2/2} dx \\ &= e^{-a^2/2} \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-x^2/2 - ax} dx \\ &\leq e^{-a^2/2} \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-x^2/2} dx = \frac{1}{2}e^{-a^2/2}. \end{aligned} \quad (3.16)$$

$\blacksquare$

**Lemma 3.12 (Large displacements)** *Let  $(B_t)_{t \geq 0}$  be Brownian motion. Then*

$$\mathbb{P}\left[\sup_{s \in [0, t]} |B_s| \geq a\right] \leq 2e^{-a^2/(2t)}. \quad (3.17)$$

**Proof** Let  $N$  denote a standard normal random variable. We estimate, using Lemmas 3.10 and 3.11,

$$\begin{aligned} \mathbb{P}\left[\sup_{s \in [0, t]} |B_s| \geq a\right] &\leq 2\mathbb{P}\left[\sup_{s \in [0, t]} B_s \geq a\right] = 2\mathbb{P}[|B_t| > a] = 4\mathbb{P}[B_t > a] \\ &= 4\mathbb{P}[\sqrt{t}N > a] \leq 2e^{-a^2/(2t)}. \end{aligned} \tag{3.18}$$

■

### 3.4 Dual arrow configurations

By definition, we call

$$\mathbb{Z}_{\text{odd}}^2 := \{(x, t) \in \mathbb{Z}^2 : x + t \text{ is odd}\}$$

the *odd sublattice* of  $\mathbb{Z}^2$ . In Section 3.1, we showed how an i.i.d. collection  $\omega = (\omega_z)_{z \in \mathbb{Z}_{\text{even}}^2}$  of uniformly distributed  $\{-1, +1\}$ -valued random variables defines a random directed graph  $(\mathbb{Z}_{\text{even}}^2, \vec{E})$  that we called an arrow configuration. Given  $\omega$ , we define  $\hat{\omega} = (\hat{\omega}_z)_{z \in \mathbb{Z}_{\text{odd}}^2}$  by

$$\hat{\omega}_{(x, t+1)} = \omega_{(x, t)} \quad ((x, t) \in \mathbb{Z}_{\text{even}}^2).$$

We can use  $\hat{\omega}$  to define a random directed graph with vertex set  $\mathbb{Z}_{\text{odd}}^2$  and set of oriented edges

$$\vec{F} := \{(x, t), (x - \hat{\omega}_{(x, t)}, t - 1) : (x, t) \in \mathbb{Z}_{\text{odd}}^2\}.$$

We call the random directed graph  $(\mathbb{Z}_{\text{odd}}^2, \vec{F})$  the *dual arrow configuration* associated with the original (“forward”) arrow configuration  $(\mathbb{Z}_{\text{even}}^2, \vec{E})$ . The dual arrows are uniquely characterised in terms of the forward arrows by the property that dual arrows and forward arrows do not cross. See Figure 3.3 for a picture.

Recall that in general,  $\sigma_\pi$  and  $\tau_\pi$  denote the starting and final time of a path  $\pi \in \Pi(\overline{\mathbb{R}})$ . In particular, we define

$$\Pi^\downarrow := \{\pi \in \Pi(\overline{\mathbb{R}}) : \sigma_\pi = -\infty\}.$$

We call  $\Pi^\downarrow$  the space of all *downward paths*. Clearly,  $\Pi^\downarrow$  is equal to  $\Pi^\uparrow$  after a rotation over 180 degrees. When no confusion can arrive,<sup>1</sup> we will call the point

$$z_\pi := (\pi(\sigma_\pi), \sigma_\pi)$$

<sup>1</sup>We have to be careful since the intersection of  $\Pi^\uparrow$  and  $\Pi^\downarrow$  is not empty, but consists of all bi-infinite paths for which  $\sigma_\pi = -\infty$  and  $\tau_\pi = \infty$ . As we will see in a moment, however, there are no nontrivial bi-infinite paths in an arrow configuration.

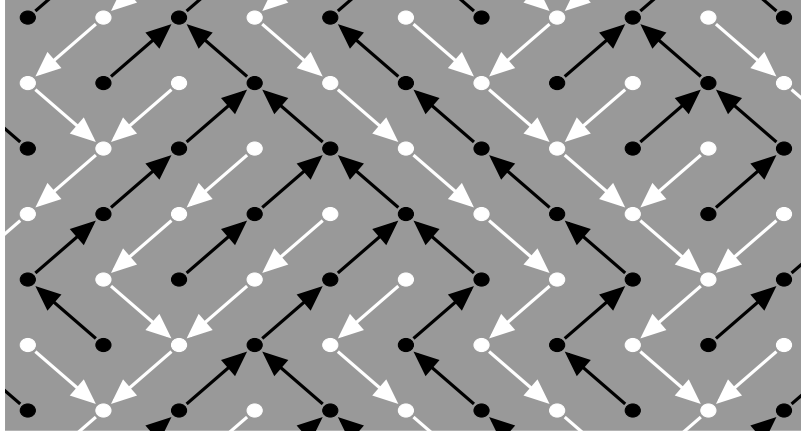


Figure 3.3: An arrow configuration (black) and its dual (white).

the *starting point* of a downward path  $\pi \in \Pi^\downarrow$ . We define a *downward path in the dual arrow configuration*  $(\mathbb{Z}_{\text{odd}}^2, \vec{F})$ , or simply a *path in  $\hat{\omega}$*  in exactly the same way as we defined upward paths in the forward arrow configuration. We let

$$\mathcal{U}' = \mathcal{U}'(\hat{\omega}) := \{\pi \in \Pi^\downarrow : \pi \text{ is a path in } \hat{\omega}\} \quad (3.19)$$

denote the set of all downward paths in the dual arrow configuration and we let  $\overline{\mathcal{U}'}$  denote the closure of  $\mathcal{U}'$  in the topology on  $\Pi^\downarrow$ .

### 3.5 The dual Brownian web

We have already introduced notation for the diffusive scaling map  $\theta_\varepsilon$  which may be applied to points  $z = (x, t)$  in space-time  $\mathcal{R}(\overline{\mathbb{R}})$ , to subsets of space-time such as paths, and even to sets of paths. We will use similar notation for the map

$$\mathcal{R}(\overline{\mathbb{R}}) \ni (x, t) \mapsto -(x, t) = (-x, -t) \in \mathcal{R}(\overline{\mathbb{R}}).$$

Thus, for any set  $A \subset \mathcal{R}(\overline{\mathbb{R}})$ , we set  $-A := \{-z : z \in A\}$ . In particular, this applies to the case that  $A = \pi \in \Pi^\uparrow$ . Then  $\Pi^\uparrow \ni \pi \mapsto -\pi \in \Pi^\downarrow$  is a bijection from  $\Pi^\uparrow$  to  $\Pi^\downarrow$ . Also, if  $\mathcal{A} \subset \Pi^\uparrow$  is a sets whose elements are paths, then we set  $-\mathcal{A} := \{-\pi : \pi \in \mathcal{A}\}$ . Using this notation, we say that  $\hat{\pi}_1, \dots, \hat{\pi}_n$  are *downward* coalescing Brownian motions starting from space-time points  $z_1, \dots, z_n$  if  $-\hat{\pi}_1, \dots, -\hat{\pi}_n$  are (usual, forward) coalescing Brownian motions starting from space-time points  $-z_1, \dots, -z_n$ . In the same way, we define countable collections of downward coalescing Brownian motions.

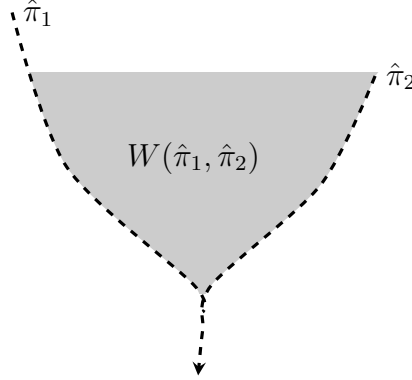


Figure 3.4: The wedge  $W(\hat{\pi}_1, \hat{\pi}_2)$  defined by the dual paths  $\hat{\pi}_1$  and  $\hat{\pi}_2$ .

Let  $\hat{\pi}_1, \hat{\pi}_2 \in \Pi^\downarrow$  be two downward paths started from space-time points  $(x_i, s_i) \in \mathbb{R}^2$  ( $i = 1, 2$ ), and let

$$\tau = \tau(\hat{\pi}_1, \hat{\pi}_2) := \sup \{t < s_1 \wedge s_2 : \hat{\pi}_1(t) = \hat{\pi}_2(t)\}$$

be their first meeting time (in the downward direction), which may be  $-\infty$ . The open set

$$W(\hat{\pi}_1, \hat{\pi}_2) := \{(x, t) : \tau < t < s_1 \wedge s_2 : \hat{\pi}_1(t) < x < \hat{\pi}_2(t)\}$$

is called the *wedge* defined by  $\hat{\pi}_1, \hat{\pi}_2$ . See Figure 3.4 for an illustration. We say that a (forward) path  $\pi \in \Pi^\uparrow$  *enters* the wedge  $W(\hat{\pi}_1, \hat{\pi}_2)$  if there exist times  $\sigma_\pi < s < t$  such that

$$(\pi(s), s) \notin \overline{W}(\hat{\pi}_1, \hat{\pi}_2) \quad \text{and} \quad (\pi(t), t) \in W(\hat{\pi}_1, \hat{\pi}_2),$$

where  $\overline{W}(\hat{\pi}_1, \hat{\pi}_2)$  denotes the closure of  $W(\hat{\pi}_1, \hat{\pi}_2)$ . In a completely analogous way, we define the first meeting time of two forward paths, the wedge defined by two forward paths, and what it means for a downward path to enter such a wedge. We make the following simple observation.

**Lemma 3.13 (Limits of wedges)** *Let  $(\hat{\pi}_i^n)_{n \geq 1}$  ( $i = 1, 2$ ) be sequences of downward paths and let  $(\pi^n)_{n \geq 1}$  be a sequence of forward paths. Assume that there exist  $\hat{\pi}_i \in \Pi^\downarrow$  ( $i = 1, 2$ ) and  $\pi \in \Pi^\uparrow$  such that*

$$\hat{\pi}_i^n \xrightarrow[n \rightarrow \infty]{} \hat{\pi}_i \quad (i = 1, 2) \quad \text{and} \quad \pi^n \xrightarrow[n \rightarrow \infty]{} \pi$$

*in the topologies on  $\Pi^\downarrow$  and  $\Pi^\uparrow$ , and that moreover*

$$\tau(\hat{\pi}_1^n, \hat{\pi}_2^n) \xrightarrow[n \rightarrow \infty]{} \tau(\hat{\pi}_1, \hat{\pi}_2).$$



Assume that for each  $n$ , the path  $\pi^n$  does not enter the wedge  $W(\pi_1^n, \pi_2^n)$ . Then the path  $\pi$  does not enter the wedge  $W(\pi_1, \pi_2)$ .

**Proof** By definition, if  $\pi$  enters the wedge  $W(\pi_1, \pi_2)$ , then there exist times  $\sigma_\pi < s < t$  such that

$$(\pi(s), s) \notin \overline{W}(\hat{\pi}_1, \hat{\pi}_2) \quad \text{and} \quad (\pi(t), t) \in W(\hat{\pi}_1, \hat{\pi}_2).$$

But then our assumptions imply that for  $n$  sufficiently large,  $\sigma_{\pi^n} < s < t$  and

$$(\pi^n(s), s) \notin \overline{W}(\hat{\pi}_1^n, \hat{\pi}_2^n) \quad \text{and} \quad (\pi^n(t), t) \in W(\hat{\pi}_1^n, \hat{\pi}_2^n),$$

which contradicts the assumption that  $\pi^n$  does not enter  $W(\pi_1^n, \pi_2^n)$ .  $\blacksquare$

**Proposition 3.14 (Dual coalescing Brownian motions)** *Let  $\mathcal{D}, \hat{\mathcal{D}}$  be countable dense subsets of  $\mathbb{R}^2$ . Then it is possible to construct a collection  $(\pi_z)_{z \in \mathcal{D}}$  of coalescing Brownian motions together with a collection  $(\hat{\pi}_z)_{z \in \hat{\mathcal{D}}}$  of downward coalescing Brownian motions in such a way that:*

- For each  $z \in \mathcal{D}$  and  $z_1, z_2 \in \hat{\mathcal{D}}$ , the path  $\pi_z$  does not enter the wedge  $W(\hat{\pi}_{z_1}, \hat{\pi}_{z_2})$ .
- For each  $z \in \hat{\mathcal{D}}$  and  $z_1, z_2 \in \mathcal{D}$ , the downward path  $\hat{\pi}_z$  does not enter the wedge  $W(\pi_{z_1}, \pi_{z_2})$ .

The proof of Proposition 3.14 makes use of the following simple lemma.

**Lemma 3.15 (Tightness of joint law)** *Let  $\mathcal{X}, \mathcal{Y}$  be Polish spaces, let  $(X_n, Y_n)_{n \geq 1}$  be a sequence of random variables with values in  $\mathcal{X} \times \mathcal{Y}$ , and let  $X$  and  $Y$  be random variables with values in  $\mathcal{X}$  and  $\mathcal{Y}$ , respectively. Assume that*

$$\mathbb{P}[X_n \in \cdot] \xrightarrow{n \rightarrow \infty} \mathbb{P}[X \in \cdot] \quad \text{and} \quad \mathbb{P}[Y_n \in \cdot] \xrightarrow{n \rightarrow \infty} \mathbb{P}[Y \in \cdot]$$

Then the probability laws

$$(\mathbb{P}[(X_n, Y_n) \in \cdot])_{n \geq 1}$$

are tight.

**Proof** The convergence of the marginal laws implies that the probability laws

$$(\mathbb{P}[X_n \in \cdot])_{n \geq 1} \quad \text{and} \quad (\mathbb{P}[Y_n \in \cdot])_{n \geq 1}$$

are tight, so for each  $\varepsilon > 0$ , there exist compact sets  $C \subset \mathcal{X}$  and  $K \subset \mathcal{Y}$  such that

$$\sup_{n \geq 1} \mathbb{P}[X_n \notin C] \leq \varepsilon \quad \text{and} \quad \sup_{n \geq 1} \mathbb{P}[Y_n \notin K] \leq \varepsilon$$

Then  $C \times K$  is compact and

$$\sup_{n \geq 1} \mathbb{P}[(X_n, Y_n) \notin C \times K] \leq 2\varepsilon.$$

Since  $\varepsilon > 0$  is arbitrary, it follows that the laws of  $(X_n, Y_n)$  are tight.  $\blacksquare$

**Proof of Proposition 3.14 (sketch)** Let  $\mathcal{U}$  be the collection of paths in an arrow configuration and let  $\mathcal{U}'$  be the collection of downward paths in the associated dual arrow configuration. Let  $\varepsilon_n$  be positive constants tending to zero. For each  $z \in \mathcal{D}$ , choose  $z_n \in \mathbb{Z}_{\text{even}}^2$  such that  $\theta_{\varepsilon_n}(z_n) \rightarrow z$ , and for each  $z \in \hat{\mathcal{D}}$ , choose  $z^n \in \mathbb{Z}_{\text{odd}}^2$  such that  $\theta_{\varepsilon_n}(z^n) \rightarrow z$ . For each  $z \in \mathcal{D}$  and  $n \geq 1$ , let  $R_z^n \in \mathcal{U}$  be the unique forward path starting at  $z_n$ , let  $\hat{R}_z^n \in \mathcal{U}'$  be the unique downward path starting at  $z^n$ , and let

$$\pi_z^n := \theta_{\varepsilon_n}(R_z^n) \quad \text{and} \quad \hat{\pi}_z^n := \theta_{\varepsilon_n}(\hat{R}_z^n)$$

denote the associated diffusively rescaled paths. We claim that

$$\begin{aligned} \mathbb{P}[(\pi_z^n)_{z \in \mathcal{D}} \in \cdot] &\xrightarrow[n \rightarrow \infty]{} \mathbb{P}[(\pi_z)_{z \in \mathcal{D}} \in \cdot], \\ \mathbb{P}[(\hat{\pi}_z^n)_{z \in \hat{\mathcal{D}}} \in \cdot] &\xrightarrow[n \rightarrow \infty]{} \mathbb{P}[(\hat{\pi}_z)_{z \in \hat{\mathcal{D}}} \in \cdot], \end{aligned}$$

where  $\Rightarrow$  denotes weak convergence of probability laws on the spaces  $(\Pi^\uparrow)^\mathcal{D}$  and  $(\Pi^\downarrow)^\mathcal{D}$ , respectively, which are equipped with the product topology, and  $(\pi_z)_{z \in \mathcal{D}}$  is a collection of coalescing Brownian motions while  $(\hat{\pi}_z)_{z \in \hat{\mathcal{D}}}$  is a collection of downward coalescing Brownian motions. Indeed, to prove this, by the definition of the product topology, it suffices to prove convergence of finite dimensional distributions. But this has already been done in Proposition 3.4.

In fact, using Exercise 3.6, we can strengthen our previous claim in a sense that also includes convergence of meeting times. More precisely, one can show that

$$\begin{aligned} \mathbb{P}[(\pi_z^n)_{z \in \mathcal{D}}, (\tau(\pi_{z_1}^n, \pi_{z_2}^n))_{(z_1, z_2) \in \mathcal{D}^2} \in \cdot] \\ \xrightarrow[n \rightarrow \infty]{} \mathbb{P}[(\pi_z)_{z \in \mathcal{D}}, (\tau(\pi_{z_1}, \pi_{z_2}))_{(z_1, z_2) \in \mathcal{D}^2} \in \cdot], \end{aligned}$$

and similarly for the collection of downward paths.

By Lemma 3.15, going to a subsequence if necessary, we can assume that the joint law of the random variables

$$(\pi_z^n)_{z \in \mathcal{D}}, \quad (\tau(\pi_{z_1}^n, \pi_{z_2}^n))_{(z_1, z_2) \in \mathcal{D}^2}, \quad (\hat{\pi}_z^n)_{z \in \hat{\mathcal{D}}}, \quad (\tau(\hat{\pi}_{z_1}^n, \hat{\pi}_{z_2}^n))_{(z_1, z_2) \in \mathcal{D}^2}$$

converges weakly. Then we can use Skorohod's representation theorem (Theorem 2.11) to couple our random variables so that the convergence is almost sure, i.e., we can find a coupling such that

$$\pi_z^n \xrightarrow[n \rightarrow \infty]{} \pi_z \text{ a.s.} \quad \text{and} \quad \tau(\pi_{z_1}^n, \pi_{z_2}^n) \xrightarrow[n \rightarrow \infty]{} \tau(\pi_{z_1}, \pi_{z_2}) \text{ a.s.}$$

for all  $z, z_1, z_2 \in \mathcal{D}$ , and likewise for downward paths. Since paths of  $\mathcal{U}$  do not enter wedges of  $\mathcal{U}'$  and vice versa, we can use Lemma 3.13 to conclude that the same is true for the limit object.  $\blacksquare$

**Theorem 3.16 (Wedge characterisation of the Brownian web)** *Let  $\mathcal{D}, \hat{\mathcal{D}}$  be countable dense subsets of  $\mathbb{R}^2$ , let  $(\pi_z)_{z \in \mathcal{D}}$  be a collection of coalescing Brownian motions started from  $\mathcal{D}$ , and let  $(\hat{\pi}_z)_{z \in \hat{\mathcal{D}}}$  be a collection of downward coalescing Brownian motions started from  $\hat{\mathcal{D}}$ . Assume that paths in  $(\pi_z)_{z \in \mathcal{D}}$  do not enter wedges of  $(\hat{\pi}_z)_{z \in \hat{\mathcal{D}}}$ . Let*

$$\begin{aligned} \mathcal{W}_- &:= \overline{\{\pi_z : z \in \mathcal{D}\}}, \\ \mathcal{W}_+ &:= \{\pi \in \Pi^\uparrow : \pi \text{ does not enter wedges of } (\hat{\pi}_z)_{z \in \hat{\mathcal{D}}}\}. \end{aligned}$$

Then  $\mathcal{W}_- = \mathcal{W}_+$ .

**Proof (sketch)** To prove the inclusion  $\mathcal{W}_- \subset \mathcal{W}_+$ , let  $\pi \in \mathcal{W}_-$ . Then there exists  $z_n \in \mathcal{D}$  such that  $\pi_{z_n} \rightarrow \pi$  as  $n \rightarrow \infty$ . Let  $z^1, z^2 \in \hat{\mathcal{D}}$ . By assumption,  $\pi_{z_n}$  does not enter the wedge  $W(\hat{\pi}_{z^1}, \hat{\pi}_{z^2})$  for any  $n \geq 1$ . By Lemma 3.13, it follows that  $\pi$  does not enter  $W(\hat{\pi}_{z^1}, \hat{\pi}_{z^2})$ . This completes the proof that  $\mathcal{W}_- \subset \mathcal{W}_+$ .

Before we continue, we note that our assumptions imply that the forward paths do not cross downward paths, in the sense that if  $z = (x, s) \in \mathcal{D}$  and  $z' = (y, u) \in \hat{\mathcal{D}}$  satisfy  $s < u$ , then  $\pi_z(s) < \hat{\pi}_{z'}(s)$  implies  $\pi_z(t) \leq \hat{\pi}_{z'}(t)$  for all  $t \in [s, u]$ . Indeed, we can always choose some  $z'' = (y', u') \in \hat{\mathcal{D}}$  with  $u \leq u'$  such that  $\hat{\pi}_{z'}(u) < \hat{\pi}_{z''}(u)$  and the meeting time  $\tau(\hat{\pi}_{z'}, \hat{\pi}_{z''})$  is less than  $s$ . Then  $\pi_z(t) > \hat{\pi}_{z'}(t)$  for some  $t \in (s, u]$  would imply that  $\pi_z$  enters the wedge  $W(\hat{\pi}_{z'}, \hat{\pi}_{z''})$ , contradicting our assumptions.

We now prove that  $\mathcal{W}_+ \subset \mathcal{W}_-$ . Let  $\pi \in \mathcal{W}_+$ . By Lemma 3.9 we can without loss of generality assume that  $\pi(t) \in \mathbb{R}$  for all  $t \in I_\pi$ . Fix  $\sigma_\pi < t_1 < \dots < t_m$  and  $\varepsilon > 0$ . We claim that there exists a  $z = (x, s) \in \mathcal{D}$  such that  $\sigma_\pi < s < t_1$  and  $|\pi_z(t_i) - \pi(t)| \leq \varepsilon$  for all  $i = 1, \dots, m$ . To see this, for each  $i = 1, \dots, m$ , we choose  $z_\pm^i = (x_\pm^i, t_\pm^i) \in \hat{\mathcal{D}}$  such that  $t_\pm^i > t_i$  and

$$\pi(t_i) - \varepsilon < \hat{\pi}_{z_-^i}(t_i) < \pi(t_i) < \hat{\pi}_{z_+^i}(t_i) < \pi(t_i) + \varepsilon.$$

Since  $\pi$  does not enter the wedge  $W(\hat{\pi}_{z_-^i}, \hat{\pi}_{z_+^i})$ , the meeting time of  $\hat{\pi}_{z_-^i}$  and  $\hat{\pi}_{z_+^i}$  must satisfy

$$\tau(\hat{\pi}_{z_-^i}, \hat{\pi}_{z_+^i}) \leq \sigma_\pi,$$

and we have  $\hat{\pi}_{z_-^i}(t) \leq \pi(t) \leq \hat{\pi}_{z_+^i}(t)$  for all  $t \in [\sigma_\pi, t_i]$ . We can now choose  $z = (x, s) \in \mathcal{D}$  such that  $\sigma_\pi < s < t_1$  and

$$\sup_{1 \leq i \leq m} \hat{\pi}_{z_-^i}(t_1) < \pi_z(t_1) < \inf_{1 \leq i \leq m} \hat{\pi}_{z_+^i}(t_1).$$

Since the path  $\pi_z$  cannot cross any of the downward paths  $\hat{\pi}_{z_\pm^i}$ , we must have

$$\hat{\pi}_{z_-}(t_i) < \pi_z(t_i) < \hat{\pi}_{z_+}(t_i) \quad (1 \leq i \leq m)$$

and hence  $|\pi_z(t_i) - \pi(t)| \leq \varepsilon$  for all  $i = 1, \dots, m$ , proving our claim.

Now let  $\varepsilon_n > 0$  satisfy  $\varepsilon_n \rightarrow 0$  and let  $\sigma_\pi < t_1 < \dots < t_m$ . By what we have just proved, for each  $n$  there exists a  $z_n \in \mathcal{D}$  such that  $|\pi_{z_n}(t_i) - \pi(t)| \leq \varepsilon$  for all  $i = 1, \dots, m$ . By Proposition 3.7, the closure of  $\{\pi_z : z \in \mathcal{D}\}$  is compact, so we can find a convergent subsequence. It follows that there exists a  $\pi' \in \mathcal{W}_-$  such that  $\pi'(t_i) = \pi(t_i)$  for all  $i = 1, \dots, m$ . Now let  $\{t_i : i \in \mathbb{N}\} \subset (\sigma_\pi, \infty)$  be countable and dense. By what we have just proved, for each  $m$ , there exists a  $\pi_m \in \mathcal{W}_-$  such that  $\pi_m(t_i) = \pi(t_i)$  for all  $i = 1, \dots, m$ . Since  $\mathcal{W}_-$  is compact, we can find a convergent subsequence, the limit of which must be the path  $\pi$ . This proves that  $\mathcal{W}_+ \subset \mathcal{W}_-$ . ■

### 3.6 Convergence to the Brownian web

**Proposition 3.17 (Tightness of rescaled arrow configurations)** *Let  $\mathcal{U}$  be the set of all paths in an arrow configurations and let  $\overline{\mathcal{U}}$  be its closure. Let  $\varepsilon_n > 0$  be positive constants such that  $\varepsilon_n \rightarrow 0$ . The the probability laws*

$$(\mathbb{P}[\theta_{\varepsilon_n}(\overline{\mathcal{U}}) \in \cdot])_{n \geq 1}$$

on  $\mathcal{K}(\Pi^\dagger)$  are tight.

**Proof (crude sketch)** One needs to check the tightness criterion of Proposition 2.29. This is very similar to the proof of Proposition 3.7. One uses convergence of finite dimensional distributions (Proposition 3.4) and then uses a grid as in the proof of Proposition 3.7 to estimate the event in Proposition 2.29. We refer to [FINR04, Prop. B2] and [SSS16, Prop. 6.6.4] for details. ■

Let  $\mathcal{D}, \hat{\mathcal{D}}$  be countable dense subsets of  $\mathbb{R}^2$ . By Proposition 3.14, we can construct a collection  $(\pi_z)_{z \in \mathcal{D}}$  of coalescing Brownian motions starting from  $\mathcal{D}$  and a collection  $(\hat{\pi}_z)_{z \in \hat{\mathcal{D}}}$  of downward coalescing Brownian motions starting from  $\hat{\mathcal{D}}$  such that paths in  $(\pi_z)_{z \in \mathcal{D}}$  do not enter wedges of  $(\hat{\pi}_z)_{z \in \hat{\mathcal{D}}}$  and vice versa. We call the pair  $(\mathcal{W}, \hat{\mathcal{W}})$  defined as

$$\mathcal{W} := \overline{\{\pi_z : z \in \mathcal{D}\}} \quad \text{and} \quad \hat{\mathcal{W}} := \overline{\{\hat{\pi}_z : z \in \hat{\mathcal{D}}\}} \quad (3.20)$$

the *double Brownian web*.

**Lemma 3.18 (Double Brownian web)** *The law of the random variable  $(\mathcal{W}, \hat{\mathcal{W}})$  does not depend on the choice of the countable dense sets  $\mathcal{D}, \hat{\mathcal{D}} \subset \mathbb{R}^2$ .*

**Proof** The analogue statement for the Brownian web has already been proved as part of the proof of Theorem 3.8, around (3.12). The statement for a single web does, as far as I can see, not automatically imply the statement for the double Brownian web, but one can adapt the argument given at (3.12). Here we give an alternative argument that also reproofs the statement for a single web and does not depend on the earlier argument.

Let  $\mathcal{D}, \mathcal{D}', \hat{\mathcal{D}}$  be countable dense subsets of  $\mathbb{R}^2$ . Let  $(\pi_z)_{z \in \mathcal{D}}$  be a collection of coalescing Brownian motions starting from  $\mathcal{D}$ , let  $(\pi'_z)_{z \in \mathcal{D}'}$  be a collection of coalescing Brownian motions starting from  $\mathcal{D}'$ , and let  $(\hat{\pi}_z)_{z \in \hat{\mathcal{D}}}$  be a collection of downward coalescing Brownian motions starting from  $\hat{\mathcal{D}}$ . By Proposition 3.14, we can couple  $(\pi_z)_{z \in \mathcal{D}}$  to  $(\hat{\pi}_z)_{z \in \hat{\mathcal{D}}}$  in such a way that paths in  $(\pi_z)_{z \in \mathcal{D}}$  do not enter wedges of  $(\hat{\pi}_z)_{z \in \hat{\mathcal{D}}}$  and vice versa. Similarly, we can couple  $(\pi'_z)_{z \in \mathcal{D}'}$  to  $(\hat{\pi}_z)_{z \in \hat{\mathcal{D}}}$  in such a way that paths in  $(\pi'_z)_{z \in \mathcal{D}'}$  do not enter wedges of  $(\hat{\pi}_z)_{z \in \hat{\mathcal{D}}}$  and vice versa. We can then couple all three collections  $(\pi_z)_{z \in \mathcal{D}}$ ,  $(\pi'_z)_{z \in \mathcal{D}'}$ , and  $(\hat{\pi}_z)_{z \in \hat{\mathcal{D}}}$  in such a way that the joint law of  $(\pi_z)_{z \in \mathcal{D}}$  and  $(\hat{\pi}_z)_{z \in \hat{\mathcal{D}}}$  is as before and the joint law of  $(\pi'_z)_{z \in \mathcal{D}'}$  and  $(\hat{\pi}_z)_{z \in \hat{\mathcal{D}}}$  is also as before. For example, this can be achieved by making  $(\pi_z)_{z \in \mathcal{D}}$  and  $(\pi'_z)_{z \in \mathcal{D}'}$  conditionally independent given  $(\hat{\pi}_z)_{z \in \hat{\mathcal{D}}}$ , and with the same conditional laws as before.

For this coupling, let  $(\mathcal{W}, \hat{\mathcal{W}})$  be defined using  $\mathcal{D}, \hat{\mathcal{D}}$  and let  $(\mathcal{W}', \hat{\mathcal{W}})$  be defined using  $\mathcal{D}', \hat{\mathcal{D}}$ . Then Theorem 3.16 tells us that

$$\mathcal{W} = \{ \pi \in \Pi^\dagger : \pi \text{ does not enter wedges of } (\hat{\pi}_z)_{z \in \hat{\mathcal{D}}} \} = \mathcal{W}' \quad \text{a.s.}$$

It follows that the joint law of  $(\mathcal{W}, \hat{\mathcal{W}})$  is the same as the joint law of  $(\mathcal{W}', \hat{\mathcal{W}})$ . In the same way, we can also replace  $\hat{\mathcal{D}}$  by another countable dense subset of  $\mathbb{R}^2$  without changing the law of the double Brownian web.  $\blacksquare$

The following theorem, which is the main result of this chapter, implies in particular the convergence in (3.5).

**Theorem 3.19 (Approximation of the double Brownian web)** *Let  $\mathcal{U}$  be the set of paths in an arrow configuration and let  $\mathcal{U}'$  be the set of downward paths in the associated dual arrow configuration. Then*

$$\mathbb{P}[\theta_\varepsilon(\bar{\mathcal{U}}, \bar{\mathcal{U}}') \in \cdot] \xrightarrow{\varepsilon \rightarrow 0} \mathbb{P}[(\mathcal{W}, \hat{\mathcal{W}}) \in \cdot], \quad (3.21)$$

where  $\Rightarrow$  denotes weak convergence of probability laws on the space  $\mathcal{K}(\Pi^\dagger) \times \mathcal{K}(\Pi^\perp)$ , and  $(\mathcal{W}, \hat{\mathcal{W}})$  is the double Brownian web.

**Proof** Fix countable dense sets  $\mathcal{D}, \hat{\mathcal{D}} \subset \mathbb{R}^2$  and define  $(\mathcal{W}, \hat{\mathcal{W}})$  as in (3.20). It suffices to prove convergence along any sequence  $\varepsilon_n$  of positive constants tending to zero. It follows from Proposition 3.17 (compare Lemma 3.15) that the laws

$$(\mathbb{P}[\theta_{\varepsilon_n}(\bar{\mathcal{U}}, \bar{\mathcal{U}}') \in \cdot])_{n \geq 1}$$

are tight, so by going to a subsequence, we may assume that they converge to some limit law  $\mathbb{P}[(\mathcal{V}, \hat{\mathcal{V}}) \in \cdot]$ . By Lemma 2.2, it suffices to show that each such subsequential limit is equal to  $\mathbb{P}[(\mathcal{W}, \hat{\mathcal{W}}) \in \cdot]$ .

As in the proof of Proposition 3.14, for each  $z \in \mathcal{D}$ , we choose  $z_n \in \mathbb{Z}_{\text{even}}^2$  such that  $\theta_{\varepsilon_n}(z_n) \rightarrow z$ , and for each  $z \in \hat{\mathcal{D}}$ , we choose  $z^n \in \mathbb{Z}_{\text{odd}}^2$  such that  $\theta_{\varepsilon_n}(z^n) \rightarrow z$ . For each  $z \in \mathcal{D}$  and  $n \geq 1$ , we let  $R_z^n \in \mathcal{U}$  be the unique forward path starting at  $z_n$ , we let  $\hat{R}_z^n \in \mathcal{U}'$  be the unique downward path starting at  $z^n$ , and we let

$$\pi_z^n := \theta_{\varepsilon_n}(R_z^n) \quad \text{and} \quad \hat{\pi}_z^n := \theta_{\varepsilon_n}(\hat{R}_z^n)$$

denote the associated diffusively rescaled paths. In the proof of Proposition 3.14, we have shown that

$$\begin{aligned} & \mathbb{P}[\left((\pi_z^n)_{z \in \mathcal{D}}, (\tau(\pi_{z_1}^n, \pi_{z_2}^n))_{(z_1, z_2) \in \mathcal{D}^2}\right) \in \cdot] \\ & \xrightarrow{n \rightarrow \infty} \mathbb{P}[\left((\pi_z)_{z \in \mathcal{D}}, (\tau(\pi_{z_1}, \pi_{z_2}))_{(z_1, z_2) \in \mathcal{D}^2}\right) \in \cdot], \end{aligned}$$

and similarly for the collection of downward paths. We argued there that going to a subsequence if necessary and using Skorohod's representation theorem, we can couple our random variables such that

$$\pi_z^n \xrightarrow[n \rightarrow \infty]{} \pi_z \text{ a.s.} \quad \text{and} \quad \tau(\pi_{z_1}^n, \pi_{z_2}^n) \xrightarrow[n \rightarrow \infty]{} \tau(\pi_{z_1}, \pi_{z_2}) \text{ a.s.}$$

for all  $z, z_1, z_2 \in \mathcal{D}$ , and likewise for downward paths. We can extend this argument to obtain that moreover

$$\theta_{\varepsilon_n}(\bar{\mathcal{U}}, \bar{\mathcal{U}}') \xrightarrow[n \rightarrow \infty]{} (\mathcal{V}, \hat{\mathcal{V}}) \quad \text{a.s.}$$

in the topology on  $\mathcal{K}(\Pi^\uparrow) \times \mathcal{K}(\Pi^\downarrow)$  for some random compact sets  $\mathcal{V} \subset \Pi^\uparrow$  and  $\hat{\mathcal{V}} \subset \Pi^\downarrow$ . We will show that for this particular coupling,  $(\mathcal{V}, \hat{\mathcal{V}}) = (\mathcal{W}, \hat{\mathcal{W}})$  a.s., where the latter is defined in terms of  $(\pi_z)_{z \in \mathcal{D}}$  and  $(\hat{\pi}_z)_{z \in \hat{\mathcal{D}}}$ . This shows that all subsequential limit laws are the same and hence by Lemma 2.2 that the original sequence converges.

By symmetry between forward and dual webs, it suffices to prove that  $\mathcal{V} = \mathcal{W}$ . We will prove that  $\mathcal{W}_- \subset \mathcal{V} \subset \mathcal{W}_+$ , where  $\mathcal{W}_-$  and  $\mathcal{W}_+$  are defined as in Theorem 3.16. Since  $\mathcal{W} = \mathcal{W}_- = \mathcal{W}_+$ , the claim then follows.

Since  $\mathcal{V}$  is closed, to prove that  $\mathcal{W}_- \subset \mathcal{V}$ , it suffices to prove that  $\pi_z \in \mathcal{V}$  for all  $z \in \mathcal{D}$ . Recalling Lemma 2.14, this is obvious since  $\pi_z^n \in \theta_{\varepsilon_n}(\mathcal{U})$  for all  $n$  while  $\pi_z^n \rightarrow \pi_z$  a.s. and  $\theta_{\varepsilon_n}(\mathcal{U}) \rightarrow \mathcal{V}$  a.s.

To prove that  $\mathcal{V} \subset \mathcal{W}_+$ , we need to show that paths  $\pi \in \mathcal{V}$  do not enter wedges of  $(\hat{\pi}_z)_{z \in \hat{\mathcal{D}}}$ . By Lemma 2.14, for each  $\pi \in \mathcal{V}$ , there exist  $\pi^n \in \theta_{\varepsilon_n}(\mathcal{U})$  such that  $\pi^n \rightarrow \pi$ . To see that  $\pi$  does not enter any wedge  $W(\hat{\pi}_{z_1}, \hat{\pi}_{z_2})$  of  $(\hat{\pi}_z)_{z \in \hat{\mathcal{D}}}$ , we use that for each  $n$ , the path  $\pi^n$  does not enter the wedge  $W(\hat{\pi}_{z_1}^n, \hat{\pi}_{z_2}^n)$ . By our assumptions, the discrete paths  $\hat{\pi}_{z_i}^n$  ( $i = 1, 2$ ) converge a.s. to  $\hat{\pi}_{z_i}$  ( $i = 1, 2$ ) and moreover their meeting times converge a.s., so we can use Lemma 3.13 to conclude that  $\pi$  does not enter  $W(\hat{\pi}_{z_1}, \hat{\pi}_{z_2})$ .  $\blacksquare$

### 3.7 The coalescing point set

Let  $\mathcal{W}$  be a Brownian web. For each closed set  $A \subset \mathbb{R}$ , we define a process  $(\xi_t^A)_{t \geq 0}$  by

$$\xi_t^A := \{\pi(t) : \pi \in \mathcal{W}, \sigma_\pi = 0, \pi(0) \in A\} \quad (t \geq 0).$$

If  $A$  is a finite set, then by Theorem 3.8, the Brownian web  $\mathcal{W}$  contains a unique path  $\pi_{(x,0)}$  for each  $x \in A$ , and these paths are distributed as coalescing Brownian motions. More generally, we can loosely interpret  $(\xi_t^A)_{t \geq 0}$  as a collection of coalescing Brownian motions started from every point in  $A$ . We will see in a moment that no matter what  $A$  is, for each  $t > 0$ , the set  $\xi_t^A$  is already locally finite. Since clearly,  $A \subset B$  implies  $\xi_t^A \subset \xi_t^B$ , it suffices to prove the statement for  $\xi_t^{\mathbb{R}}$ . Roughly speaking, the following result says that if we start particles performing coalescing Brownian motions from each point on the real line, then at each positive time there are only locally finitely many particles left.

**Proposition 3.20 (Density of the coalescing point set)** *One has*

$$\mathbb{E}[|\xi_t^{\mathbb{R}} \cap [a, b]|] = \frac{b-a}{\sqrt{\pi t}} \quad (a < b, t > 0).$$

**Proof** We first calculate the probability that  $\xi_t^{\mathbb{R}} \cap [a, b] \neq \emptyset$ . We construct  $(\mathcal{W}, \hat{\mathcal{W}})$  from collections  $(\pi_z)_{z \in \mathcal{D}}$  and  $(\hat{\pi}_z)_{z \in \hat{\mathcal{D}}}$  of forward and downward coalescing Brownian motions, so that paths in  $(\pi_z)_{z \in \mathcal{D}}$  do not enter wedges of  $(\hat{\pi}_z)_{z \in \hat{\mathcal{D}}}$  and vice versa. We choose  $\hat{\mathcal{D}}$  such that  $(a, t), (b, t) \in \hat{\mathcal{D}}$ . Let

$$\tau_{a,b} = \tau(\hat{\pi}_{(a,t)}, \hat{\pi}_{(b,t)})$$

be the first meeting time of the downward paths started at  $(a, t)$  and  $(b, t)$ . We claim that<sup>2</sup>

$$\xi_t^{\mathbb{R}} \cap (a, b) \neq \emptyset \quad \text{implies} \quad \tau_{a,b} \leq 0 \quad \text{implies} \quad \xi_t^{\mathbb{R}} \cap [a, b] \neq \emptyset.$$

Indeed, if  $\tau_{a,b} > 0$ , then the paths  $\hat{\pi}_{(a,t)}$  and  $\hat{\pi}_{(b,t)}$  form a wedge that prevents paths in  $\mathcal{W}$  starting at time zero from passing between  $(a, t)$  and  $(b, t)$ , proving the first implication. On the other hand, if  $\tau_{a,b} \leq 0$ , then for each time  $s > 0$  we can find some  $x$  such that  $\hat{\pi}_{(a,t)}(s) < x < \hat{\pi}_{(b,t)}(s)$ . The web  $\mathcal{W}$  must contain a path  $\pi$  starting at  $(x, s)$  and since such a path cannot cross the downward paths  $\hat{\pi}_{(a,t)}$  and  $\hat{\pi}_{(b,t)}$ , it must satisfy  $a \leq \pi(t) \leq b$ . We can construct such a path  $\pi^s$  with starting time  $s$  for each  $s > 0$ , so using the compactness of  $\mathcal{W}$ , we see that  $\mathcal{W}$  must also contain a path  $\pi^0$  starting at time zero such that  $a \leq \pi(t) \leq b$ , proving the second implication.

The difference  $(B_1(s) - B_2(s))_{s \geq 0}$  of two Brownian motions is equally distributed with  $(\sqrt{2}B(s))_{s \geq 0}$ , where  $(B(s))_{t \geq 0}$  is a single Brownian motion. Therefore, using the reflection principle,

$$\begin{aligned} \mathbb{P}[\tau_{a,b} \leq 0] &= \mathbb{P}\left[\sup_{0 \leq s \leq t} (B_2(s) - B_1(s)) \leq b - a\right] \\ &= \mathbb{P}\left[\sup_{0 \leq s \leq t} B(s) \leq \frac{b-a}{\sqrt{2}}\right] = \frac{1}{\sqrt{2\pi t}} \int_{-\frac{b-a}{\sqrt{2}}}^{\frac{b-a}{\sqrt{2}}} e^{-x^2/2t} dx. \end{aligned}$$

In particular, this implies that

$$\mathbb{P}[x \in \xi_t^{\mathbb{R}}] = \lim_{\varepsilon \rightarrow 0} \mathbb{P}[\xi_t^{\mathbb{R}} \cap (x - \varepsilon, x + \varepsilon) \neq \emptyset] = 0 \quad (x \in \mathbb{R}, t > 0),$$

and hence

$$\mathbb{P}[\xi_t^{\mathbb{R}} \cap (a, b) \neq \emptyset] = \mathbb{P}[\xi_t^{\mathbb{R}} \cap [a, b] \neq \emptyset] = \mathbb{P}[\tau_{a,b} \leq 0].$$

Now

$$\begin{aligned} \mathbb{E}[|\xi_t^{\mathbb{R}} \cap [0, 1]|] &= \lim_{n \rightarrow \infty} \sum_{i=1}^{2^n} \mathbb{P}[\xi_t^{\mathbb{R}} \cap [(i-1)2^{-n}, i2^{-n}] \neq \emptyset] \\ &= \lim_{\varepsilon \rightarrow 0} \varepsilon^{-1} \frac{1}{\sqrt{2\pi t}} \int_{-\varepsilon/\sqrt{2}}^{\varepsilon/\sqrt{2}} e^{-x^2/2t} dx = \frac{1}{\sqrt{\pi t}}. \end{aligned}$$

A similar formula holds for the expectation of  $|\xi_t^{\mathbb{R}} \cap [0, r]|$  for any  $r > 0$  and the general result follows by translation invariance.  $\blacksquare$

We conclude this section with some useful consequences of Proposition 3.20. In the following lemma, we let  $\Pi^\dagger := \Pi^\uparrow \cap \Pi^\downarrow$  denote the space of all

<sup>2</sup>In fact, with a bit extra work, one can show that  $\tau_{a,b} \leq 0$  is equivalent to  $\xi_t^{\mathbb{R}} \cap [a, b] \neq \emptyset$ . In the context of the Brownian net, we will prove a similar statement in Corollary 4.6 below.



bi-infinite paths and we let  $\Pi^\dagger(\mathbb{R}) := \Pi^\dagger \cap \Pi(\mathbb{R})$  denote the space bi-infinite paths with values in  $\mathbb{R}$  (as opposed to  $\overline{\mathbb{R}}$ ).

**Lemma 3.21 (No bi-infinite paths)** *Let  $\mathcal{W}$  be a Brownian web. Then  $\mathcal{W} \cap \Pi^\dagger(\mathbb{R}) = \emptyset$  a.s.*

**Proof** We start by observing that

$$\mathbb{P}[\mathcal{W} \cap \Pi^\dagger(\mathbb{R}) \neq \emptyset] \leq \lim_{n \rightarrow \infty} \mathbb{P}[\exists \pi \in \mathcal{W} \text{ s.t. } \sigma_\pi = -\infty, \pi(0) \in [-n, n]],$$

where by Lemma 3.9 the inequality is in fact an equality. Now Proposition 3.20 gives

$$\begin{aligned} & \mathbb{P}[\exists \pi \in \mathcal{W} \text{ s.t. } \sigma_\pi = -\infty, \pi(0) \in [-n, n]] \\ & \leq \lim_{t \rightarrow \infty} \mathbb{P}[\exists \pi \in \mathcal{W} \text{ s.t. } \sigma_\pi \leq -t, \pi(0) \in [-n, n]] = \lim_{t \rightarrow \infty} \frac{2n}{\sqrt{\pi t}} = 0, \end{aligned}$$

Here again, with a bit of extra work, one can show that the inequality is in fact an equality, but we do not presently need this.  $\blacksquare$

**Lemma 3.22 (Coalescence of paths)** *Almost surely, for all paths  $\pi, \pi' \in \mathcal{W}$ , if  $\pi(t) = \pi'(t)$  for some  $t > \sigma_\pi \vee \sigma_{\pi'}$ , then  $\pi(u) = \pi'(u)$  for all  $u \geq t$ .*

**Proof** By Lemma 3.9, it suffices to prove the statement under the additional assumption that  $\pi(t) = \pi'(t) \in \mathbb{R}$ . Let  $\mathcal{T} \subset \mathbb{R}$  be countable and dense. If  $t > \sigma_\pi \vee \sigma_{\pi'}$ , then there exist  $r, s \in \mathcal{T}$  with  $\sigma_\pi \vee \sigma_{\pi'} < r < s \leq t$  and the paths obtained from  $\pi$  and  $\pi'$  by cutting off the piece before time  $r$  are also paths in the Brownian web. Therefore, it suffices to prove for deterministic  $r < s$  that if two paths  $\pi, \pi' \in \mathcal{W}$  with  $\sigma_\pi = \sigma_{\pi'} = r$  satisfy  $\pi(t) = \pi'(t)$  for some  $t \geq s$ , then  $\pi(u) = \pi'(u)$  for all  $u \geq t$ .

By Proposition 3.20, the random set

$$A := \{\pi(s) : \pi \in \mathcal{W}, \sigma_\pi = r\} \cap \mathbb{R}$$

is locally finite. We claim that for each  $x \in A$ , there exists a unique path  $\pi_{(x,s)} \in \mathcal{W}(x, s)$ , and conditional on  $A$ , the collection of paths

$$(\pi_{(x,s)})_{x \in A}$$

is distributed as coalescing Brownian motions. Indeed, this follows from the fact (which can easily be proved using discrete approximation) that restrictions of the Brownian web to disjoint parts of space-time are independent. As a result, the random set  $A$  is independent of  $\mathcal{W}(\mathbb{R} \times [s, \infty))$ , so after

we condition on  $A$ , paths started from a countable collection of fixed points  $(x, s)$  with  $x \in \mathbb{R}$  will be distributed as coalescing Brownian motions.

The statement we want to prove now follows from the fact that if two coalescing Brownian motions meet at some random time, then they coalesce, i.e., the two paths are equal from that time onwards. ■

**Lemma 3.23 (Strong convergence of paths)** *Let  $\mathcal{W}$  be a Brownian web. Then almost surely, for all  $\pi_n, \pi \in \mathcal{W}$  such that  $\pi_n \rightarrow \pi$ , there exist times  $t_n > \sigma_{\pi_n} \vee \sigma_\pi$  such that  $t_n \rightarrow \sigma_\pi$  and  $\pi_n(t) = \pi(t)$  for all  $t \geq t_n$ .*

**Proof** By Lemma 3.9, it suffices to prove the statement under the additional assumption that  $\pi(t) \in \mathbb{R}$ . Proposition 3.20 tells us that for each deterministic  $s < t$ , the set

$$A_{s,t} := \{\pi(t) : \pi \in \mathcal{W}, \sigma_\pi \leq s\} \cap \mathbb{R}$$

is a.s. a locally finite subset of  $\mathbb{R}$ . Let  $\mathcal{T}$  be a countable dense subset of  $\mathbb{R}$ . Then almost surely,  $A_{s,t}$  is locally finite for all  $s, t \in \mathcal{T}$  with  $s < t$ . Now if  $\pi_n, \pi \in \mathcal{W}$  satisfy  $\pi_n \rightarrow \pi$ , then for each  $s, t \in \mathcal{T}$  with  $\sigma_\pi < s < t$ , we have for  $n$  sufficiently large that  $\sigma_{\pi_n} < s$  and hence  $\pi_n(t), \pi(t) \in A_{s,t}$ . Since  $\pi_n(t) \rightarrow \pi(t)$  and since  $A_{s,t}$  is locally finite, it follows that  $\pi_n(t) = \pi(t)$  for  $n$  sufficiently large. By Lemma 3.22,  $\pi_n(t) = \pi(t)$  implies  $\pi_n(u) = \pi(u)$  for all  $u \geq t$ . Since  $\mathcal{T}$  is dense, we can choose  $t$  as close to  $\sigma_\pi$  as we wish, and hence the statement of the lemma follows. ■

**Lemma 3.24 (Paths do not enter wedges)** *Let  $(\mathcal{W}, \hat{\mathcal{W}})$  be a Brownian web and its dual. Then almost surely, for each  $\pi \in \mathcal{W}$  and  $\hat{\pi}_1, \hat{\pi}_2 \in \hat{\mathcal{W}}(\mathbb{R}^2)$ , the path  $\pi$  does not enter the wedge  $W(\hat{\pi}_1, \hat{\pi}_2)$ .*

**Proof (sketch)** Let  $\hat{\mathcal{D}} \subset \mathbb{R}^2$  be countable and dense. Then paths  $\pi \in \mathcal{W}$  do not enter wedges  $W(\hat{\pi}_1, \hat{\pi}_2)$  with  $\hat{\pi}_1, \hat{\pi}_2 \in \hat{\mathcal{W}}(\hat{\mathcal{D}})$  by Theorem 3.16. Applying Lemma 3.23 to the dual web  $\hat{\mathcal{W}}$ , it is easy to extend the statement to paths  $\hat{\pi}_1, \hat{\pi}_2 \in \hat{\mathcal{W}}(\mathbb{R}^2)$ . ■

### 3.8 Special points

We have defined the Brownian web  $\mathcal{W}$  as the closure of  $\{\pi_z : z \in \mathcal{D}\}$ , where  $(\pi_z)_{z \in \mathcal{D}}$  is a collection of coalescing Brownian motions started from a countable dense set  $\mathcal{D} \subset \mathbb{R}^2$ . Here  $\{\pi_z : z \in \mathcal{D}\}$  is precompact by Proposition 3.7 and hence  $\mathcal{W}$  is a compact subset of  $\Pi^\uparrow$ . Using compactness and the fact

that  $\mathcal{D}$  is dense, we see that for each  $z \in \mathbb{R}^2$ , there exists at least one path  $\pi \in \mathcal{W}$  that starts at  $z$ . For each  $z \in \mathbb{R}^2$ , we let

$$m_{\text{out}}(z) := |\mathcal{W}(z)|$$

denote the number of paths in  $\mathcal{W}$  that start at  $z$ . In Theorem 3.8, we have proved that  $m_{\text{out}}(z) = 1$  a.s. for each deterministic  $z \in \mathbb{R}^2$ . In this section, we will prove that in spite of this, almost surely, there exist points  $z$  with  $m_{\text{out}}(z) = 2$  and even  $m_{\text{out}}(z) = 3$ . The key to understanding this is (again) duality.

We say that a path  $\pi \in \mathcal{W}$  *enters* a point  $z = (x, t) \in \mathbb{R}^2$  if  $\sigma_\pi < t$  and  $\pi(t) = x$ . We call two paths  $\pi, \pi'$  entering  $z$  *equivalent* if there exists a  $\sigma_\pi \vee \sigma_{\pi'} \leq s < t$  such that  $\pi(r) = \pi'(r)$  for all  $s \leq r \leq t$ . This obviously defines an equivalence relation on the set of all paths  $\pi \in \mathcal{W}$  entering  $z$ . We let  $m_{\text{in}}(z)$  denote the number of equivalence classes of paths in  $\mathcal{W}$  entering  $z$ . We call  $(m_{\text{in}}(z), m_{\text{out}}(z))$  the *type* of a point  $z \in \mathbb{R}^2$ .

**Theorem 3.25 (Special points of the Brownian web)** *Let  $\mathcal{W}$  be a Brownian web. Then almost surely, all points in  $\mathbb{R}^2$  are of one of the following types:*

$$(0, 1), \quad (0, 2), \quad (0, 3), \quad (1, 1), \quad (1, 2), \quad (2, 1),$$

*and all these types occur. For each deterministic  $t \in \mathbb{R}$ , almost surely, all points in  $\mathbb{R} \times \{t\}$  are of one of the following types:*

$$(0, 1), \quad (0, 2), \quad (1, 1),$$

*and all these types occur. A deterministic point  $(x, t) \in \mathbb{R}^2$  is almost surely of type  $(0, 1)$ .*

The proof of Theorem 3.25 is based on the following lemma, which is of independent interest.

**Lemma 3.26 (Types of points in dual web)** *Let  $(\hat{m}_{\text{in}}(z), \hat{m}_{\text{out}}(z))$  denote the type of a point  $z \in \mathbb{R}^2$  in the dual Brownian web  $\hat{\mathcal{W}}$ . Then for each  $z \in \mathbb{R}^2$ ,*

$$m_{\text{out}}(z) = \hat{m}_{\text{in}}(z) + 1 \quad \text{and} \quad \hat{m}_{\text{out}}(z) = m_{\text{in}}(z) + 1.$$

**Proof (crude sketch)** By symmetry, it suffices to prove that  $m_{\text{out}}(z) = \hat{m}_{\text{in}}(z) + 1$ . If there is an incoming path in  $\hat{\mathcal{W}}$  at  $z$ , then forward paths started on either side of such a dual path cannot coalesce until the starting time of the dual path, since otherwise the dual path would enter the wedge defined by these forward paths. As a result, since the incoming paths divide the area just above  $z$  into  $\hat{m}_{\text{in}}(z) + 1$  regions, approaching the point  $z$  from

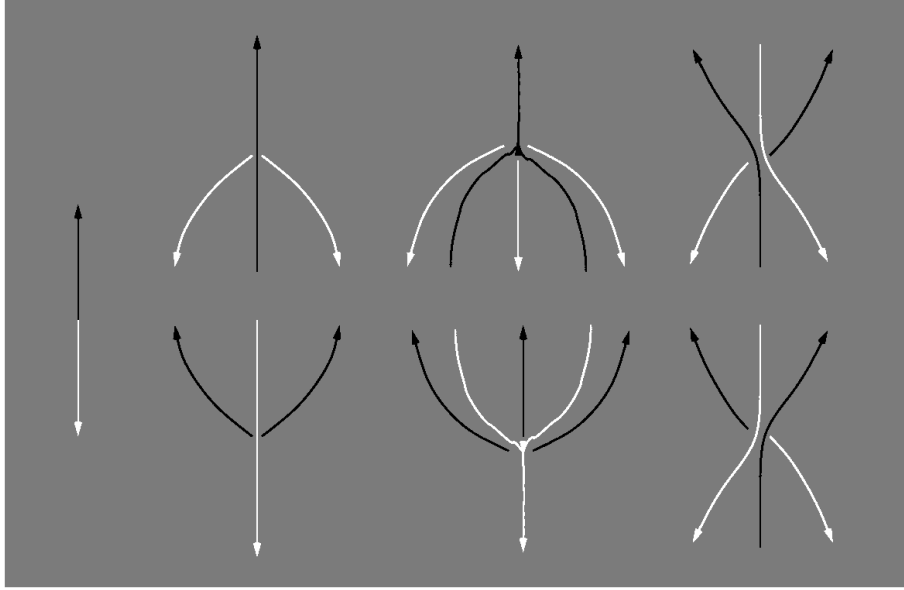


Figure 3.5: Possible types of points in the Brownian web and its dual.

different directions, using the compactness of  $\mathcal{W}$ , we see that there are at least  $\hat{m}_{\text{in}}(z) + 1$  distinct paths in  $\mathcal{W}$  starting at  $z$ . On the other hand, if there are two outgoing paths in  $\mathcal{W}$  at  $z$ , then any dual path that is started between these paths must stay between these forward paths and pass through  $z$ . Therefore,  $\hat{m}_{\text{in}} \geq m_{\text{out}} - 1$ . Together with our earlier claim that  $m_{\text{out}}(z) \geq \hat{m}_{\text{in}}(z) + 1$ , this proves the claim. ■

**Proof of Theorem 3.25 (crude sketch)** It is clear that there exist points  $z$  with  $\hat{m}_{\text{in}}(z) = 1$  and  $\hat{m}_{\text{in}}(z) = 2$ . On the other hand, using the fact mentioned at the end of the proof of Lemma 3.26 that for each  $\pi \in \mathcal{W}$  and  $t > \sigma_\pi$ , there exists a  $z = (x, s) \in \mathcal{D}$  with  $s < t$  such that  $\pi(u) = \pi_z(u)$  for all  $u \geq t$ , it is easy to see that a deterministic point  $z$  almost surely has  $\hat{m}_{\text{in}}(z) = 0$ . Using the same fact, one moreover obtains that there are only countably many points  $z$  with  $\hat{m}_{\text{in}}(z) = 2$  and it is not too hard to show that these points have  $m_{\text{in}}(z) = 0$ .

To see that there exist points with  $m_{\text{in}}(z) = 1 = \hat{m}_{\text{in}}(z)$ , we observe that in an arrow configuration, disjoint parts of space-time are independent. This property carries over to the limit which has the consequence that dual paths do not “see” forward paths until they hit them. In fact, it is known that dual paths are reflected off forward paths by Skorohod reflection. At deterministic times, however, we do not see such points since two Brownian motions started in the forward and downward directions have zero probability

to be at a deterministic time at the same position.

These arguments show that all the mentioned types of points exist, and no other types of points can exist. ■

### 3.9 Some historical notes

The Brownian web originated from Arratia's PhD thesis [Arr79] and a subsequent unfinished manuscript [Arr81]. The topic remained dormant until the work of Tóth and Werner [TW98] who used the Brownian web to study a form of one-dimensional self-repellent random walk. They classified all types of special points. Together with Soucaliuc [STW00] they also proved that forward and dual paths interact through Skorohod reflection. Fontes, Isopi, Newman and Stein got interested in the Brownian web motivated by a one-dimensional model in mathematical physics [FINS01], which led Fontes, Isopi, Newman and Ravishankar [FINR04] to study this object in more detail. In particular, they were the first to give the Brownian web its name, view it as a compact set of paths, and prove convergence with respect to the Hausdorff topology. Wedges were first introduced in the framework of the Brownian net in [SS08]. A more detailed account of the history of the Brownian web can be found in [SSS16].



# Chapter 4

## The Brownian net

### 4.1 Adding branching and deaths

As in Chapter 3, we let  $\mathbb{Z}_{\text{even}}^2$  and  $\mathbb{Z}_{\text{odd}}^2$  denote the even and odd sublattices of  $\mathbb{Z}^2$ . Generalising the set-up of Chapter 3, let  $\omega = (\omega_z)_{z \in \mathbb{Z}_{\text{even}}^2}$  be an i.i.d. collection of random variables that take values in the subsets of  $\{-1, +1\}$ . We can use  $\omega$  to define a random directed graph with vertex set  $\mathbb{Z}_{\text{even}}^2$  and set of oriented edges

$$\vec{E} := \{((x, t), (x + y, t + 1)) : (x, t) \in \mathbb{Z}_{\text{even}}^2, y \in \omega_{(x, t)}\}.$$

We call the random directed graph  $(\mathbb{Z}_{\text{even}}^2, \vec{E})$  an *arrow configuration*. In particular, when  $\omega_z$  takes the values  $\{-1\}$  and  $\{+1\}$  with equal probabilities, this is an arrow configuration as defined in Section 3.1. In the present chapter, we look at sequences  $\omega^n$  of arrow configurations where  $\omega^n = (\omega_z^n)_{z \in \mathbb{Z}_{\text{even}}^2}$ , for each  $n \geq 1$ , is an i.i.d. collection with common law

$$\begin{aligned} \mathbb{P}[\omega_z^n = \{-1\}] &= l_n, & \mathbb{P}[\omega_z^n = \{+1\}] &= r_n, \\ \mathbb{P}[\omega_z^n = \{-1, +1\}] &= b_n, & \mathbb{P}[\omega_z^n = \emptyset] &= d_n. \end{aligned} \tag{4.1}$$

Here  $l_n$  is the probability that at a given point  $z \in \mathbb{Z}_{\text{even}}^2$ , there starts (only) an arrow to the left,  $r_n$  is the probability of an arrow to the right,  $b_n$  is the branching probability, i.e., the probability that both arrows are present, and  $d_n$  is the death probability, i.e., the probability that no arrows are present.

Recall that  $\sigma_\pi$  and  $\tau_\pi$  denote the starting time and final time of a path  $\pi \in \Pi(\overline{\mathbb{R}})$ . Generalising our definition from Section 3.1, we say that  $\pi$  is a *path in the arrow configuration*  $\omega^n$  if  $\pi \in \Pi(\overline{\mathbb{R}})$  has following properties:

- (i)  $(\pi(t), t) \in \mathbb{Z}_{\text{even}}^2$  ( $t \in \mathbb{Z}$ ,  $t \geq \sigma_\pi$ ),

$$(ii) \quad \pi(t+1) - \pi(t) \in \omega_{(\pi(t), t)} \quad (t \in \mathbb{Z}, t \geq \sigma_\pi),$$

$$(iii) \quad \pi(t+s) = (1-s)\pi(t) + s\pi(t+1) \quad (0 \leq s \leq 1, t \in \mathbb{Z}, t \geq \sigma_\pi).$$

We let  $\mathcal{V}_n$  denote the set of all paths in  $\omega^n$ . Note that even in the special case when  $l_n = r_n = \frac{1}{2}$  and  $b_n = d_n = 0$ , this is not quite the same object as the set  $\mathcal{U}$  defined in Section 3.1, since we allow paths to end at some final time  $\tau_\pi < \infty$ . We let  $\overline{\mathcal{V}}_n$  denote the closure of  $\mathcal{V}_n$  in  $\Pi(\overline{\mathbb{R}})$ . In this chapter, we will sketch a proof of the following theorem. Recall that  $\theta_\varepsilon$  denotes the diffusive scaling map defined in (3.2).

**Theorem 4.1 (The Brownian net with killing)** *Let  $\varepsilon_n$  be positive constants tending to zero and let  $\alpha \in \mathbb{R}$  and  $\beta, \delta \in [0, \infty)$ . Let  $\omega_n$  be arrow configurations with probabilities  $l_n, r_n, b_n, d_n$  satisfying*

$$\varepsilon_n^{-1}(r_n - l_n) \xrightarrow[n \rightarrow \infty]{} \alpha, \quad \varepsilon_n^{-1}b_n \xrightarrow[n \rightarrow \infty]{} \beta, \quad \text{and} \quad \varepsilon_n^{-2}d_n \xrightarrow[n \rightarrow \infty]{} \delta.$$

*Let  $\mathcal{V}_n$  be the set of paths in the arrow configuration  $\omega_n$ . Then*

$$\mathbb{P}[\theta_{\varepsilon_n}(\overline{\mathcal{V}}_n) \in \cdot] \xRightarrow[n \rightarrow \infty]{} \mathbb{P}[\mathcal{N} \in \cdot], \quad (4.2)$$

*where  $\Rightarrow$  denotes weak convergence of probability laws on the space  $\mathcal{K}(\Pi(\overline{\mathbb{R}}))$  of compact sets of paths, equipped with the Hausdorff topology, and  $\mathcal{N}$  is a random compact subset of  $\Pi(\overline{\mathbb{R}})$ , whose law only depends on the parameters  $\alpha, \beta, \delta$ .*

For most of the chapter, we will be concerned with the case that  $d_n = 0$  for all  $n$ , and hence also  $\delta = 0$ . This will allow us to work with the space  $\Pi^\uparrow$  of upward paths as we are used to from Chapter 4. In Section 4.6, we will briefly indicate how the arguments can be generalised to allow for a positive death probability. For simplicity, in what follows, we will moreover focus on the case that  $\alpha = 0$  and  $\beta = 1$ . In this case, the limiting object in (4.2) is known as the *standard Brownian net*.

## 4.2 Left and right paths

We consider a sequence  $\omega^n$  of arrow configurations as in the previous section with

$$d_n = 0, \quad \varepsilon_n^{-1}(r_n - l_n) \xrightarrow[n \rightarrow \infty]{} 0 \quad \text{and} \quad \varepsilon_n^{-1}b_n \xrightarrow[n \rightarrow \infty]{} 1. \quad (4.3)$$

We define  $\mathcal{V}_n$  as in the previous section and set  $\mathcal{U}_n := \mathcal{V}_n \cap \Pi^\uparrow$ . Since the death probability is zero,  $\mathcal{V}_n$  can simply be recovered from  $\mathcal{U}_n$  by adding



all shortened paths, that are cut off at an arbitrary time in  $\mathbb{Z}$ . Thus, all information is contained in the set  $\mathcal{U}_n$  and we can continue to work with the space  $\Pi^\uparrow$  that we are used to from the previous chapter.

By definition, a *left path* in  $\omega^n$  is a path  $\pi \in \mathcal{U}_n$  that satisfies

$$\pi(t+1) = \pi(t) - 1 \quad \text{if } \omega_{(\pi(t),t)}^n = \{-1, +1\},$$

i.e., left paths always turn left at branching points. Similarly, we define right paths as paths in  $\mathcal{U}_n$  that always turn right at branching points. We let  $\mathcal{U}_n^l$  and  $\mathcal{U}_n^r$  denote the collections of all left and right paths in  $\mathcal{U}_n$ , respectively. We claim that

$$\begin{aligned} \mathbb{P}[\theta_{\varepsilon_n}(\bar{\mathcal{U}}_n^l) \in \cdot] &\xrightarrow{n \rightarrow \infty} \mathbb{P}[\mathcal{W}^l \in \cdot], \\ \mathbb{P}[\theta_{\varepsilon_n}(\bar{\mathcal{U}}_n^r) \in \cdot] &\xrightarrow{n \rightarrow \infty} \mathbb{P}[\mathcal{W}^r \in \cdot], \end{aligned}$$

where  $\mathcal{W}^l$  and  $\mathcal{W}^r$  are Brownian webs with drift  $-1$  and  $+1$ , respectively, i.e., these are objects that are defined exactly in the same way as the Brownian web, except that the coalescing standard Brownian motions are replaced by coalescing Brownian motions with drift  $-1$  and  $+1$ , respectively.

Indeed, letting  $L_z^n$  and  $R_z^n$  denote the unique left and right paths in  $\mathcal{U}_n$  starting from a point  $z \in \mathbb{Z}_{\text{even}}^2$ , we observe that

$$\begin{aligned} \mathbb{E}[L_z^n(t+1) - L_z^n(t)] &= r_n - l_n - b_n \sim -\varepsilon_n, \\ \mathbb{E}[R_z^n(t+1) - R_z^n(t)] &= r_n - l_n + b_n \sim +\varepsilon_n \end{aligned}$$

as  $n \rightarrow \infty$ , which is easily seen to imply that  $L_z^n$  and  $R_z^n$  converge after diffusive rescaling to Brownian motions with drift  $-1$  and  $+1$ , respectively.

A more tricky question is how to describe the scaling limit of the joint law of  $L_z^n$  and  $R_z^n$ . We will use the following proposition to describe the interaction between a single left and right path.

**Proposition 4.2 (Left-right SDE)** *Let  $\mathbb{R}_{\leq}^2 := \{(l, r) \in \mathbb{R}^2 : l \leq r\}$ . Then for each initial state  $(L_0, R_0) \in \mathbb{R}_{\leq}^2$ , there exists an  $\mathbb{R}_{\leq}^2$ -valued weak solution  $(L_t, R_t)_{t \geq 0}$  to the stochastic differential equation (SDE)*

$$\begin{aligned} dL_t &= 1_{\{L_t \neq R_t\}} dB_t^l + 1_{\{L_t = R_t\}} dB_t^s - dt, \\ dR_t &= 1_{\{L_t \neq R_t\}} dB_t^r + 1_{\{L_t = R_t\}} dB_t^s + dt, \end{aligned} \tag{4.4}$$

where  $B^l, B^r, B^s$  are independent Brownian motions. Moreover,  $\mathbb{R}_{\leq}^2$ -valued solutions to (4.4) are unique in law.

In words, the SDE (4.4) says that  $L_t$  and  $R_t$  are Brownian motions with drift  $-1$  and  $+1$ , that evolve independently when they are apart but are driven by the same Brownian motion when they are on the same position.

**Proof of Proposition 4.2 (crude sketch)** This has been proved in [SS08, Lemma 2.2]. It has been conjectured that solutions to (4.4) are not pathwise unique. Instead, existence and weak uniqueness of solutions to (4.4) are proved in [SS08] via a random time change. Let  $\tilde{B}^1, \tilde{B}^r, \tilde{B}^s$  be independent Brownian motions, and consider the equations:

$$\begin{aligned} \text{(i)} \quad dL_t &= d\tilde{B}_{T_t}^1 + d\tilde{B}_{S_t}^s - dt, \\ \text{(ii)} \quad dR_t &= d\tilde{B}_{T_t}^r + d\tilde{B}_{S_t}^s + dt, \\ \text{(iii)} \quad T_t + S_t &= t \\ \text{(iv)} \quad \int_0^t 1_{\{L_s < R_s\}} dS_s &= 0, \end{aligned}$$

where  $S, T$  are nonnegative, nondecreasing functions that are adapted to the filtration generated by the Brownian motions  $\tilde{B}^1, \tilde{B}^r, \tilde{B}^s$ . Condition (iv) says that  $S_t$  increases only at times  $t$  when  $L_t = R_t$ . In fact, it is shown in the proof of [SS08, Lemma 2.2] that the conditions (i)–(iv) imply that

$$\begin{aligned} \text{(iii)'} \quad T_t &= \int_0^t 1_{\{L_s < R_s\}} ds \\ \text{(iv)'} \quad S_t &= \int_0^t 1_{\{L_s = R_s\}} ds. \end{aligned}$$

Condition (i) can now informally be interpreted as follows: during an infinitesimal time interval  $[t, t + dt]$ , we can imagine that either  $L_s < R_s$  for all  $s \in [t, t + dt]$  or  $L_s = R_s$  for all  $s \in [t, t + dt]$ . In the first case,  $L_{t+dt} - L_t = \tilde{B}_{T_t+dt}^1 - \tilde{B}_{T_t}^1 - dt$ , while in the second case  $L_{t+dt} - L_t = \tilde{B}_{S_t+dt}^s - \tilde{B}_{S_t}^s - dt$ . This is similar to (4.4), except that we “stop” the Brownian motions  $\tilde{B}^1$  and  $\tilde{B}^s$  at times when we do not need them to “steer” the process  $L$ , and start reading them off again at the time when we last stopped them as soon as we need them again.

Using standard techniques, the equation (i)–(iv) can be transformed into a SDE with immediate Skorohod reflection, which has an almost surely unique pathwise solution. In these arguments, the condition that  $L_t \leq R_t$  for all  $t \geq 0$  is essential; without this condition, uniqueness would not hold. It then follows that  $\mathbb{R}_{\leq}^2$ -valued solutions to (i)–(iv) are also pathwise unique. Again by standard techniques, this yields existence and distributional uniqueness of weak solutions to (4.4). For the details, we refer to [SS08, Lemma 2.2]. ■

As a side result of the proof of [SS08, Lemma 2.2], one obtains that  $L$  and  $R$  spend positive Lebesgue time together each time they meet. More precisely, if  $I := \{t \geq 0 : L_t = R_t\}$ , and  $\mu$  is the measure on  $[0, \infty)$  whose

density with respect to the Lebesgue measure is the indicator function  $1_I$ , then  $I$  is the support of  $\mu$ ; see [SS08, Prop. 3.1(b)]. One can moreover show that the set  $I$  is nowhere dense, i.e., each open time interval contains smaller open intervals on which  $L_t < R_t$ . We say that  $L$  and  $R$  interact via a form of *sticky interaction*.

There is nothing special about time zero, so generalising slightly, we can use Proposition 4.2 to describe the joint law of a single left path  $(L_t)_{t \geq s}$  and right path  $(R_t)_{t \geq s}$ , started at the same time  $s \in \mathbb{R}$  from an initial state such that  $L_s \leq R_s$ . We need to generalise this to multiple paths, started at different times. The different starting times are easy to deal with: we simply run the process until the next time one or more new left or right paths are started. At this time, we start a new process, conditionally independent given the old one, with left or right paths started at the positions of the old ones and the new ones. In view of this, our main task is to describe the joint law of multiple left and right paths, all started at the same time.

To describe the joint law of multiple paths, we use the following rules:

- (i) All paths evolve independently when they are in disjoint parts of space.
- (ii) As soon as a two left paths meet, they coalesce, and similarly for right paths.
- (iii) After the first time a left and a right path meet, the left path stays on the left of the right path and their interaction is described by the left-right SDE (4.4).

Let us say that a time is a *coalescence time* if for a given pair of two left or two right paths, this is the first time they meet. Let us further say that a time is a *crossing time* if for a given pair consisting of a left and a right path, this is the first time they meet, while before this time, the left path is on the right of the right path. In view of rules (ii) and (iii), if we start with finitely many left and right paths, then there can be only finitely many coalescence times and crossing times. By a restart argument, it then suffices to describe the joint law of multiple left and right paths up to the first coalescence or crossing time.

We claim that by rule (i), it now suffices to describe only the joint law of a single left and right path. Imagine, for example, that initially we have a collection of left and right paths that are ordered from left to right as  $LRLRLRRRLR$ . We can then group left and right paths that immediately follow after each other (in this order), leaving the remaining paths as singletons, as follows:

$$\{LR\}\{L\}\{LR\}\{LR\}\{R\}\{R\}\{LR\}.$$

Here paths in each group stay away from each other until the next coalescence or crossing time, and hence evolve independently up to that moment. In this way, we define a collection of *left-right coalescing Brownian motions*, starting from space-time points  $(z_1, \dots, z_n, z'_1, \dots, z'_m)$ , where  $(z_1, \dots, z_n)$  are the starting points of left paths and  $(z'_1, \dots, z'_m)$  are the starting points of right paths. We cite the following result from [SS08, Prop. 5.2].

**Proposition 4.3 (Convergence of finite dimensional distributions)**

Let  $\varepsilon_k > 0$  satisfy  $\varepsilon_k \rightarrow 0$ . Fix  $n, m \geq 1$  and for each  $k$ , let  $z_1^k, \dots, z_n^k \in \mathbb{Z}_{\text{even}}^2$  and  $z'_1{}^k, \dots, z'_m{}^k \in \mathbb{Z}_{\text{even}}^2$ . Assume that

$$\theta_{\varepsilon_k}(z_1^k, \dots, z_n^k, z'_1{}^k, \dots, z'_m{}^k) \xrightarrow[k \rightarrow \infty]{} (z_1, \dots, z_n, z'_1, \dots, z'_m) \in (\mathbb{R}^2)^{n+m}.$$

Let  $\omega_k$  be arrow configurations with probabilities  $l_k, r_k, b_k, d_k$  as in 94.1) satisfying (4.3). Let  $\pi_1^k, \dots, \pi_n^k$  denote the unique left paths in  $\omega_k$  starting from  $z_1^k, \dots, z_n^k$  and let  $\pi'_1{}^k, \dots, \pi'_m{}^k$  denote the unique right paths in  $\omega_k$  starting from  $z'_1{}^k, \dots, z'_m{}^k$ . Then

$$\mathbb{P}[\theta_{\varepsilon_k}(\pi_1^k, \dots, \pi_n^k, \pi'_1{}^k, \dots, \pi'_m{}^k) \in \cdot] \xrightarrow[k \rightarrow \infty]{} \mathbb{P}[(\pi_1, \dots, \pi_n, \pi'_1, \dots, \pi'_m) \in \cdot],$$

where  $\Rightarrow$  denotes weak convergence of probability measures on  $(\Pi^\uparrow)^{n+m}$ , and  $\pi_1, \dots, \pi_n, \pi'_1, \dots, \pi'_m$  is a collection of left-right coalescing Brownian motions starting from  $z_1, \dots, z_n, z'_1, \dots, z'_m$ .

**Proof (crude idea)** The proof, which can be found in [SS08], is based on the time-changed reformulation of the left-right SDE, i.e., formulas (i)–(iv) from the proof of Proposition 4.2. One can give similar formulas for a discrete left and right path, where now the driving processes are random walks instead of Brownian motions. One then uses the fact that rescaled random walks converge to Brownian motions, shows that any subsequential limit satisfies (i)–(iv) with respect to these Brownian motions, and finally uses that (i)–(iv) have a pathwise unique solution. ■

### 4.3 The left-right Brownian web

It is clear that left and right random paths in an arrow configuration are consistent in the sense of Kolmogorov's extension theorem, and hence by Proposition 4.3 the same must be true for left-right coalescing Brownian motions. In view of this, we can construct a collection  $(\pi_z, \pi'_{z'})_{(z, z') \in \mathcal{D} \times \mathcal{D}'}$  of

left and right coalescing Brownian motions started from two countable dense sets  $\mathcal{D}, \mathcal{D}' \subset \mathbb{R}^2$ . We set

$$\mathcal{W}^l := \overline{\{\pi_z : z \in \mathcal{D}\}} \quad \text{and} \quad \mathcal{W}^r := \overline{\{\pi'_{z'} : z' \in \mathcal{D}'\}}.$$

Since  $(\pi_z)_{z \in \mathcal{D}}$  is a collection of coalescing drifted Brownian motions, Theorem 3.8 implies that  $\mathcal{W}^l$  is a *Brownian web with drift  $-1$* , i.e., if  $\mathcal{W}$  is a standard Brownian web, then

$$\mathcal{W}^* := \{\pi^* : \pi \in \mathcal{W}\} \quad \text{with} \quad \pi^*(t) := \pi(t) - t \quad (t \geq \sigma_\pi)$$

is equally distributed with  $\mathcal{W}^l$ . Likewise,  $\mathcal{W}^r$  is a *Brownian web with drift  $+1$* . We call the pair  $(\mathcal{W}^l, \mathcal{W}^r)$  a *left-right Brownian web*. We let  $\hat{\mathcal{W}}^l, \hat{\mathcal{W}}^r$  denote the dual Brownian webs associated with  $\mathcal{W}^l, \mathcal{W}^r$ . As a side result of the following theorem, we obtain that the joint law of  $\mathcal{W}^l, \mathcal{W}^r$  does not depend on the choice of the countable dense sets  $\mathcal{D}, \mathcal{D}'$ .

**Theorem 4.4 (Approximation of the left-right Brownian web)** *Let  $\varepsilon_k > 0$  satisfy  $\varepsilon_k \rightarrow 0$  and let  $\omega_k$  be arrow configurations with probabilities  $l_k, r_k, b_k, d_k$  as in (4.1) satisfying (4.3). Let  $\bar{\mathcal{U}}_k^l, \bar{\mathcal{U}}_k^r$  be the closures of the sets of left and right paths in  $\omega_k$  and let  $\hat{\mathcal{U}}_k^l, \hat{\mathcal{U}}_k^r$  be the closures of the sets of dual left and right paths in  $\omega_k$ . Then*

$$\mathbb{P}[\theta_{\varepsilon_k}(\bar{\mathcal{U}}_k^l, \bar{\mathcal{U}}_k^r, \hat{\mathcal{U}}_k^l, \hat{\mathcal{U}}_k^r) \in \cdot] \xrightarrow[k \rightarrow \infty]{} \mathbb{P}[(\mathcal{W}^l, \mathcal{W}^r, \hat{\mathcal{W}}^l, \hat{\mathcal{W}}^r) \in \cdot].$$

**Proof** Convergence of  $(\bar{\mathcal{U}}_k^l, \hat{\mathcal{U}}_k^l)$  to  $(\mathcal{W}^l, \hat{\mathcal{W}}^l)$  and of  $(\bar{\mathcal{U}}_k^r, \hat{\mathcal{U}}_k^r)$  to  $(\mathcal{W}^r, \hat{\mathcal{W}}^r)$  follows from Theorem 3.19. Using Lemma 3.15, it follows that the laws of the random variables  $\theta_{\varepsilon_k}(\bar{\mathcal{U}}_k^l, \bar{\mathcal{U}}_k^r, \hat{\mathcal{U}}_k^l, \hat{\mathcal{U}}_k^r)$  are tight, so by going to a subsequence, we may assume that they converge in law to some random variable  $(\mathcal{V}^l, \mathcal{V}^r, \hat{\mathcal{V}}^l, \hat{\mathcal{V}}^r)$ . In view of Lemma 2.2, it suffices to show that  $(\mathcal{V}^l, \mathcal{V}^r, \hat{\mathcal{V}}^l, \hat{\mathcal{V}}^r)$  is equal in law to  $(\mathcal{W}^l, \mathcal{W}^r, \hat{\mathcal{W}}^l, \hat{\mathcal{W}}^r)$ . Since a web is a.s. uniquely determined by its dual, it suffices to show that  $(\mathcal{V}^l, \mathcal{V}^r)$  is equal in law to  $(\mathcal{W}^l, \mathcal{W}^r)$ . By Theorem 3.19,  $\mathcal{V}^l$  is a Brownian web with drift  $-1$  and  $\mathcal{V}^r$  is a Brownian web with drift  $+1$ . Therefore, by Theorem 3.8, we know that at each deterministic  $z \in \mathbb{R}^2$ , the sets  $\mathcal{V}^l(z)$  and  $\mathcal{V}^r(z)$  almost surely contain a single path. To show that  $(\mathcal{V}^l, \mathcal{V}^r)$  is equal in law to  $(\mathcal{W}^l, \mathcal{W}^r)$ , by Theorem 3.8, it suffices to show that  $(\mathcal{V}^l, \mathcal{V}^r)$  has the right finite dimensional distributions, i.e., we must show that the left and right paths started from finitely many points are distributed as coalescing left and right Brownian motions. This follows from Proposition 4.3, so the proof is complete.  $\blacksquare$

## 4.4 The hopping and wedge constructions

Ultimately, we are not interested in left and right paths only, but in the scaling limit of the sets  $\mathcal{U}_k$  of all paths in the arrow configurations  $\omega_k$ . Let  $\pi_1^l, \pi_2^r, \pi_3^l, \dots$  be a finite sequence of paths that are alternatively taken from  $\mathcal{W}^l$  and  $\mathcal{W}^r$ , such that

$$\sigma_{\pi_1^l} < \sigma_{\pi_2^r} < \sigma_{\pi_3^l} < \dots$$

and

$$\pi_2^r(\sigma_{\pi_2^r}) < \pi_1^l(\sigma_{\pi_2^r}), \quad \pi_2^r(\sigma_{\pi_3^l}) < \pi_3^l(\sigma_{\pi_3^l}), \dots$$

i.e., the second path, which is a right path, is started on the left of the first path, which is a left path, and then the third path, which is a left path, is started on the right of the second path and so on. Let us also assume that

$$\tau(\pi_1^l, \pi_2^r) < \sigma_{\pi_3^l}, \quad \tau(\pi_2^r, \pi_3^l) < \sigma_{\pi_4^r}, \dots$$

i.e., we start the third path only after the first meeting time of the first two paths and so on. Then we can define a path  $\pi$  with starting time  $\sigma_\pi := \sigma_{\pi_1^l}$  by

$$\pi(t) := \begin{cases} \pi_1^l(t) & (\sigma_{\pi_1^l} \leq t \leq \tau(\pi_1^l, \pi_2^r)), \\ \pi_2^r(t) & (\tau(\pi_1^l, \pi_2^r) \leq t \leq \tau(\pi_2^r, \pi_3^l)), \\ \pi_3^l(t) & (\tau(\pi_2^r, \pi_3^l) \leq t \leq \tau(\pi_3^l, \pi_4^r)), \end{cases}$$

and so on, i.e., we start by following the path  $\pi_1^l$ , then “hop” onto the path  $\pi_2^r$  at the first time when  $\pi_1^l$  meets  $\pi_2^r$ , and so on, until we arrive at the last path in our finite sequence, which we follow till time  $+\infty$ . We fix a countable dense set  $\mathcal{D} \subset \mathbb{R}^2$  and let

$$\mathcal{N}_- := \text{the closure of } \left\{ \pi : \pi \text{ is obtained by hopping} \right. \\ \left. \text{between paths in } (\pi_z^l)_{z \in \mathcal{D}} \text{ and } (\pi_z^r)_{z \in \mathcal{D}} \right\}.$$

The set  $\mathcal{N}_-$  will play the role of the lower bound in the proof of convergence to the Brownian net, similar to the set  $\mathcal{W}_-$  in the proof of Theorem 3.19.

We also need an upper bound. This will again involve wedges and be very similar to what we did for the Brownian web. Let  $(\mathcal{W}^l, \mathcal{W}^r)$  be a left-right Brownian web and let  $\hat{\mathcal{W}}^l, \hat{\mathcal{W}}^r$  be the associated dual webs. It is not hard to see that  $-\hat{\mathcal{W}}^l$  is equally distributed with  $\mathcal{W}^l$  (both are Brownian webs with drift  $-1$ ) and  $-\hat{\mathcal{W}}^r$  is equally distributed with  $\mathcal{W}^r$ . In fact, one can check (this is most easily seen using finite approximation) that  $(-\hat{\mathcal{W}}^l, -\hat{\mathcal{W}}^r)$  is equally distributed with  $(\mathcal{W}^l, \mathcal{W}^r)$ . We now define

$$\mathcal{N}_+ := \left\{ \pi \in \Pi^\uparrow : \pi \text{ does not enter wedges} \right. \\ \left. \text{of the form } W(\hat{\pi}_{z_1}^r, \hat{\pi}_{z_2}^l) \text{ with } z_1, z_2 \in \mathcal{D} \right\}.$$

Note that here the left boundary of the wedge is formed by a dual right path and the right boundary is a dual left path. Because of the drift, these paths may fail to meet so the wedge may be infinite in size. In particular, the fact that paths do not enter wedges of this form implies that paths in  $\mathcal{N}_+$  do not cross dual left paths from right to left, or dual right paths from left to right. The following theorem is similar to Theorem 3.16 (and in fact historically predates it). We call the compact set  $\mathcal{N} := \mathcal{N}_- = \mathcal{N}_+$  from the following theorem the *Brownian net*.

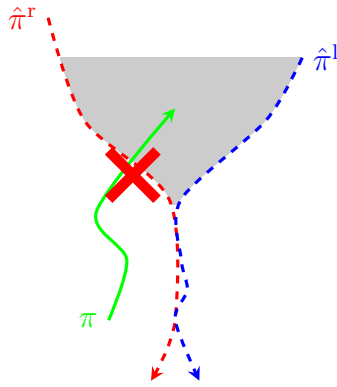


Figure 4.1: Illustration of the definition of the set  $\mathcal{N}_+$ . Paths  $\pi \in \mathcal{N}_+$  cannot enter wedges  $W(\hat{\pi}^r, \hat{\pi}^l)$  defined by a dual right and left path.

**Theorem 4.5 (Wedge characterisation of the Brownian net)** *Let  $\mathcal{D}$  be a countable dense subset of  $\mathbb{R}^2$  and let  $\mathcal{N}_-$  and  $\mathcal{N}_+$  be defined in terms of a left-right Brownian web  $(\mathcal{W}^l, \mathcal{W}^r)$  and its dual as above. Then  $\mathcal{N}_- = \mathcal{N}_+$ .*

**Proof (sketch)** The first step is to prove that paths in  $\mathcal{W}^l$  or  $\mathcal{W}^r$  cannot enter wedges of the form  $W(\hat{\pi}_z^r, \hat{\pi}_z^l)$ . One way to see this is to use finite approximation and Theorem 4.4. The same is then true for paths that are constructed by hopping between left and right paths at their first meeting times, from which we conclude that  $\mathcal{N}_- \subset \mathcal{N}_+$ . The fact that left or right paths cannot enter wedges also implies that forward left paths cannot cross dual right paths from left to right, and forward right paths cannot cross dual left paths from right to left.

The next step is similar to the proof of Theorem 3.16. We fix  $\pi \in \mathcal{N}_+$ ,  $\sigma_\pi < t_1 < \dots < t_m$ , and  $\varepsilon > 0$ . We claim that we can construct a path  $\pi^{\text{hop}}$  by hopping finitely often between paths in  $(\pi_z^l)_{z \in \mathcal{D}}$  and  $(\pi_z^r)_{z \in \mathcal{D}}$ , such that  $\sigma_\pi < \sigma_{\pi^{\text{hop}}} < t_1$  and  $|\pi^{\text{hop}}(t_i) - \pi(t_i)| \leq \varepsilon$  for all  $i = 1, \dots, m$ . To see this, for

each  $i = 1, \dots, m$ , we choose  $z_{\pm}^i = (x_{\pm}^i, t_{\pm}^i) \in \mathcal{D}$  such that  $t_{\pm}^i > t_i$  and

$$\pi(t_i) - \varepsilon < \hat{\pi}_{z_-}^r(t_i) < \pi(t_i) < \hat{\pi}_{z_+}^l(t_i) < \pi(t_i) + \varepsilon.$$

Since  $\pi$  does not enter the wedge  $W(\hat{\pi}_{z_-}^r, \hat{\pi}_{z_+}^l)$ , the meeting time of  $\hat{\pi}_{z_-}^r$  and  $\hat{\pi}_{z_+}^l$  must satisfy

$$\tau(\hat{\pi}_{z_-}^r, \hat{\pi}_{z_+}^l) \leq \sigma_{\pi},$$

and we have  $\hat{\pi}_{z_-}^r(t) \leq \pi(t) \leq \hat{\pi}_{z_+}^l(t)$  for all  $t \in [\sigma_{\pi}, t_i]$ . We can now choose  $z = (x, s) \in \mathcal{D}$  such that  $\sigma_{\pi} < s < t_1$  and

$$\sup_{1 \leq i \leq m} \hat{\pi}_{z_-}^r(t_1) < \pi_z^l(t_1) < \inf_{1 \leq i \leq m} \hat{\pi}_{z_+}^l(t_1).$$

The forward left path  $\pi_z^l$  cannot cross any of the left downward paths  $\hat{\pi}_{z_+}^l$ , but it can cross the right downward paths  $\hat{\pi}_{z_-}^r$ . Just before it does so, however, we can hop onto a cleverly chosen forward right path and continue until it threatens to cross one of the left downward paths  $\hat{\pi}_{z_+}^l$ . Just before it does, we can again hop onto a left path, and so on. Using the equicontinuity of  $\mathcal{W}^l$  and  $\mathcal{W}^r$ , one can prove that with a finite number of hoppings, one can steer the hopping path so that it stays between the bounding dual right and left paths and hence satisfies  $|\pi^{\text{hop}}(t_i) - \pi(t_i)| \leq \varepsilon$  for all  $i = 1, \dots, m$ .

The rest of the proof is now the same as in the proof of Theorem 3.16. ■

The proof of Theorem 4.5 has a useful corollary.

**Corollary 4.6 (Connections in the Brownian net)** *Let  $\mathcal{N}$  be a standard Brownian net. Then almost surely, for all  $a, b, s, t, x \in \mathbb{R}$  with  $a \leq b$  and  $s < t$ , there exists a path  $\pi \in \mathcal{N}(x, s)$  such that  $\pi(t) \in [a, b]$  if and only if there exist paths  $\hat{\pi}^r \in \hat{\mathcal{W}}^r(a, t)$  and  $\hat{\pi}^l \in \hat{\mathcal{W}}^l(b, t)$  such that*

$$\tau(\hat{\pi}^r, \hat{\pi}^l) \leq s \quad \text{and} \quad \hat{\pi}^r(s) \leq x \leq \hat{\pi}^l(s). \quad (4.5)$$

**Proof** If there exist paths  $\hat{\pi}^r$  and  $\hat{\pi}^l$  with the described properties such that moreover  $\hat{\pi}^r(s) < x < \hat{\pi}^l(s)$ , then by the argument in the proof of Theorem 4.5, there exists a path  $\pi \in \mathcal{N}_-(x, s)$  such that  $\pi(t) \in [a, b]$ . Using the compactness of  $\mathcal{N}$ , we can relax the strict inequalities in  $\hat{\pi}^r(s) < x < \hat{\pi}^l(s)$  to inequalities.

Conversely, if there exists a path  $\pi \in \mathcal{N}_+(x, s)$  such that  $\pi(t) \in [a, b]$ , then for each  $\varepsilon > 0$ , paths  $\hat{\pi}^r \in \hat{\mathcal{W}}^r(a - \varepsilon, t)$  and  $\hat{\pi}^l \in \hat{\mathcal{W}}^l(b + \varepsilon, t)$  must satisfy (4.5). Letting  $\varepsilon \rightarrow 0$ , using the compactness of  $\mathcal{W}^r$  and  $\mathcal{W}^l$ , the claim follows. ■



**Theorem 4.7 (Convergence to the Brownian net)** *Let  $\varepsilon_n$  be positive constants tending to zero and let  $\mathcal{U}_n$  be the set of paths in arrow configurations  $\omega_n$  for which the probabilities in (4.1) satisfy  $d_n = 0$ ,  $\varepsilon_n^{-1}(r_n - l_n) \rightarrow 0$ , and  $\varepsilon_n^{-1}b_n \rightarrow 1$ . Then*

$$\mathbb{P}[\theta_{\varepsilon_n}(\overline{\mathcal{U}}_n) \in \cdot] \xrightarrow[n \rightarrow \infty]{} \mathbb{P}[\mathcal{N} \in \cdot],$$

where  $\mathcal{N} := \mathcal{N}_- = \mathcal{N}_+$  is defined as in Theorem 4.5.

**Proof** This is very similar to the proof of Theorem 3.19. One first proves tightness, which in fact follows easily from the convergence of the collections of left and right paths and the fact that all paths starting from a point must stay between the left and right path starting from such a point. It then suffices to prove that all subsequential limit laws are equal. By going to a subsequence, we can assume that all left and right paths and dual left and right paths starting from some countable dense set  $\mathcal{D}$  also converge in law, and also their meeting times. We can then use Skorohods representation theorem to construct a coupling such that the convergence is almost sure. We then use the paths starting from the set  $\mathcal{D}$  to construct sets  $\mathcal{N}_-$  and  $\mathcal{N}_+$ . The proof then consists of showing that the limit  $\mathcal{N}$  of the set of all paths satisfies  $\mathcal{N}_- \subset \mathcal{N} \subset \mathcal{N}_+$ . The lower bound follows from the fact that if discrete left and right paths converge to left and right paths that cross, then the approximating discrete paths must for  $n$  large enough also cross. The discrete path constructed by hopping between these paths is then certainly an element of  $\mathcal{U}_n$ , and hence  $\mathcal{N}_- \subset \mathcal{N}$ . The upper bound follows in the same way as in the proof of Theorem 3.19, by showing that if the approximating discrete paths do not enter wedges, then this property must be preserved in the limit.  $\blacksquare$

## 4.5 The density of the net

Let  $\mathcal{N}$  be a Brownian net. For each closed set  $A \subset \mathbb{R}$ , we define a process  $(\xi_t^A)_{t \geq 0}$  by

$$\xi_t^A := \{\pi(t) : \pi \in \mathcal{N}, \sigma_\pi = 0, \pi(0) \in A\} \quad (t \geq 0).$$

We call  $(\xi_t^A)_{t \geq 0}$  the *branching-coalescing point set*. Compared to the coalescing point set introduced in Section 3.7, it is a more complicated process, even when  $A$  is a set consisting of a single point. We will study  $(\xi_t^A)_{t \geq 0}$  in more detail in the next chapter. In the present section, we only state a result, with a sketch of the proof, that similar to the coalescing point set from Section 3.7, the branching-coalescing point set *comes down from infinity*, in the

sense that for each deterministic  $t > 0$ , the set  $\xi_t^A$  is a.s. locally finite. As we will see in the next chapter, however, this result holds only for deterministic times: there exists a random dense set of times at which  $\xi_t^A$  is not locally finite.

**Proposition 4.8 (Density of the Brownian net)** *The branching-coalescing point set satisfies*

$$\mathbb{E}[|\xi_t^{\mathbb{R}} \cap [a, b]|] = (b - a) \cdot \left( \frac{e^{-t}}{\sqrt{\pi t}} + 2\Phi(\sqrt{2t}) \right), \quad (4.6)$$

where  $\Phi(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-y^2/2} dy$  is the distribution function of the normal distribution.

**Proof (sketch)** The proof is very similar to the proof of Proposition 3.20. For  $a, b \in \mathbb{R}$  with  $a < b$ , let  $\hat{\pi}_{(a,t)}^r$  and  $\hat{\pi}_{(b,t)}^l$  be the dual right and left paths started from  $(a, t)$  and  $(b, t)$ , and let  $\tau_{a,b} := \tau(\hat{\pi}_{(a,t)}^r, \hat{\pi}_{(b,t)}^l)$  be their first meeting time. Then by Corollary 4.6,

$$\xi_t^{\mathbb{R}} \cap [a, b] \neq \emptyset \quad \text{if and only if} \quad \tau_{a,b} \leq 0.$$

The proof now proceeds in a very similar way to the proof of Proposition 3.20, except that one needs to calculate the probability that a Brownian motion with negative drift stays, up to some finite time horizon, below a given constant. For the details, we refer to [SS08, Prop. 1.12]. ■

As a consequence of Proposition 4.8, we obtain the following result.

**Lemma 4.9 (No incoming paths at typical points)** *Let  $\mathcal{N}$  be a standard Brownian net and let  $(x, t) \in \mathbb{R}^2$ . Then almost surely, there exist no  $\pi \in \mathcal{N}$  with  $\sigma_\pi < t$  and  $\pi(t) = x$ .*

**Proof** Let us write  $\xi_{s,t}^{\mathbb{R}} := \{\pi(t) : \pi \in \mathcal{N}, \sigma_\pi \leq s\}$  and let  $\varepsilon_n$  be positive constants converging to zero. By Proposition 4.8 and translation invariance,  $\mathbb{P}[x \in \xi_{t-\varepsilon_n, t}^{\mathbb{R}}] = 0$  for all  $n$ . ■

## 4.6 Marking constructions

The construction of the Brownian net based on the left-right SDE (4.4) and wedges closely follows the original construction of the Brownian net in [SS08]. In this section, we outline a completely different *marking construction* due to Newman, Ravishankar, and Schertzer [NRS10] and further developed in

[SSS14]. This approach not only naturally leads to an alternative construction of the Brownian net but can more generally be used to construct the scaling limit of branching and coalescing random walks that moreover die with a positive probability, as in Theorem 4.1. The most general marking construction, which also involves killing, was developed in [NRS15] and used to study perturbations of the voter model in [NRS17].

In Section 3.8 we have seen that there exist random space-time points  $z$  of type  $(1, 2)$ . These are points such that  $m_{\text{in}}(z) = 1 = \hat{m}_{\text{in}}(z)$  and consequently  $m_{\text{in}}(z) = 2$ . At such points, the incoming path in the Brownian web continues as one of the outgoing paths (either the left or the right one) while the other outgoing path is not the continuation of an incoming path. It turns out there are uncountably many such points. Indeed, if we fix a forward and a dual path, then conditional on the forward path, the law of the dual path is described by a Brownian motion with Skorohod reflection off the forward path; this result is originally due to Soucaliuc, Tóth, and Werner [STW00]. For a Brownian motion reflected off a continuous path, it is well-known that the set of times when the Brownian motion is at the same position as the path has Lebesgue measure zero, but is nevertheless uncountable. There is a natural measure on this set, which is the *reflection local time measure*, and which is the limit of the Lebesgue measure on the set of times when the paths are within distance  $\varepsilon$ , rescaled by  $\varepsilon^{-1}$ .

The idea is now to construct a Brownian net from a Brownian web by allowing paths at some of the points of type  $(1, 2)$  to continue along both outgoing paths. More precisely, the idea is to first construct a Poisson point set whose intensity measure is precisely the reflection local time measure between (all) forward and dual paths, and then turn the points in this Poisson point set (which by construction are all of type  $(1, 2)$ ) into branching points. We refer to [NRS15, Thm 5.5] or [SSS14, Thm 4.6] for details.

It is also possible to go back: a *branching point* of a Brownian net is a point  $(x, t) \in \mathbb{R}^2$  such that there exist paths  $\pi, \pi' \in \mathcal{N}$  with  $\sigma_\pi = \sigma_{\pi'} < t$  and  $\pi(s) = \pi'(s)$  for all  $s \in [\sigma_\pi, t]$ , but there exists an  $\varepsilon > 0$  such that  $\pi(u) < \pi'(u)$  for all  $t < u < t + \varepsilon$ . It can be shown that the set of branching points is countable. We can modify a Brownian net by turning branching points in points of type  $(1, 2)$ , where we decide in an i.i.d. fashion which of the outgoing paths will be the continuation of the incoming path. The result is a Brownian web; see [SSS14, Thm 4.4].

By a similar, but simpler Poisson construction, one can also construct the *Brownian net with killing*, which is the scaling limit of branching and coalescing random walks that moreover die with a positive probability as in Theorem 4.1. In this case, one needs a Poisson marking of the points of type  $(1, 1)$ . The set of all points  $z \in \mathbb{R}^2$  that are of type  $(1, 1)$  can naturally be

equipped with a measure  $\mu$  such that for each  $\pi \in \mathcal{W}$  and  $\sigma_\pi < s < u$ , one has

$$\mu(\{(x, t) \in \mathbb{R}^2 : s < t < u, x = \pi(t)\}) = u - s. \quad (4.7)$$

One can now construct a Poisson point set with intensity  $\delta\mu$ , where  $\delta \geq 0$  is a constant. By construction, all such points are of type  $(1, 1)$ . One can now “cut” the paths passing through these points, resulting in a Brownian web in which Brownian paths are “killed” with rate  $\delta$ . Combining this with the marking construction of the Brownian net, one can also obtain a Brownian net with killing rate  $\delta$ . Note that Proposition 4.8 says that at deterministic times (and hence also at almost all times w.r.t. Lebesgue measure), the branching-coalescing point set consists of locally finitely many points. The effect of the Poisson marking of points of type  $(1, 1)$  is then simply that these points are killed with Poisson rate  $\delta$ .

# Chapter 5

## The branching-coalescing point set

### 5.1 Brownian net related Markov processes

In Section 1.4, we informally introduced two Markov processes, taking values in the closed subsets of the real line, that are the diffusive scaling limits of branching and coalescing random walks and their dual biased voter models, in the limit when the branching rate is small. With the help of the Brownian net, we can now formally define these processes and study their properties.

Given a standard Brownian net, for each  $s, t \in \mathbb{R}$  with  $s \leq t$ , we define random maps acting on compact subsets of  $\overline{\mathbb{R}}$  by

$$\begin{aligned}\mathbf{X}_{s,t}(A) &:= \{\pi(t) : \pi \in \mathcal{N}(A \times \{s\})\}, \\ \mathbf{Y}_{t,s}(A) &:= \{x \in \overline{\mathbb{R}} : \exists \pi \in \mathcal{N}(x, s) \text{ s.t. } \pi(t) \in A\}.\end{aligned}$$

Note that these maps are *additive* in the sense that

$$\mathbf{X}_{s,t}(\emptyset) = \emptyset \quad \text{and} \quad \mathbf{X}_{s,t}(A \cup B) = \mathbf{X}_{s,t}(A) \cup \mathbf{X}_{s,t}(B), \quad (5.1)$$

and similarly for  $\mathbf{Y}_{t,s}$ .

**Lemma 5.1 (Stochastic flows)** *For each  $t \in \mathbb{R}$ , the maps  $\mathbf{X}_{t,t}$  and  $\mathbf{Y}_{t,t}$  are the identity maps. For each  $s \leq t \leq u$ , one has*

$$\mathbf{X}_{t,u} \circ \mathbf{X}_{s,t} = \mathbf{X}_{s,u} \text{ a.s.} \quad \text{and} \quad \mathbf{Y}_{t,s} \circ \mathbf{Y}_{u,t} = \mathbf{Y}_{u,s} \text{ a.s.} \quad (5.2)$$

*Moreover, for all  $t_0 < \dots < t_n$ , the maps  $\mathbf{X}_{t_0,t_1}, \dots, \mathbf{X}_{t_{n-1},t_n}$  are independent, and so are  $\mathbf{Y}_{t_n,t_{n-1}}, \dots, \mathbf{Y}_{t_1,t_0}$ .*

**Proof (crude sketch)** The statement about independence follows from the fact that restrictions of the Brownian net to disjoint time intervals are independent, which can be derived using discrete approximation (Theorem 4.7).

The fact that  $\mathbf{X}_{t,t}$  and  $\mathbf{Y}_{t,t}$  are the identity maps is immediate from the definitions. It therefore suffices to prove (5.2) under the additional assumption that  $s < t < u$ . The claim will now follow provided we can show that for deterministic  $s < t < u$ , almost surely, if there exist  $\pi, \pi' \in \mathcal{N}$  with  $\sigma_\pi = s$ ,  $\sigma_{\pi'} = t$ , and  $\pi(t) = \pi'(t)$ , then the path  $\pi''$  defined by  $\sigma_{\pi''} := s$ ,  $\pi'' := \pi'$  on  $[s, t]$ , and  $\pi'' := \pi'$  on  $[t, \infty)$  satisfies  $\pi'' \in \mathcal{N}$ .

We first note that the analogue statement is true for the Brownian web  $\mathcal{W}$  by the fact, stated in Theorem 3.25, that at deterministic times almost surely all points are of type  $(0, 1)$ ,  $(0, 2)$ , or  $(1, 1)$ . In particular, whenever there is an incoming path, there is a unique outgoing path. Note, however, that even for the Brownian web, because of the existence of points of type  $(1, 2)$ , the statement only holds a.s. for deterministic times, and not almost surely for all times (including random times).

For the Brownian net  $\mathcal{N}$ , at deterministic times, all points must be either of type  $(0, 1)$ ,  $(0, 2)$ , or  $(1, 1)$  in the left web  $\mathcal{W}^l$ , and also in the right web  $\mathcal{W}^r$ . It is known [SSS09, Thm 1.7] that at deterministic times, all points of the same type in the left web as in the right web. Moreover, if a point is of type  $(1, 1)$  in the left and right webs, then the concatenation of any incoming net path and outgoing net path is again a path in the net [SSS09, Thm 1.12(d)].  $\blacksquare$

Below, we let  $\mathcal{K}(\overline{\mathbb{R}})$  denote the space of compact subsets of  $\overline{\mathbb{R}}$ . We equip  $\mathcal{K}(\overline{\mathbb{R}})$  with a topology such that the induced topology on  $\mathcal{K}_+(\overline{\mathbb{R}})$  is the Hausdorff topology and  $\emptyset$  is an isolated point, i.e.,  $\{\emptyset\}$  is both open and closed as a subset of  $\mathcal{K}(\overline{\mathbb{R}})$ .

**Lemma 5.2 (Markov processes)** *For each  $s \in \mathbb{R}$  and compact  $A \subset \overline{\mathbb{R}}$ , setting*

$$\mathcal{X}_t := \mathbf{X}_{s,s+t}(A) \quad \text{and} \quad \mathcal{Y}_t := \mathbf{Y}_{s,s-t}(A) \quad (t \geq 0)$$

*defines time-homogeneous Markov processes  $(\mathcal{X}_t)_{t \geq 0}$  and  $(\mathcal{Y}_t)_{t \geq 0}$  with state space  $\mathcal{K}(\overline{\mathbb{R}})$ . The process  $(\mathcal{X}_t)_{t \geq 0}$  has continuous sample paths.*

**Proof** We will only prove the statement for  $(\mathcal{X}_t)_{t \geq 0}$ . The proof for  $(\mathcal{Y}_t)_{t \geq 0}$  is almost completely the same. By translation invariance, it suffices to prove the statement for  $s = 0$ . We fix  $A_0 \in \mathcal{K}(\overline{\mathbb{R}})$  and define  $\mathcal{X}_t := \mathbf{X}_{0,t}(A_0)$  ( $t \geq 0$ ). We will show that  $(\mathcal{X}_t)_{t \geq 0}$  is a time-homogeneous Markov process with state space  $\mathcal{K}(\overline{\mathbb{R}})$  and continuous sample paths.

For each  $t \geq 0$ , we define a probability kernel  $P_t$  on the space  $\mathcal{K}(\overline{\mathbb{R}})$  by

$$P_t(A, \cdot) := \mathbb{P}[\mathbf{X}_{s,s+t}(A) \in \cdot,] \quad (A \in \mathcal{K}(\overline{\mathbb{R}}), t \geq 0, s \in \mathbb{R}),$$

where the definition does not depend on the choice of  $s$  by the translation-invariance of the Brownian net. Since by Lemma 5.1,  $(\mathbf{X}_{s,t})_{s \leq t}$  is a stochastic flow with independent increments, for each  $0 < t_1 < \dots < t_n$  and measurable sets  $\mathcal{A}_1, \dots, \mathcal{A}_n \subset \mathcal{K}(\overline{\mathbb{R}})$ ,

$$\begin{aligned} & \mathbb{P}[\mathcal{X}_{t_1} \in \mathcal{A}_1, \dots, \mathcal{X}_{t_n} \in \mathcal{A}_n] \\ &= \int_{\mathcal{A}_1 \times \dots \times \mathcal{A}_n} \mathbb{P}[\mathbf{X}_{0,t_1}(A_0) \in dA_1, \dots, \mathbf{X}_{t_{n-1},t_n}(A_{n-1}) \in dA_n] \\ &= \int_{\mathcal{A}_1} P_{t_1}(A_0, dA_1) \cdots \int_{\mathcal{A}_n} P_{t_n}(A_{n-1}, dA_n), \end{aligned}$$

which shows that  $(\mathcal{X}_t)_{t \geq 0}$  is a time-homogeneous Markov process with transition kernels  $(P_t)_{t \geq 0}$ .

To prove that  $(\mathcal{X}_t)_{t \geq 0}$  has continuous sample paths, we must show that

$$t_n \rightarrow t \quad \text{implies} \quad \mathbf{X}_{0,t_n}(A_0) \rightarrow \mathbf{X}_{0,t}(A_0),$$

where the convergence on the right is in the topology on  $\mathcal{K}(\overline{\mathbb{R}})$ . The case  $A_0 = \emptyset$  is trivial, so from now on we assume that  $A_0 \neq \emptyset$ . We have to show that the sets  $K_n := \mathbf{X}_{0,t_n}(A_0)$  and  $K := \mathbf{X}_{0,t}(A_0)$  satisfy the convergence criterion (2.4). For each  $x \in K$ , there exists a path  $\pi \in \mathcal{N}(A_0 \times \{0\})$ . Then  $\pi(t_n) \in K_n$  satisfy  $\pi(t_n) \rightarrow \pi(t)$ , which shows that  $K$  is contained in the set of all limit points of sequences  $x_n \in K_n$ . To complete the proof, we must show that if a sequence  $x_n \in K_n$  has a subsequential limit  $x$ , then  $x \in K$ . Each  $x_n \in K$  is of the form  $x_n = \pi_n(t_n)$  for some  $\pi_n \in \mathcal{N}(A_0 \times \{0\})$ . By the compactness of  $\mathcal{N}$ , by going to a further subsequence if necessary, we can assume that  $\pi_n \rightarrow \pi \in \mathcal{N}$ . By Lemma 2.23, this means that  $\pi_n \rightarrow \pi$  locally uniformly. It follows that  $\pi \in \mathcal{N}(A_0 \times \{0\})$  and hence  $x = \lim_{n \rightarrow \infty} \pi_n(t_n) = \pi(t) \in K$ , where in the last step we have used Lemma 2.13 (ii).  $\blacksquare$

**Remark** The process  $(\mathcal{Y}_t)_{t \geq 0}$  does not have continuous sample paths, since it may happen that  $\mathcal{Y}_0 \neq \emptyset$  while  $\mathcal{Y}_t = \emptyset$  for some  $t > 0$ , and  $\emptyset$  is an isolated point of  $\mathcal{K}(\overline{\mathbb{R}})$ . In fact, one can check that the only discontinuities of  $(\mathcal{Y}_t)_{t \geq 0}$  are due to this sort of (local) extinctions and the sample paths of  $(\mathcal{Y}_t)_{t \geq 0}$  are in fact left-continuous.

**Remark** It has been shown in [SS08, Thm 1.11] that  $(\mathcal{X}_t)_{t \geq 0}$  is a Feller process, which means that the map

$$\mathcal{K}(\overline{\mathbb{R}}) \ni (x, t) \mapsto P_t(x, \cdot)$$

is continuous with respect to weak convergence of probability measures on  $\mathcal{K}(\overline{\mathbb{R}})$ . Since by Lemma 2.16, the state space  $\mathcal{K}(\overline{\mathbb{R}})$  is compact, standard

theory now tells us that the Feller semigroup  $(P_t)_{t \geq 0}$  is uniquely characterised by its generator  $G$ , which is a closed linear operator defined on a dense subspace of the space of all continuous real functions on  $\mathcal{K}(\overline{\mathbb{R}})$ , equipped with the supremum norm. It is an open problem to give an explicit characterisation of  $G$ , in particular, to find some explicit linear operator  $G'$  defined on a dense set of functions such that  $G$  is the closure of  $G'$ .

Making our informal definitions from Chapter 1 precise, we call the Markov process  $(\mathcal{X}_t)_{t \geq 0}$  the *branching-coalescing point set* and we call  $(\mathcal{Y}_t)_{t \geq 0}$  the *expanding interval process*. The following lemma, together with the additive property (5.1) of the stochastic flow  $(\mathbf{Y}_{s,t})_{s \leq t}$ , shows that our present definition of the expanding interval process coincides with the definition for special initial states given in Chapter 1.

**Lemma 5.3 (Expanding interval process)** *Let  $a, b \in \mathbb{R}$  satisfy  $a < b$  and let  $B^l, B^r$  be independent Brownian motions. Set*

$$L_t := B_t^l - t, \quad R_t := B_t^r + t, \quad \text{and} \quad \tau := \inf\{t > 0 : L_t = R_t\}.$$

Then setting

$$\mathcal{Y}_t := \begin{cases} [L_t, R_t] & \text{if } t \leq \tau, \\ \emptyset & \text{otherwise} \end{cases}$$

defines an expanding interval process  $(\mathcal{Y}_t)_{t \geq 0}$ .

**Remark** The expanding interval process can also be defined for a Brownian net with killing. The behaviour of this process is much less trivial than in the case without killing, since even if the initial state is an interval, the set  $\mathcal{Y}_t$  at positive times a.s. contains infinitely many (small) holes. Processes of this and more general form have been studied in [NRS17].

**Proof of Lemma 5.3** Let  $\mathcal{N}$  be a standard Brownian net with associated left-right Brownian web  $(\mathcal{W}^l, \mathcal{W}^r)$  and dual left-right Brownian web  $(\hat{\mathcal{W}}^l, \hat{\mathcal{W}}^r)$ . Let  $\hat{\pi}^r \in \hat{\mathcal{W}}^r(a, 0)$  and  $\hat{\pi}^l \in \hat{\mathcal{W}}^l(b, 0)$  be the a.s. unique dual right and left paths starting from  $(a, 0)$  and  $(b, 0)$  and let  $\tau := \tau(\hat{\pi}^r, \hat{\pi}^l)$  be their first meeting time. For  $t \geq 0$ , let

$$\mathcal{Y}_t := \mathbf{Y}_{0,-t}([a, b]) = \{x \in \overline{\mathbb{R}} : \exists \pi \in \mathcal{N} \text{ s.t. } \pi(0) \in [a, b]\}.$$

Then Corollary 4.6 implies that

$$\mathcal{Y}_t = \begin{cases} [\hat{\pi}^r(-t), \hat{\pi}^l(-t)] & \text{if } \tau \leq t, \\ \emptyset & \text{otherwise.} \end{cases}$$



Since  $\hat{\pi}^r$  and  $\hat{\pi}^l$  are drifted Brownian motions, the claim follows.  $\blacksquare$

In Chapter 1, we introduced the branching-coalescing point set via its duality to the expanding interval process described in Theorem 1.5. There, we stated the duality only in the case  $\mathcal{Y}_0$  was a finite union of intervals of positive length. We can now state and prove the duality for general initial states  $\mathcal{Y}_0$ .

**Theorem 5.4 (Duality of continuum processes)** *The branching-coalescing point set  $(\mathcal{X}_t)_{t \geq 0}$  and the expanding interval process  $(\mathcal{Y}_t)_{t \geq 0}$  started in arbitrary deterministic initial states  $\mathcal{X}_0, \mathcal{Y}_0 \in \mathcal{K}(\overline{\mathbb{R}})$  satisfy*

$$\mathbb{P}[\mathcal{X}_t \cap \mathcal{Y}_0 \neq \emptyset] = \mathbb{P}[\mathcal{X}_0 \cap \mathcal{Y}_t \neq \emptyset] \quad (t \geq 0). \quad (5.3)$$

**Proof** We fix  $t \geq 0$  and define

$$\mathcal{X}_s := \mathbf{X}_{0,s}(\mathcal{X}_0) \quad \text{and} \quad \mathcal{Y}_s := \mathbf{Y}_{t,t-s}(\mathcal{Y}_0) \quad (s \geq 0).$$

Then the claim follows from the observation that

$$\begin{aligned} \mathcal{X}_t \cap \mathcal{Y}_0 \neq \emptyset &\Leftrightarrow \mathbf{X}_{0,t}(\mathcal{X}_0) \cap \mathcal{Y}_0 \neq \emptyset \\ &\Leftrightarrow \exists \pi \in \mathcal{N}(\mathcal{X}_0 \times \{0\}) \text{ s.t. } \pi(t) \in \mathcal{Y}_0 \\ &\Leftrightarrow \mathcal{X}_0 \cap \mathbf{Y}_{t,0}(\mathcal{Y}_0) \neq \emptyset \Leftrightarrow \mathcal{X}_0 \cap \mathcal{Y}_t \neq \emptyset \end{aligned}$$

$\blacksquare$

**Remark** In Section 1.4, we informally stated Theorems 1.4 and 1.6, which said that the branching-coalescing point set and the expanding interval process are the scaling limits of diffusively rescaled continuous-time branching and coalescing random walks with small branching rates, and their dual voter models with a small bias. We now have all the tools available to state and prove precise versions of these theorems. Recall that in Sections 1.2 and 1.3 we constructed branching and coalescing random walks and their dual biased voter models from a graphical representation with two kinds of arrows, which represent coalescing random walk jumps and branching events. In Section 1.3, we defined *open paths* in such a graphical representation. Similar to Theorem 4.7, one can show that the collection of all half-infinite open paths in such a graphical representation, properly rescaled, converges to a Brownian net. The proof is almost completely the same as the proof of Theorem 4.7. Using this, one can also prove convergence of branching and coalescing random walks and biased voter models to the branching-coalescing point set and the expanding interval process, in an appropriate sense. For brevity, we skip the details.

**Remark** The proof of Theorem 4.7 essentially uses the nearest-neighbour nature of the random walks and non-crossing properties of random walk paths. Nevertheless, it is believed that even if we allow random walk jumps and infections between points in  $\mathbb{Z}$  that are at distance two or more, the collection of all half-infinite open paths should still converge to the Brownian net, provided the jump rates satisfy a suitable moment condition. In the context of the Brownian web, this has been proved in [NRS05, BMSV06]. For the Brownian net, it is an open problem.

## 5.2 The branching-coalescing point set

In this section, we study some of the properties of the branching-coalescing point set.

**Proposition 5.5 (Finiteness of the branching-coalescing point set)**

*Let  $(\mathcal{X}_t)_{t \geq 0}$  be a branching-coalescing point set started in a compact set  $\mathcal{X}_0 \subset \mathbb{R}$ . Then for each  $t > 0$ , the set  $\mathcal{X}_t$  is a.s. finite. But there almost surely exists a dense set of times  $t$  at which  $\mathcal{X}_t$  is infinite.*

**Proof (crude sketch)** Finiteness at deterministic times follows from Proposition 4.8 and the fact that if  $\mathcal{X}_0 \subset [a, b]$  for some compact interval  $[a, b]$ , then  $\mathcal{X}_t \subset [\pi^l(t), \pi^r(t)]$  ( $t \geq 0$ ), where  $\pi^l, \pi^r$  are the a.s. unique left and right paths started in  $(a, 0)$  and  $(b, 0)$ .

To see why there exists a dense set of times  $t$  at which  $\mathcal{X}_t$  is infinite, it is instructive to consider the case that  $a = b$ . We have seen in Section 4.2 that the interaction between left and right paths is described by the left-right SDE (4.4), which describes a form of sticky interaction. It has been shown in [SS08, Prop. 3.1(a)] that the set of times when a left and right paths are on the same location has positive Lebesgue measure but is nowhere dense, which means that each time interval of positive length contains (small) open time intervals during which the two processes are not at the same position.

For the branching-coalescing point set started in a single point  $\mathcal{X}_0 = \{x\}$ , this means that although for small times, most of the time  $\mathcal{X}_t$  consists of a single point, there is a dense set of small time intervals during which  $\mathcal{X}_t$  has split into at least two points that however quickly coalesce again. On a smaller scale, these points also split into two points, so one can inductively find open time intervals, nested inside each other, during which  $\mathcal{X}_t$  contains at least 2, 4, 8, 16,  $\dots$  points. At the intersection of infinitely many of such nested time intervals, one finds random times at which  $\mathcal{X}_t$  is infinite. ■

**Remark** For the branching-coalescing point set started in  $\mathcal{X}_0 = \overline{\mathbb{R}}$  it is shown in [SSS09, Prop. 3.14] that there even exists a dense set of times when  $\mathcal{X}_t$

does not contain isolated points.

**Remark** It is an interesting open problem to describe the excursions of the branching-coalescing point set away from the set of singletons by means of an excursion measure, similar to the theory of excursions of Brownian motion. In [SSS16, Conjecture 6.8.1], a precise conjecture is formulated about the structure of these excursions.

We recall from Lemma 1.1 that product measure with intensity  $p$  is a reversible law for branching and coalescing continuous-time random walks with parameter  $p$ . In the scaling limit of Theorem 1.6, this product measure converges to the law of a Poisson point process with intensity 2. The following proposition says that this is a reversible law for the branching-coalescing point process.

**Lemma 5.6 (Reversible law)** *The law of a Poisson point process with intensity 2 is a reversible invariant law for the branching-coalescing point process.*

**Proof (sketch)** Let  $\varepsilon_n$  be positive constants tending to zero and let  $\omega^n$  be an arrow configuration with probabilities of a left arrow, right arrow, branching or death given by  $l_n = (1 - \varepsilon)/2$ ,  $r_n = (1 - \varepsilon)/2$ ,  $b_n = \varepsilon$ , and  $d_n = 0$ . Let  $\mathcal{U}_n$  be the set of paths in  $\omega_n$  and let  $\mathcal{U}_n(*, -\infty)$  be the set of paths  $\pi \in \mathcal{U}_n$  with starting point  $(*, -\infty)$ . It is shown in [SS08, Prop. 1.14(a)] that  $A := \{\pi(0) : \pi \in \mathcal{U}_n(*, -\infty)\}$  is a random subset of  $\mathbb{Z}_{\text{even}}$  such that independently, each  $i \in \mathbb{Z}_{\text{even}}$  is included in  $A$  with probability  $4\varepsilon_n/(1 + \varepsilon_n)^2$ . Taking the limit, using Theorem 4.7, it follows that the set  $\{\pi(0) : \pi \in \mathcal{N}(*, -\infty)\}$  stochastically dominates a Poisson point set with intensity 2. On the other hand, letting  $t \rightarrow \infty$  in formula (4.6) of Proposition 4.8, it is easy to see that the expected intensity of the point process  $\{\pi(0) : \pi \in \mathcal{N}(*, -\infty)\}$  is at most two, so we can conclude that  $\{\pi(0) : \pi \in \mathcal{N}(*, -\infty)\}$  is a Poisson point set with intensity 2.

Using the definition of the branching-coalescing point process, it follows that the law of a Poisson point process with intensity 2 is an invariant law for the branching-coalescing point process. It is moreover shown in [SS08, Prop. 1.14(a)] that  $\mathcal{U}_n(*, -\infty)$  and  $-\mathcal{U}_n(*, -\infty)$  are equal in law. Taking the limit, it follows that also  $\mathcal{N}(*, -\infty)$  and  $-\mathcal{N}(*, -\infty)$  are equal in law, and hence the Poissonian invariant law of the branching-coalescing point process is reversible.

In this proof, we have used approximation with discrete-time branching and coalescing random walks, but one could equally use approximation with continuous-time branching and coalescing random walks defined by a graphi-

cal representation as in Section 1.2 and then use Lemma 1.1 instead of [SS08, Prop. 1.14].  $\blacksquare$

**Proposition 5.7 (Ergodicity)** *Let  $(\mathcal{X}_t)_{t \geq 0}$  be a branching-coalescing point set started in any nonempty initial state  $\mathcal{X}_0$ . Then  $\mathcal{X}_t$  converges in law as  $t \rightarrow \infty$  to a Poisson point set with intensity 2.*

**Proof (crude sketch)** We only prove the statement when  $\mathcal{X}_0 = \{x\}$  consists of a single point. Then  $\mathcal{X}_t$  is equal in law to  $\mathbf{X}_{-t,0}(\{x\})$ . Let  $\pi_{(x,-t)}^l$  and  $\pi_{(x,-t)}^r$  be the left and right paths started at  $(x, -t)$ . Then

$$\mathbf{X}_{-t,0}(\{x\}) = \mathbf{X}_{-t,0}(\overline{\mathbb{R}}) \cap [\pi_{(x,-t)}^l(0), \pi_{(x,-t)}^r(0)].$$

Here  $\mathbf{X}_{-t,0}(\overline{\mathbb{R}})$  almost surely converges as  $t \rightarrow \infty$  to  $\{\pi(0) : \pi \in \mathcal{N}(*, -\infty)\}$ , which is a Poisson point set with intensity 2, and  $\pi_{(x,-t)}^l(0) \rightarrow -\infty$  a.s. and  $\pi_{(x,-t)}^r(0) \rightarrow +\infty$  a.s. We refer to [SS08, Prop. 1.15(iii)] for details.  $\blacksquare$

### 5.3 The branching-killing process revisited

In this final section we return to the biased annihilating branching process (BAB) introduced in Section 1.1. We outline a very rough idea of a possible proof of Conjecture 1.7. We first introduce an auxiliary process.

Let  $\mathbb{Z} + \frac{1}{2} := \{k + \frac{1}{2} : k \in \mathbb{Z}\}$  and  $\frac{1}{2}\mathbb{Z} := \{\frac{1}{2}k : k \in \mathbb{Z}\}$ . Let

$$\vec{E} := \{(i, j) \in (\frac{1}{2}\mathbb{Z})^2 : |i - j| = \frac{1}{2}\}$$

For each  $(i, j) \in \vec{E}$ , we define a *coalescing random walk map*  $\mathbf{rw}_{ij}$  as in (1.7), except that this map acts on configurations  $z$  of zeros and ones on  $\frac{1}{2}\mathbb{Z}$ . Moreover, for each  $i \in \mathbb{Z} + \frac{1}{2}$ , we define a *splitting map*  $\mathbf{split}_i$  by

$$\mathbf{split}_i(z)(k) := \begin{cases} 0 & \text{if } k = i \\ 1 & \text{if } k \in \{i - \frac{1}{2}, i + \frac{1}{2}\}, z(i) = 1 \\ z(k) & \text{otherwise.} \end{cases}$$

In words, the effect of this map is that if there is a particle at  $i$ , then this particle splits into two particles that are placed on the sites  $i - \frac{1}{2}$  and  $i + \frac{1}{2}$  and coalesce with any particles that may already be present on these sites. We will be interested in the interacting particle system  $(Z_t)_{t \geq 0}$  with generator

given by

$$\begin{aligned}
G_{\text{cov}}f(z) &:= \frac{1}{2}p \sum_{\substack{(i,j) \in \vec{E} \\ i \in \mathbb{Z}}} \{f(\mathbf{rw}_{ij}(z)) - f(z)\} \\
&\quad + \frac{1}{2}(1-p) \sum_{\substack{(i,j) \in \vec{F} \\ i \in \mathbb{Z} + \frac{1}{2}}} \{f(\mathbf{rw}_{ij}(z)) - f(z)\} \\
&\quad + p \sum_{i \in \mathbb{Z} + \frac{1}{2}} \{f(\mathbf{split}_i(z)) - f(z)\}.
\end{aligned}$$

For reasons that will become clear later, we call  $(Z_t)_{t \geq 0}$  the *covering branching-coalescent*. We can represent the covering branching-coalescent with a graphical representation as in Section 1.2, where as in that section, we represent the application of a coalescing random walk map  $\mathbf{rw}_{ij}$  at time  $t$  by an arrow with a blocking symbol at its tail that points from  $(i, t)$  to  $(j, t)$ . Similarly, we represent the application of a splitting map  $\mathbf{split}_i$  at time  $t$  by two arrows with blocking symbols at their tails, that point from  $(i, t)$  to  $(i - \frac{1}{2}, t)$  and to  $(i + \frac{1}{2}, t)$ . It is easy to check from this construction that the interacting particle system  $(Z_t)_{t \geq 0}$  is additive in the sense of (1.10).

For the graphical representation of  $(Z_t)_{t \geq 0}$ , we define open paths as in Section 1.3 and let  $\mathcal{U}^p$  denote the set of all half infinite or infinite open paths, i.e., these are functions  $\gamma : [s, \infty) \rightarrow \frac{1}{2}\mathbb{Z}$  with  $s \in \mathbb{R}$  or  $\gamma : \mathbb{R} \rightarrow \frac{1}{2}\mathbb{Z}$  that satisfy the conditions listed in Section 1.3. We let  $\overline{\mathcal{U}}^p$  denote the closure of  $\mathcal{U}^p$  in  $\Pi^\uparrow$  and we let  $\Theta_p : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  denote the scaling map

$$\Theta_p(x, t) := (px, \frac{1}{2}p^3t),$$

which we extend continuously to the squeezed space  $\mathcal{R}(\overline{\mathbb{R}})$ . We make the following conjecture.

**Conjecture 5.8 (Convergence of the covering branching-coalescent)**

*Let  $\mathcal{U}^p$  be the collection of half-infinite or infinite open paths in the graphical representation of the covering branching-coalescent with parameter  $p$ . Then*

$$\mathbb{P}[\Theta_p(\overline{\mathcal{U}}^p) \in \cdot] \xrightarrow[p \rightarrow 0]{} \mathbb{P}[\mathcal{N} \in \cdot],$$

where  $\mathcal{N}$  is the standard Brownian net.

**Proof (idea)** Because of the nearest-neighbour nature of the covering branching-coalescent, open paths in its graphical representation have a non-crossing property: two open paths that change their order must for some

time coincide. As a result, all open paths starting at a given point are contained between the left-most and right-most open paths starting at such a point. Because of this, known techniques for proving convergence to the Brownian net should be available.

To see that the limit is the standard Brownian net, it should therefore suffice to check that rescaled right-most paths converge to Brownian motions with drift one. Let  $(R_t)_{t \geq 0}$  be a right-most path in the graphical representation. Then

$$M_t := R_t - \frac{1}{2}p \int_0^t 1_{\{R_s \in \mathbb{Z} + \frac{1}{2}\}} ds$$

is a martingale, since the only asymmetric jumps are due to the splitting, which happens with rate  $p$  when  $R_t \in \mathbb{Z} + \frac{1}{2}$  and leads to an increment of size  $\frac{1}{2}$ . The quadratic variation of this martingale is

$$\langle M \rangle_t = \frac{1}{4}p \int_0^t 1_{\{R_s \in \mathbb{Z}\}} ds + \frac{1}{4} \int_0^t 1_{\{R_s \in \mathbb{Z} + \frac{1}{2}\}} ds,$$

since  $R_t$  makes random walk jumps with rate  $p$  while it is at an integer position, and, due to the combined effect of random walk jumps and splitting, jumps with rate  $(1 - p) + p = 1$  while it is at a half integer position, and the square of the jump size is  $1/4$  for all types of jumps. When  $t$  is large,  $R_t$  spends approximately a  $p$  fraction of its time at half integer positions, so we see that

$$\mathbb{E}[R_t] \approx \frac{1}{2}p^2 t \quad \text{and} \quad \text{Var}(R_t) \approx \frac{1}{2}pt \quad \text{for large } t,$$

which gives

$$\mathbb{E}[pR_{2t/p^3}] \approx t \quad \text{and} \quad \text{Var}(pR_{2t/p^3}) \approx t,$$

i.e., after rescaling with  $\Theta_p$ , right-most paths should converge to Brownian motions with drift  $+1$ .  $\blacksquare$

We next turn our attention to the scaling limit of the BAB, which is much more challenging. It is easy to see that the covering branching-coalescent and the BAB, started with a single particle, behave very similarly up to the first splitting of a particle of the covering branching-coalescent or the first time the BAB has three particles. Indeed, as long as the BAB has at most two particles, it behaves essentially in the same way as the covering branching-coalescent, when we interpret two particles of the BAB at positions  $i$  and  $i + 1$  as a single particle of the covering branching-coalescent at  $i + \frac{1}{2}$ .

To compare the two processes after the first splitting of a particle of the covering branching-coalescent, we must determine the “effective” branching rate of both processes. Particles of the covering branching-coalescent can split only when they are at half integer positions, in which case the splitting

rate is  $p$ . After splitting, a particle has been replaced by two particles at distance 1 from each other. If we ignore further branching, then the distance between these particles is a martingale with absorption at zero. Therefore, the probability that the two particles manage to separate to a distance  $L$  is approximately  $1/L$ . When the BAB contains two particles next to each other, which corresponds to a particle in the covering branching-coalescent at a half integer site, then these two particles produce with rate  $p$  a third particle directly the left or right of the two existing particles. When  $p$  is small, briefly after this, either the middle particle or one of the outer particles dies. If the middle particle dies, then this results in two particles at distance 2 of each other. When  $p$  is small, the distance between these particles is approximately a martingale as long as there is at least one free site between them. In fact, we can approximate them by two particles on  $\frac{1}{2}\mathbb{Z}$  that behave as in the covering branching-coalescent, except that when they come within distance  $3/2$  of each other, very quickly, with equal probabilities, their distance either increases again to 2 or the two particles coalesce. We see from this that effectively, the distance is a martingale with absorption at 1. As a result, the probability that the two particles manage to separate to a distance  $L$  is approximately  $1/(L - 1)$ .

We see from this that a particle in the BAB produces three particles at neighbouring positions at the same rate at which the covering branching-coalescent splits; these three particles then have a probability  $1/2$  to kill the middle one, after which the two remaining particles have roughly the same probability to separate to a large distance  $L$  as the two particles of the covering branching-coalescent. The upshot of this is that the scaling limit of the BAB should be similar to the scaling limit of the covering branching-coalescent, except that the “effective” branching rate (or equivalently the drift of the right-most particle) should be a factor  $1/2$  smaller. We can compensate this by scaling space by another factor  $1/2$  and time by  $(1/2)^2$ . This motivates Conjecture 1.7, which we can now formulate more precisely by replacing the space  $\text{Cl}(\mathbb{R})$  of closed subsets of  $\mathbb{R}$  by the space  $\mathcal{K}(\overline{\mathbb{R}})$  of compact subsets of  $\overline{\mathbb{R}}$ , and specifying that in (1.33) we mean weak convergence of probability measures on the space of continuous functions from  $[0, \infty)$  into  $\mathcal{K}(\overline{\mathbb{R}})$ , equipped with the topology of locally uniform convergence.

## 5.4 Relevant separation points

In this final section, we collect some ideas that could possibly be used to prove Conjecture 1.7. The main idea is to show that in a space-time box with spatial size of order  $p^{-1}$  and temporal size of order  $p^{-3}$ , the BAB can be

closely controlled in terms of the covering branching-coalescent, in the sense that for each particle of the BAB, there is a nearby particle of the covering branching-coalescent. Moreover, at each instance where a particle of the covering branching-coalescent splits, the corresponding particle of the BAB has a probability of  $1/2$  to also split, while with the remaining probability it stays close to one of the outgoing particles of the covering branching-coalescent. In the limit, space-time points where a particle of the branching-coalescing point set splits into two particles correspond to *separation points* of the Brownian net, which we now discuss.

By definition, for given times  $s < u$ , an  $(s, u)$ -relevant separation point of the Brownian net  $\mathcal{N}$  is a space-time point  $(x, t) \in \mathbb{R}^2$  with  $s < t < u$  such that there exist paths  $\pi, \pi' \in \mathcal{N}$  with  $\sigma_\pi = s = \sigma_{\pi'}$ ,  $\pi = \pi'$  on  $(s, t)$ ,  $\pi(t) = x = \pi'(t)$ , and  $\pi < \pi'$  on  $(t, u)$ . We simply call  $(x, t)$  a *separation point* if it is an  $(s, u)$ -relevant separation point for some  $s < t < u$ . The following lemma gives an alternative characterisation of  $(s, u)$ -relevant separation points that is surprisingly symmetric with respect to webs and dual webs. See Figure 5.1 for an illustration.

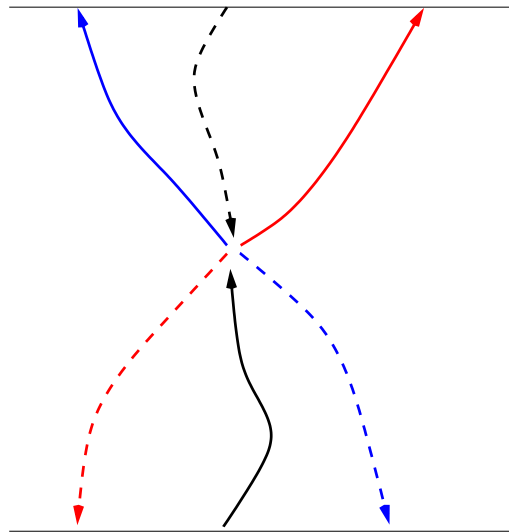


Figure 5.1: A relevant separation point

**Lemma 5.9 (Relevant separation points)** *Let  $\mathcal{N}$  be a standard Brownian net, let  $(\mathcal{W}^l, \mathcal{W}^r)$  be the associated left-right Brownian web, and let  $(\hat{\mathcal{W}}^l, \hat{\mathcal{W}}^r)$  be their associated dual Brownian webs. Then for each  $s, u \in \mathbb{R}$  with  $s < u$ , a point  $z = (x, t) \in (s, u) \times \mathbb{R}$  is an  $(s, u)$ -relevant separation point if and only if*



if there exist  $\pi^l \in \mathcal{W}^l(z)$ ,  $\pi^r \in \mathcal{W}^r(z)$ ,  $\hat{\pi}^l \in \hat{\mathcal{W}}^l(z)$ , and  $\hat{\pi}^r \in \hat{\mathcal{W}}^r(z)$ , such that  $\pi^l < \pi^r$  on  $(t, u)$  and  $\hat{\pi}^r < \hat{\pi}^l$  on  $(s, t)$ .

**Proof (sketch)** This follows roughly from the same arguments as Corollary 4.6, using moreover [SSS09, Thm 1.12(d)] to see that the concatenation of an incoming path of  $\mathcal{N}$  with either  $\pi^l$  or  $\pi^r$  is a path in  $\mathcal{N}$ . ■

We cite the following fact from [SSS09, Prop. 2.9]

**Proposition 5.10 (Density of relevant points)** *For each deterministic  $s, u \in \mathbb{R}$  with  $s < t$ , almost surely, the set of  $(s, u)$ -relevant separation points is a locally finite subset of  $\mathbb{R}^2$ .*

**Proof (rough idea)** By calculating the probability of the event that a left and right path, started closely together at time  $t$ , do not meet during the time interval  $(t, u)$ , and of the independent event that a dual left and dual right path, started closely together at time  $t$ , do not meet during the time interval  $(s, u)$ , it is possible to derive an explicit formula for the density of  $(s, u)$ -relevant separation points as a function of  $t \in (s, u)$ . This density diverges as  $t$  approaches  $s$  or  $u$ , but in an integrable way, so that the total number of  $(s, u)$ -relevant separation points is finite in each compact subset of  $\mathbb{R} \times [s, u]$ . ■

In particular, if we start the covering branching-coalescent with a single particle and run it up to a final time of order  $p^{-3}$ , then in view of Proposition 5.10, for small  $p$ , there should only be finitely many space-time points where the splitting of a particle has a significant effect on the state of the process at the final time.<sup>1</sup> The hope is that we only need to give a precise description of the coupling between the BAB and its covering branching-coalescent at this finite collection of relevant separation points, while (hopefully) in the rest of space-time more crude arguments suffice to show that the BAB cannot wander too far away from its covering branching-coalescent. In particular, for this last statement, one only needs upper bounds on the drift of the right-most particle, which are easier to obtain than lower bounds.

---

<sup>1</sup>Here we are implicitly using that separation points of the Brownian net arise as limits of a single branching point of the approximating branching-coalescent. For the usual discrete approximation of the Brownian net, this has indeed been proved in [SSS14, Prop. 6.14].



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