Interacting Particle Systems

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Contents

These lecture notes are based on the second half of the master course *Introduction to Statistical Mechanics* that I taught together with Beatrice De Tillière at University Dauphine - PSL during fall 2021 (Master 2 MATH-PSL).

Many misprints are most likely to be present: I will be very happy to have any feedback/corrections (toninelli@ceremade.dauphine.fr)!

Chapter 1

INTRODUCING IPS

Interacting particle systems (IPS for short) are systems composed of a large or infinite number of particles living on a lattice and evolving as a whole as a Markov process.
To define an IPS we have to choose a (finite or infinite) lattice, namely a countable vertex set Λ and edge set E, and a finite local state space S. In these notes we will deal always with the case in which S contains only two possible states, more precisely $S = \{0, 1\}$ (or $S = \{\pm 1\}$ for the stochastic Ising model) and $\Lambda = \mathbb{Z}^d$. We will denote by $\eta \in S^\Lambda$ the configurations and by $(\eta_t)_{t \geqslant 0}$ the Markov process on the space S^Λ with elementary moves corresponding to the modification of the configuration on a finite number of sites. Moves occur at a rate depending on the configuration on a certain neighbourhood of the to-be-updated sites *so dynamics is random and follows local rules*. Due to these interactions, the single particle evolution is not Markovian.

There is a strong connection among IPS and statistical mechanics. As you have learned in the first half of the course, statistical mechanics studies the collective behavior of systems composed of a large number of particles (atoms, molecules, droplets, grains...) with the goal to understand the macroscopic laws by using a probabilistic model encoding the microscopic interactions. In particular, you learned that via statistical mechanics we can explain the fascinating phenomenon of phase transitions: a small variation of the parameter tuning the intensity of the interactions may correspond a non smooth variation fo the macroscopic laws. For example the Ising model accounts for the fact that ferromagnetic materials display a critical temperature above which the model is in the disordered paramagnetic phase and below which it is in the ferromagnetic phase featuring a long range order. For the Ising model this corresponds to the fact that below the critical temperature the Gibbs measure that describes the equilibrium of the model is not unique: there are two Gibbs measures corresponding to the state with positive and with negative magnetisation.

IPS were first introduced with the aim of defining a stochastic dynamics whose large time measure would coincide with the Gibbs measure of statistical mechanics models. More precisely they were constructed to address a key issue for statistical mechanics: the one of efficiently sampling the Gibbs measure in the presence of a phase transition (i.e. when they display complicate long range correlations). In fact the first IPS to be introduced and studied (starting from the seminal works of Dobrushin and Spitzer in the 70's) is the *stochastic Ising model*, an IPS whose stationary distributions, as we shall see, coincide with the Gibbs measures for the Ising model.
After these pioneering works the field of IPS rapidly expanded with the introduction of many other models. It was immediately clear that the strength of these models was not only that they allow to sample the equilibrium Gibbs measure, but also to *follow the evolution of physics systems out of equilibrium* (either in the pre-asymptotic regime of approach to equilibrium or for systems constantly driven out of equilibrium, e.g. by some boundary conditions). Indeed, though real systems evolve according to deterministic laws, a stochastic description is well suited in the presence of a large number of microscopic components, due to the fact that following the deterministic laws is impossible and the knowledge of the initial configuration inevitably contains some alea. Furthermore, IPS rapidly turned out to have interesting applications as *models of collective complex behavior* in many other fields besides physics, including *biology* (models for spread of infections), *social sciences* (e.g. opinion dynamics models) and *economics*.
So, though IPS were born as auxiliary models in the framework of statistical mechanics, they rapidly evolved as an independent field at the border among probability theory and various fields of applied mathematics.

In these 5 lectures my aim is to give an introduction to IPS. *change here* We will start by constructing the processes today, then we will focus on two models: the stochastic Ising model and the contact process. Studying these two models we will have the occasion to meet some of the tools that have been developed for IPS in particular coupling and duality, and to review the basic issues: determining the large time behavior, the invariant laws and their domain of attraction, the speed of convergence to equilibrium, ...
Many important facets of the IPS field will not be covered by this mini-course. A crucial missing part are *scaling limits*, which link the evolution of the microscopic discrete stochastic IPS with some macroscopic continuous equations. These are either PDE or stochastic PDE depending on whether one is looking at the law of large numbers or the central limit theorem scaling. This

part has been in particularly very much developed for the so called exclusion type IPS, those for which particles are locally conserved, namely elementary moves correspond to jump of particles. The interested reader may have a look at the classic books [Spo91] or [KL99] to have an idea of the vastness of this subject.

This first lecture will probably be the more boring one because our main job today is to give a somewhat formal construction of the Markov process. So to give it a more funny start and motivate you I will first carry you through an overview of different examples of IPS.

1.1 An informal definition of the most popular IPS

1.1 Contact process (CP)

CP is a model of spread of infection. The on-site configuration space is $S = \{0, 1\}$, with 0 (resp. 1) representing healthy (resp. infected) individuals. Here

- Infected individuals become healthy after an exponential time of mean 1, independently of the others
- an healthy individual at site x in configuration η becomes infected after an exponential time of mean $1/(\lambda N_x(\eta))$ with $\lambda \geqslant 0$ a parameter that is called the *infection rate* and N_x the number of infected nearest neighbours of x.

It is easily seen that if the initial configuration contains only healthy individuals we will always have only healthy individuals, namely the measure concentrated on the completely empty configuration, that we shall call δ_0, is an invariant law.

Q. What happens if we start with some infections?
the answer depends on the value of the infection rate λ. We will see that for any $d \geqslant 1$, CP on $\Lambda = \mathbb{Z}^d$ undergoes a phase transition, namely there exists $\lambda_c(d)$ such that

- for $\lambda < \lambda_c(d)$: δ_0 is the unique invariant measure and all initial measures are attracted to δ_0;
- for $\lambda > \lambda_c(d)$: there are other invariant measures besides δ_0. We shall see that an important role is played by the measure towards which the process is attracted starting from all infected individuals.

1.2 Voter model (VM)

VM is a model of opinion spread. $S = \{0, 1\}$, with 0 and 1 representing voters for two different parties, say 0 is a republican voter, 1 a democrat voter. Here the dynamics evolves as follows: after an exponential time of mean 1, the voter at site x chooses uniformly at random one of its neighbours and adopts its opinion.

From the above definition it follows immediately that for VM both δ_0 and δ_1 are invariant measures, where we denote by δ_0 (resp. δ_1) the measure concentrated on the configuration with all sites 0 (resp. all sites 1).

Q. What happens if $\Lambda = \mathbb{Z}^d$ and we start from a mixture of opinions? can we preserve a mixture of opinions or are we deemed to a totalitarian situation?

The answer strongly depends on the spatial dimension

- for $d = 1, 2$: δ_0 and δ_1 are the only two extremal invariant measures: the process is always attracted to a single opinion state
- for $d \geqslant 3$ there is a whole family of extremal invariant measures (that are ergodic under translations): a mixture of opinions can survive.

1.3 The Stochastic Ising model (SIM)

SIM is a model for magnetism that has been introduced in 1963 by Glauber and very much studied since the seminal works of Dobrushin in the 1970s. The usual convention is to let the onsite space state be $S = \{+1, -1\}$ [1]. Here sites represents atoms in a ferromagnetic material, e.g. iron, and ± 1 are the two possible orientations (up and down) of the spin on each atom. The elementary moves of the dynamics are spin flips and the rates are chosen to take into account the fact that a spin "prefers" to be aligned with its nearest neighbours. More precisely the spin at site x in configuration η changes its value with rate

$$e^{-\beta \sum_{y \sim x} \eta(x)\eta(y)} \equiv e^{-2d\beta + 2\beta \tilde{N}_x(\eta)}$$

where $\tilde{N}_x(\eta)$ is the number of spins n.n. to x and with spin not aligned with x and β corresponds to the inverse temperature. Notice that

- the larger β, the strongest the bias to align spins

[1] One can of course easily rephrase the model by letting $S = \{0, 1\}$

- the higher the number of non aligned neighbours, the highest the flip rate
- if $\beta = 0$ (= infinite temperature) SIM is an independent spin dynamics with a unique invariant measures, the product measure with $\mu_x(+1) = \mu_x(-1) = 1/2$
- δ_{+1} and δ_{-1} are no more invariant laws.

We will see that the invariant measures of SIM coincide with the Gibbs measure of the Ising model. Then, the results on the Ising model that were presented in the first part of the course imply that

- in $d = 1$ SIM has a unique invariant measure ;
- for $d \geqslant 2$ there is $\beta_c(d)$ separating the regime ($\beta < \beta_c$) in which we have a unique invariant measure and the regime ($\beta > \beta_c$) in which uniqueness is broken;
- the limit as $t \to \infty$ of the expectation under the process for $\eta_t(0)$ starting from the up configuration (i.e. from η s.t. $\eta(x) = 1$ for all $x \in \Lambda$), corresponds to the spontaneous magnetisation of the Ising model.

An alternative interpretation of the Ising model is as a model for collective decision making. Each site is a person that has to decide his (binary) state. It does so according to a utility function: if we set $\beta > 0$ it is more advantageous to make the same choice as the neighbour we take, instead for $\beta < 0$ it is more advantageous to make the opposite decision [2].

1.4 Friedrickson-Andersen 1 spin facilitated model (FA-1f) and other KCM

FA-1f is an IPS used to model the liquid glass transition, occurring for dense, low temperature liquids when we approach the dynamical arrest to the amorphous solid glass state. Here $S = \{0, 1\}$: 0 represents facilitating sites, i.e. regions that are not dense and thus facilitate motion, 1 represent highly packed regions. The dynamics evolves as follows: each site waits the ring of an exponential clock of mean time one and then "tries" to update its value. I say "tries", because when the clock on site x rings, before updating the configuration at x we have to check whether a certain *local constraint* is satisfied: at least 1 of the nearest neighbours of x should be empty. Then

- if the constraint is satisfies the configuration at x is updated to 0 at rate q and to 1 at rate $1 - q$ and we go to the next clock ring

[2] In the physics interpretation the choice $\beta < 0$ is also meaningful: it models antiferromagnetic materials

- otherwise no update occurs and we go to the next clock ring

Notice that

- the rate to update the configuration on a given site does not depend in the configuration on that site, but only on the state of its neighbours (at variance with SIM);
- the completely filled configuration is blocked, so δ_1 is an invariant measure;
- the completely empty configuration is not an invariant measure (unless $q = 1$);

FA-1f model belongs to a class of IPS called the *kinetically constrained models or KCM*. These can be obtained by varying the choice of the constraint that allows the update (changing the neighbourhood, changing the threshold value..). The only requirement is that the constraint has finite range and does not depend on the configuration on the to-be-updated site. For example, two other very much studied KCM are

- the East model on $\mathbb{Z}$ for which the constraint to update x requires $x + 1$ to be empty
- the FA-2f model on $\mathbb{Z}^d$ with $d \geqslant 2$ for which the constraint to update x requires at least 2 empty nearest neighbours.

Exercise 1. *FA-1f has another invariant measure besides δ_1. Try to guess which one.*
[**Hint.** *The measure you are looking for is also an invariant measure for East model and FA-2f model and actually for any other KCM, namely it is constraint independent (but it depends on q).*
If you don't manage find the invariant measure, I suggest retrying after Exercise 14.]

1.5 Simple Symmetric Exclusion process (SSEP)

SSEP is a models in which particles can move and never disappear (it is conservative). $S = \{0, 1\}$, 1 are particles and 0 are empty sites. After an exponential time of mean 1, a particle chooses uniformly at random a nearest neighbours and "tries" to jump there. I say "tries" because it has to check whether the arrival site is empty (i.e. to satisfy the exclusion constraint). If it is the case the jump occurs, otherwise the particle does not change position. Here

- δ_0 and δ_1 are invariant measures;
- for any density parameter $\rho \in [0, 1]$ the product measure with probability ρ that a site is filled is also an invariant measure.

The name of this model comes from the following feature:

- simple = jumps to nearest neighbours;
- symmetric= equal rate to jump to any of the empty nearest neighbours;
- exclusion : occupancy by a multiple number of particle is not allowed

Several variations of SSEP have been considered: long jumps, non symmetric rates (ASEP), totally asymmetric rates (TASEP), ...

1.6 Other notable examples

- Potts model: this is a more sophisticated version of SIM with $q \geqslant 2$ states. Potts model for $q = 2$ corresponds to SIM;
- the biased voter model. As for VM $S = \{0, 1\}$. Here $1 \to 0$ with rate equal to the fraction of 0 neighbours but $0 \to 1$ with rate $(1+s)$ times the fraction of 1 neighbour with $s > 0$. This model is relevant as a model of evolution of two genetic types. At rate one an organism dies and it is replaced by a clone of one of its nearest neighbour chosen randomly but not uniformly (as for VM) but with a bias favouring type 1. Here, even starting with a single 1 if s is sufficiently high 1's might survive.
- reaction diffusion models: coalescing random walks (walkers evolving as independent r.w. that coalesce if the meet), branching and coalescing r.w., ...

1.2 Setting some notation

In this course we will focus on *spin IPS*. We call *spin IPS* or stochastic Glauber dynamics an IPS with an on-site binary configuration space, $|S| = 2$, and a dynamics such that each elementary transition involves a single site variable that changes its value. All the examples of the previous section are spin models, except for SSEP (whose transitions involve two neighbouring sites), Potts with $q \geqslant 2$ (the on-site configuration space is not binary) and reaction diffusion models.

Recall that Λ is the countable vertex set of the lattice, which in the cases we will consider will be $\mathbb{Z}^d$. We denote by X the configuration space, $X := S^\Lambda$, and use the greek letters σ and η to denote configurations, i.e. elements of X. Given a lattice site $x \in \Lambda$ and configuration η, we denote by $\eta(x)$ the configuration at site x. We also adopt the notation $x \sim y$ to say that

$|x - y| = 1$, or there is an edge on the graph from x to y.

For spin IPS the possible elementary moves are of the type $\eta \to \eta^x$ where η^x is the configuration with the variable at x changed:

- if the onsite configuration space is $S = \{0, 1\}$ we let

$$\eta^x(y) = \begin{cases} 1 - \eta(y) & \text{if } y = x \\ \eta(y) & \text{if } y \neq x \end{cases} \tag{1.2.1}$$

- if the onsite configuration space is $\{-1, +1\}$ (SIM) we let

$$\eta^x(y) = \begin{cases} -\eta(y) & \text{if } y = x \\ \eta(y) & \text{if } y \neq x \end{cases} \tag{1.2.2}$$

For SSEP transitions involve the exchange of the variables on two sites. Here the elementary moves are of the type $\eta \to \eta^{x,y}$ with

$$\eta^{xy}(z) = \begin{cases} \eta(y) & \text{if } z = x \\ \eta(x) & \text{if } z = y \\ \eta(z) & \text{otherwise} \end{cases} \tag{1.2.3}$$

With this notation, we can summarise the informal definitions of the previous section by saying that when the system is in configuration η, it flips to η^x after an exponential time of mean $1/r(x, \eta)$ with

$$r(x, \eta) = e^{-\beta \sum_{y, y \sim x} \eta(x)\eta(y)} \quad \text{for SIM at inverse temperature } \beta \tag{1.2.4}$$

$$r(x, \eta) = \begin{cases} 1 & \text{if } \eta(x) = 1 \\ \lambda \sum_{y, y \sim x} \eta(y) & \text{if } \eta(x) = 0 \end{cases} \quad \text{for CP of infection rate } \lambda \tag{1.2.5}$$

$$r(x, \eta) = \frac{1}{2d} \sum_{y, y \sim x} 1_{\eta(y) \neq \eta(x)} \quad \text{for VM} \tag{1.2.6}$$

and

$$r(x, \eta) = c_x(\eta)(q\eta(x) + (1 - q)(1 - \eta(x)) \quad \text{for KCM of parameter } q \tag{1.2.7}$$

with

$$c_x(\eta) = (1 - \prod_{y, y \sim x} \eta(y)) \quad \text{for FA-1f} \tag{1.2.8}$$

$$c_x(\eta) = (1 - \eta(x+1)) \quad \text{for East} \tag{1.2.9}$$

and for FA-2f $c_x(\eta) = 1$ if $\sum_{y:y\sim x}(1-\eta_x) \geqslant 2$ and $c_x(\eta) = 0$ otherwise. Instead, for SSEP η is updated to η^{xy} at rate $r(x, y, \eta)$ with

$$r(x, y, \eta) = \begin{cases} \frac{1}{2d} \mathbb{1}_{\eta(x) \neq \eta(y)} & \text{if } x \sim y \\ 0 & \text{otherwise} \end{cases} \tag{1.2.10}$$

Chapter 2

CONSTRUCTING IPS

We will now proceed to construct a continuous time Markov process $(\eta_t)_{t\geqslant 0}$ with $\eta_t \in X$ that evolves according the IPS dynamics informally stated in the previous chapter. We will do this for simplicity of notation only for spin IPS (for exclusion type processes like SSEP the procedure is similar). Informally, we wish to construct a Markov process that satisfies

$$P(\eta_{t+\delta} = \eta^x | \eta_t = \eta) = \delta r(x,\eta) + o(\delta) \tag{2.0.1}$$

If the lattice Λ is finite, it is not difficult to check that such a construction is feasible for any choice of the rates, provided the rates are finite. However, if $|\Lambda|$ is infinite, the process might not be well defined due to the fact that many spin flip might occur at the same time. Indeed we will see that in order for the process to be well defined we should impose proper conditions not only on the boundedness of the rates but also on their range, i.e. on their spatial support.

2.1 The finite volume case: Poisson (or graphical) construction

Let's proceed step by step and start by formally constructing a Markov process which satisfies (2.0.1) when Λ is finite.

1.1 CP on finite volume

Let's consider for simplicity the case of the contact process, whose rates are defined in (1.2.5).

Recall that $X = \{0,1\}^\Lambda$ and define a set $\mathcal{M} = \{\mathcal{H}_x\}_{x\in\Lambda} \cup \{\mathcal{I}_{x,y}\}_{x,y\in\Lambda, y\sim x}$, as follows

- $\mathcal{H}_x : X \to X$ is the transformation that heals site x namely sets its value to 0 and leaves the other sites unchanged. Namely $\mathcal{H}_x\eta(k) = \eta(k)$ if $k \neq x$ and $\mathcal{H}_x\eta(y) = 0$.

- $\mathcal{I}_{x,y} : X \to X$ is the transformation that infects y if x is infected, otherwise it does nothing. Namely $\mathcal{I}_{x,y}\eta(k) = \eta(k)$ if $k \neq y$ and $\mathcal{I}_{x,y}\eta(y) = \max(\eta(y), \eta(x))$.

We associate to each map $m \in \mathcal{M}$ a sequence of i.i.d random variables $(\sigma_m^{(k)})_{k \geqslant 1}$ that are exponentially distributed and of mean $1/r_m$, where

$$r_{\mathcal{H}_x} = 1, \;\; \forall x \in \Lambda$$

$$r_{\mathcal{I}_{x,y}} = \lambda \;\; \forall x, y \in \Lambda, y \sim x$$

Then we define for $m \in \mathcal{M}$ the random times $(t_m^{(i)})_{i \geqslant 1}$

$$t_m^{(i)} := \sum_{k=1}^{i} \sigma_k$$

that we call *arrival times* of the map m. Note that $(t_m^{(i)})_{i \geqslant 1}$ form a Poisson point set in $[0, \infty)$ of intensity $r_m dt$ with dt the Lebesgue measure [1].

We call Δ the collection over all the maps in $\mathcal{M}$ of these arrival times (which is therefore a Poisson point set on $\mathcal{M} \times [0, \infty)$), namely

$$\Delta := \cup_{m \in \mathcal{M}} \{t_i(m)\}_{i \in \mathbb{N}}.$$

We also let, for each $0 \leqslant s \leqslant t$,

$$\Delta_{s,t} := \Delta \cap (\mathcal{M} \times (s, t]).$$

With probability one, thanks to the fact that $|\mathcal{M}| < \infty$ and $r_m < \infty$ for each m, $|\Delta_{s,t}|$ is finite for $t < \infty$. Given a finite realisation of $\Delta_{s,t}$ we re-order it in increasing order of the arrival times

$$\Delta_{s,t} := \{(m_1, \tau_1), \ldots (m_n, \tau_n)\}, \quad \text{with} \;\; \tau_1 \leqslant \ldots \leqslant \tau_n$$

Let $\psi_{\Delta_{s,t}} : X \to X$ be the composition of the maps in reverse order

$$\psi_{\Delta_{s,t}}(\eta) := m_n \cdot \cdots \cdot m_1(\eta)$$

[1] For $\Omega, \mathcal{F}, \mu$ a measurable space with μ non atomic and σ-finite, a Poisson point set is a random subset such that for each A, $|\omega \cap A|$ is Poisson distributed with mean $\mu(A)$ for any A s.t. $\mu(A) < \infty$ and if A_i disjoin, then $|\omega \cap A_1|, \ldots |\omega \cap A_n|$ are independent. Sicne μ is non atomic for each $\epsilon > 0$ we can find A^ϵ s.t. $P[|\omega \cap A^\epsilon| = 1] = \mu(A^\epsilon) + O(\epsilon^2)$ and $P[|\omega \cap A^\epsilon| \geqslant 2] = O(\epsilon^2)$

with the convention $\psi_{\Delta_{s,t}} = \mathbb{1}$ if $\Delta_{s,t} = \emptyset$.

The easiest way to understand this definition is by making a drawing as in Fig. 2.1: on the column over site x I mark with a *cross* each arrival time of $\mathcal{H}_x$, and with an *arrow* from $x \to x+1$ each arrival times of $\mathcal{I}_{x,x+1}$ and with an arrow from $x \to x-1$ each arrival times of $\mathcal{I}_{x,x-1}$.
We are now ready to construct the IPS.

Theorem 2.1.1. *Let $\eta \in X$ and set*

$$\eta_t^\eta := \psi_{\Delta_{0,t}}(\eta), \ \ t \geqslant 0.$$

Then

- $(\eta_t^\eta)_{t \geqslant 0}$ *is a Markov process on the space $\mathcal{D}_X[0,\infty]$ of cadlag functions from $[0,\infty)$ to X with initial condition $\eta_0^\eta = \eta$*
- *if we denote by $\mathbb{E}^\eta$ the mean over this process it holds*

$$\lim_{t \to 0} \frac{\mathbb{E}^\eta(f(\eta_t)) - f(\eta)}{t} = \sum_{m \in \mathcal{M}} r_m(f(m(\eta)) - f(\eta)) = \sum_x r(x,\eta)(f(\eta^x) - f(\eta)). \quad (2.1.1)$$

The above theorem proves in particular that the process satisfies the informal condition (2.0.1). Here and in the following, when confusion does not arise, we let $\eta_t^\eta = \eta_t$ for simplicity of notation.

Proof. By definition η_t has paths that are cadlag (right continuous and left limited). So to prove it is a Markov process we have to prove that the Markov property holds, namely that

$$\mathbb{E}^\eta(f(\eta_t)|\mathcal{F}_s) = \mathbb{E}^{\eta_s} f(\eta_{t-s}) \quad (2.1.2)$$

where for all $s \geqslant 0$, $\mathcal{F}_s$ is the σ-algebra

$$\mathcal{F}_s := \sigma(\eta'_s : s' \in [0,s]).$$

Note that, for $s \leqslant t$, it holds by definition $\mathcal{F}_s \subset \mathcal{F}_t$ so that $(\mathcal{F}_t)_{t \geqslant 0}$ is a filtration. Thanks to the independence of the sets of arrival times on distinct time intervals and the independence from the initial configuration of the arrival times, (2.1.2) can be easily proven. We are left with

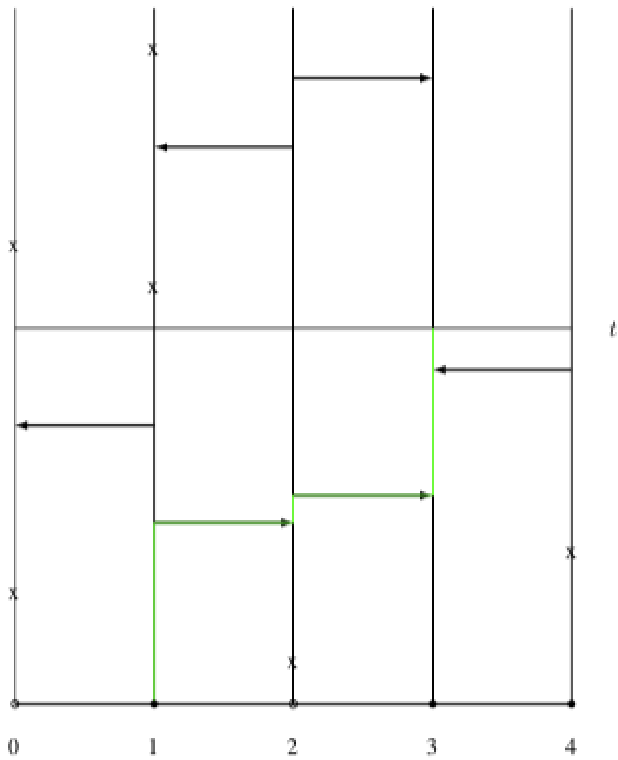

Figure 2.1: Graphical construction for CP in $d = 1$ on the finite volume $\Lambda := [0, 4]$. Here $\psi_{\Delta_{0,t}} = \mathcal{I}_{4,3}\,\mathcal{I}_{1,0}\,\mathcal{I}_{2,3}\,\mathcal{I}_{1,2}\,\mathcal{H}_4\,\mathcal{H}_0\,\mathcal{H}_2$. If we let η be the configuration depicted in the figure (were filled circles stand for infected sites and empty circles for healthy sites), we have $\psi_{\Delta_{0,t}}(\eta) = \eta'$ with $\eta'(x) = 1$ for $x \in [0, 3]$ and $\eta'(4) = 0$. We highlight in green the path of influence from $(1, 0)$ to $(3, t)$.

proving (2.1.1). From the condition on the finiteness of the sum of rates, it follows that the probability of having two arrival times in a time interval t is $O(t^2)$. This implies that

$$E^\eta(f(\eta_t)) = f(\eta) + t \sum_{m\in\mathcal{M}} r_m(f(m(\eta)) - f(\eta)) + O(t^2) \tag{2.1.3}$$

which yields (2.1.1). The validity of (2.0.1) can be easily checked by setting $f = 1_{\eta^x}$ in (2.1.1).

□

Exercise 2. *Check that you really understand the $O(t^2)$ in the formula* (2.1.3)*. If not*

1. *please revise (on your favourite probability textbook) what an exponential random variable is*
2. *now that you understand the meaning of the σ's try to extract the properties of the random variables corresponding to the arrival times that are defined as partial sums of the σ's*
3. *try again to see if you understand* (2.1.3)*.*

To be sure you fully understand what I mean by "Note that $(t_m^{(i)})_{i\geqslant 1}$ form a Poisson point set in $[0,\infty)$ of intensity $r_m dt$ " a very good reference is [Swaa] Section 1.4.

Definition 2.1.2 (Semigroup and Generator)**.** *We shall denote by $\mathbb{P}^\eta$ the law (on $\mathcal{D}_X[0,\infty]$) of the trajectory $(\eta_t^\eta)_{t\geqslant 0}$ and, as already stated in Theorem 2.1.1, by $\mathbb{E}^\eta$ the corresponding expectation.*
We also let $P_t : \mathcal{C} \to \mathbb{R}$ for $t \in [0,\infty)$ be the operator which acts on $B(X)$, the space if bounded real measurable functions[2] *on X, as follows*

$$(P_t f)(\eta) := \mathbb{E}^\eta(f(\eta_t)).$$

We also define the operator $\mathcal{L} : B(X) \to B(X)$ as

$$\mathcal{L}f(\eta) = \lim_{t\to 0} \frac{(P_t f)(\eta) - f(\eta)}{t} \tag{2.1.4}$$

Note that Theorem 2.1.1 implies that

$$\mathcal{L}f(\eta) = \sum_x r(x,\eta)(f(\eta^x) - f(\eta)). \tag{2.1.5}$$

We call $(P_t)_{t\geqslant 0}$ the semigroup of the process, and $\mathcal{L}$ the generator of the process. The definitions of these operators carry through to the infinite volume case.

[2] Measurable here means w.r.t. the Borel σ-field generated by the open subsets of X

1.2 IPS on finite volume: the general case

Exercise 3. *Show that all other IPS mentioned in Section 1.1 can be constructed on a finite volume* Λ *along the same lines as done above for CP. The difference will be the choice of the maps and of the associated rates.*

2.2 The infinite volume case: Poisson (or graphical) construction

We shall now extend the construction of the previous section to the infinite volume setting and see that it actually makes sense for all the models defined in section 1.1. Let Λ be an infinite volume, S a countable on-site configuration space, and $X = S^{\Lambda}$. Consider a countable set $\mathcal{M}$ of maps $m : X \to X$, and a set of bounded positive rates $\{r_m\}_{m\in\mathcal{M}}$. In analogy to the finite volume case we would like to construct a Markov process $(\eta_t)_{t\geqslant 0}$ with generator acting on local functions as

$$\sum_{m\in\mathcal{M}} r_m(f(m(\eta) - f(\eta)) = \sum_{x\in\Lambda} r(x,\eta)(f(\eta^x) - f(\eta)).$$

If we try to proceed as for the finite volume case, the first problem we encounter is that $\sum_m r_m = \infty$ so $\{t : (t,m) \in \Delta\}$ is dense in $\mathbb{R}$ and it is not more possible to order the elements of $\Delta_{s,t}$ according to their arrival times.

The key observation is to notice that the maps and the rates of the processes that interest us are defined in such a way that *with high probability only finitely many points of* $\Delta_{0,t}$ *are necessary to determine the value of the process at a given space time point* (x,t). Thus it will actually be possible to order these finely many relevant points according to their arrival times and proceed essentially as for the finite volume case.

In order to formalise the above observation we should introduce the notion of *path of influence*. For concreteness, we start by treating the case of CP on $\mathbb{Z}$, then we will extend the procedure to general models.

2.1 CP on $\mathbb{Z}$

Let $\mathcal{M} = \cup_{x\in\mathbb{Z}}\mathcal{H}_x \cup_{x,y\in\mathbb{Z},x\sim y} \mathcal{I}_{x,y}$ with $\mathcal{H}_x$ and $\mathcal{I}_{x,y}$ defined as for the finite volume CP. Fix $\Delta \in \mathcal{M} \times [0,\infty)$ a realisation of the Poisson point processes associated to these maps and draw the corresponding graphical construction as in Fig. 2.1 albeit for the infinite volume Λ. For any

$i, j \in \mathbb{Z}$ and $0 \leqslant s \leqslant u$ we say that there is a path of influence from (i, s) to (j, u) iff there is a path that

- grows vertically in time
- moves horizontally only following the direction of the arrows
- never meets a cross.

We denote by $(i, s) \to (j, u)$ (respectively $(i, s) \not\to (j, u)$) the event that a path of influence exists (resp. does non exist). For example, in Fig. 2.1 it holds $(2, 0) \not\to (3, t)$ and $(1, 0) \to (3, t)$. We also set, for any finite $A \subset \mathbb{Z}$

$$\xi_s^{A,u} := \{i \in \Lambda : (i, s) \to A \times \{u\}\}. \tag{2.2.1}$$

For example, in Fig. 2.1, for $s = 0$ and $A = \{3, 4\}$ it holds $\xi_s^{A,u} = \{1, 3\}$.

The following claim will play a key role

Claim 2.2.1. *For any finite $A \subset \mathbb{Z}$, it holds* [3]

$$\mathbb{E}[|\xi_s^{A,u}|] \leqslant |A| e^{2d\lambda(u-s)} \qquad 0 \leqslant s \leqslant u$$

For each $i \in \Lambda$ and $s \leqslant u$, the set

$$\{(m, t) \in \Delta_{s,u} : \mathcal{D}(m) \times \{t\} \to (i, u)\}$$

is finite almost surely.

Exercise 4. *Prove the above claim.* [**Hint.** *Use the definition of arrival times and paths of influence.*]

In view of Claim 2.2.1, for a fixed i and $0 \leqslant s \leqslant u < \infty$, we can order the relevant arrival times as

$$\{(m, t) \in \Delta_{s,u} : D(m) \times \{t\} \to (i, u)\} = \{(m_1, t_1), \dots (m_n, t_n)\}$$

with $t_1 < \cdots < t_n$. Then we define

$$\psi_{\Delta_{s,u}}(\eta)(i) = m_n \circ \cdots \circ m_1(\eta)(i)$$

[3] here the mean $\mathbb{E}$ is over the randomness in Δ

and, along the same line as for the finite volume case (Theorem 2.1.1), it follows that

$$p_t(\eta, \cdot) := \mathbb{P}(\psi_{\Delta,0,t}(\eta) \in \cdot)$$

is the transition kernel of a Markov process and it is a continuous map from $X \times [0, \infty) \to \mathcal{P}(X)$, with $\mathcal{P}(X)$ the space of probability measures on X. The continuity of the kernels implies that the Markov process correspondent to the collection of kernels $p_t(\eta, \cdot)_{t \geqslant 0}$ is Feller, namely if we let the semigroup $(P_t)_{t \geqslant 0}$ be defined as

$$P_t f(\eta) := \int_X p_t(\eta, d\eta') f(\eta')$$

it holds $P_t(f) \in \mathcal{C}(X)$ for any $f \in \mathcal{C}(X)$. As for the finite volume case we can also prove that the generator $\mathcal{L}$ correspondent to this semigroup (see Definition 2.1.2) acts on functions that depend on finitely many coordinate as

$$\mathcal{L} f(\eta) = \sum_{x \in \mathbb{Z}} r(x, \eta)(f(\eta^x) - f(\eta))$$

with $r(x, \eta)$ the rates of CP (1.2.5). Thus we have succeeded in constructing CP on the infinite volume $\mathbb{Z}$.

2.2 IPS on infinite volume: the general case

We should start by introducing the notion of *local* maps. Let $\mathcal{M}$ be a set of maps from $X \to X$. For $m \in \mathcal{M}$, we let $D(m) \subset \Lambda$ be the set of vertex whose value can be possibly changed by m, namely

$$D(m) := \{x \in \Lambda : \exists \eta \in X : \eta(x) \neq m(\eta)(x)\}.$$

Let also $R_i(m) \subset \Lambda$ be the sets of sites that are m-relevant for i, where for $j, i \in \Lambda$ we say that j is m-relevant for i if

$$\exists \eta \in X \text{ s.t. } m(\eta)(i) \neq m(\eta^j)(i).$$

For example for the maps used in Section 2.1 for CP it holds

- $D(\mathcal{H}_x) = \{x\}$, $R_y(\mathcal{H}_y) = \emptyset$, $R_z(\mathcal{H}_y) = \{z\}$ for $z \neq y$
- $D(\mathcal{I}_{x,y}) = \{y\}$,

$$R_z(\mathcal{I}_{x,y}) = \begin{cases} \{x, y\} & \text{if } z = y \\ z & \text{otherwise} \end{cases} \tag{2.2.2}$$

For the Voter Model (VM), whose rates as defined by equation (1.2.6), a natural choice of maps to make the Poisson construction is $\mathcal{M} = \{\mathcal{V}_{x,y}\}_{x\in\mathbb{Z}^d, y\sim x}$ with $r_m = 1/(2d)$ for each m and $\mathcal{V}_{x,y}$ defined as the transformation that sets the variable on y equal to $\eta(x)$ and leaves the other sites unchanged, namely

$$\mathcal{V}_{x,y}(\eta)(z) = \begin{cases} \eta(x) & \text{if } z = y \\ \eta(z) & \text{otherwise} \end{cases}. \tag{2.2.3}$$

For this maps it holds $D(\mathcal{V}_{x,y}) = \{y\}$,

$$R_z(\mathcal{V}_{x,y}) = \begin{cases} x & \text{if } z = y \\ z & \text{otherwise} \end{cases} \tag{2.2.4}$$

We say that a map m is *local* if $D(m)$ is finite and $R_i(m)$ is finite for all $i \in D(m)$.

Exercise 5. *Verify that the above maps for CP and for VM are local and that all the processes defined in Section 1.1 can be recast in term of local maps.*

We say that there is a path of influence $\gamma \subset \Lambda \times [s, u]$ from (i, s) to (j, u) (and denote this by if $(i, s) \to (j, u)$) if

- when $\gamma_{t-} \neq \gamma_t$ necessarily there exists $m \in \mathcal{M}$ with $(m, t) \in \Delta, \gamma_t \in \mathcal{D}(m), \gamma_{t-} \in \mathcal{R}_{\gamma_t}(m)$
- if $m, t \in \Delta$ for $t \in [s, u]$ and $\gamma_t \in \mathcal{D}(m)$ then $\gamma_{t-} \in \mathcal{R}_{\gamma_t}(m)$.

and we let for $A \subset \Lambda$ and $u \in \Lambda$ $(i, s) \to A \times \{u\}$ iff there is a path of influence from (i, s) to (j, u) for some $j \in A$. We also set

$$\xi_s^{A,u} := \{i \in \Lambda : (i, s) \to A \times \{u\}\}.$$

Exercise 6. *Verify that the above definition coincides with the one given previously for the specific case of CP on $\mathbb{Z}$.*

We are now ready to state two Lemma (that correspond to Claim 2.2.1 for CP on $\mathbb{Z}$) that are a key ingredient for the construction of the IPS on infinite volume

Lemma 2.2.2 (Exponential bound on paths of influence)**.** *Suppose that*

$$\sup_{i\in\Lambda} \sum_{m: i\in D(m)} r_m < \infty \tag{2.2.5}$$

and

$$K := \sup_{i\in\Lambda} \sum_{m:i\in D(m)} r_m(|R_i(m)|-1) < \infty. \tag{2.2.6}$$

Then, for any finite $A \subset \Lambda$, it holds

$$\mathbb{E}[|\xi_s^{A,u}|] \leqslant |A|e^{K(u-s)} \qquad 0 \leqslant s \leqslant u$$

Lemma 2.2.3 (Finitely many relevant clock rings)**.** *Suppose that condition* (2.2.5) *holds and furthermore*

$$K_1 := \sup_{i\in\Lambda} \sum_{m:i\in D(m)} r_m|R_i(m)| < \infty. \tag{2.2.7}$$

Then, for each $i \in \Lambda$ and $s \leqslant u$, the set

$$\{(m,t) \in \Delta_{s,u} : \mathcal{D}(m) \times \{t\} \to (i,u)\}$$

is finite almost surely.

A full proof of the two above Lemma can be found in [Swaa] Section 1.6. In view of Lemma 2.2.3, for a fixed i and $0 \leqslant s \leqslant u < \infty$, we can order the relevant arrival times as

$$\{(m,t) \in \Delta_{s,u} : D(m) \times \{t\} \to (i,u)\} = \{(m_1,t_1),\dots(m_n,t_n)\}$$

with $t_1 < \cdots < t_n$. Then, as for CP on $\mathbb{Z}$, we define

$$\psi_{\Delta,s,u}(\eta)(i) = m_n \circ \cdots \circ m_1(\eta)(i).$$

The following theorem, which concludes the Poisson construction of infinite volume IPS, can be proven using Lemma 2.2.2 and Lemma 2.2.3 along analogous lines are the one of Theorem 2.1.1 for the finite volume case (a complete proof may be found in [Swaa], see Theorem 1.15 therein).

Theorem 2.2.4. *Fix a countable collection $\mathcal{M}$ of local maps and $(r_m)_{m\in\mathcal{M}}$ non negative constants that satisfy condition* [4]

$$\sup_{x\in\mathbb{Z}^d} \sum_{m:x\in\mathcal{D}(m)} r_m(|R_x(m)|+1) < \infty). \tag{2.2.8}$$

[4] Note that condition (2.2.8) is the combination of conditions (2.2.5) and (2.2.7) (under which Lemma 2.2.3 holds) and it implies condition (2.2.6) of Lemma 2.2.2.

Let

$$P_t(\eta,\cdot) := \mathbb{P}(\psi_{\Delta,0,t}(\eta) \in \cdot)$$

with Δ *a Poisson point process on* $\mathcal{M} \times [0,\infty)$ *with intensity* $r_m dt$*. Then* $P_t(\eta,\cdot)$ *is the transition kernel of a Markov process with generator that acts on functions that depend on finitely many coordinate as*

$$\mathcal{L}f(\eta) = \sum_{m\in\mathcal{M}} r_m(f(m(\eta)) - f(\eta)).$$

Exercise 7. *Go back to exercise 5 and verify that the transition rates of the local maps that you constracted for FA-1f and SIM satisfy condition* (2.2.8).

Remark 2.2.5. *The Poisson construction provides not only a rigorous construction of IPS but also a very powerful tool to couple processes started in different initial configurations and/or evolving with different parameters. In case of processes started from different initial conditions, the idea is to couple them by the using the same realisation of the Poisson processes for the arrival times of the maps. Two questions that may be easily solved using the powerful coupling tool provided by the graphical construction are stated below in Exercise 8 and 9. An alternative rigorous construction of IPS via the generator (instead of the Poisson processes) is also possible. It may be found on Liggett's book [Lig85] or Swart's lecture notes ([Swaa] or [Swab]).*

Given $\eta, \sigma \in X$ we say that σ dominates η, and denote this as $\eta < \sigma$, if for all $x \in \Lambda$ it holds $\eta(x) \leqslant \sigma(x)$.

Exercise 8. *Consider two CP starting from two different initial configurations,* η_1, η_2 *such that* $\eta_1 < \eta_2$*. Fix* $i \in \Lambda$ *and* $t \geqslant 0$ *prove that*

$$\textit{if it holds } \mathbb{E}^{\eta_1}(\eta_t(i)) > 0 \textit{ then necessarily } \mathbb{E}^{\eta_2}(\eta_t(i)) > 0.$$

Exercise 9. *Fix* $\eta \in \{0,1\}^{\mathbb{Z}}$ *and* $0 < \lambda_1 < \lambda_2 < \infty$*. Consider two CP process started from* η*, the first one with infection rate* λ_1*, the second one with infection rate* λ_2*. Let* $\mathbb{E}_1$ *(resp.* $\mathbb{E}_2$*) be the mean over the first (resp, second) CP process. Fix* $t \geqslant 0$ *and* $x \in \Lambda$*, prove that a.s.* $\mathbb{E}_1(\eta_t(x)) \leqslant \mathbb{E}_2(\eta_t(x))$.

If you have a hard time figuring out how to do properly Exercise 8 and 9 I suggest re-trying after studying next chapter (in particular after understanding the notion of coupling and how to use this tool).

Chapter 3

SOME USEFUL TOOLS AND GENERAL RESULTS

3.1 Some additional notation

Recall that $X = S^\Lambda$ denotes the configuration space, with Λ the vertex set and S the on-site configuration space and (see Definition 2.1.2) for $\eta \in X$ and $\mu \in \mathcal{P}(X)$, we let

- $\mathbb{P}^\eta$ be the law of the IPS started at η and $\mathbb{E}^\eta$ be the corresponding expectation
- $(P_t)_{t \geqslant 0}$ be the semigroup of the Markov process, so that $\forall t \geqslant 0$ it holds $P_t f(\eta) = \mathbb{E}^\eta(f(\eta_t))$

Note that X is a compact metrizable space, with measurable structure given by the Borel σ-algebra of subsets of X. We let $\mathcal{P}(X)$ be the set of probability measures on X that we will endow with the topology of the weak convergence, i.e. $\mu_n \to \mu$ for $n \to \infty$ iff for all $f \in \mathcal{C}$ it holds $\int f d\mu_n \to \int f d\mu$, where $\mathcal{C} = \mathcal{C}(X)$ is the set of real continuous functions on X viewed as a Banach space with norm $||f|| = \sup_{\eta \in X} |f(\eta)|$. Note that $\mathcal{P}(X)$ is compact w.r.t. the topology of weak convergence because X is compact. For $\mu \in \mathcal{P}(X)$ we also let

- $\mathbb{P}^\mu$ be the law of the IPS with initial distribution μ, i.e.

$$\mathbb{P}^\mu = \int_X \mathbb{P}^\eta \mu(d\eta)$$

  and

$$\mathbb{E}^\mu(f(\eta_t)) = \int_X E^\eta(f(\eta_t))\mu(d\eta) = \int_X P_t f d\mu$$

- μP_t be the the distribution at time t of the process started from μ, i.e. the measure satisfying [1] for all $f \in \mathcal{C}$

$$\int_X f d(\mu P_t) := \int_X P_t f d\mu$$

[1]The fact that this relation determines μP_t uniquely is a consequyence of the Riesz representation theorem.

From now on we will drop the index X from the integral over the whole configuration space, namely we set for simplicity of notation $\int f d\mu := \int_X f d\mu$.

3.2 Invariant (or stationary) measures

Definition 3.2.1 (Invariant (or stationary) measures)**.** *We say that $\mu \in \mathcal{P}$ is invariant if*

$$\int P_t f d\mu = \int f d\mu, \quad \forall\, t \geqslant 0, \quad \forall\, f \in \mathcal{C}(X)$$

namely if

$$\mu P_t = \mu, \ \ \forall t \geqslant 0.$$

We denote by $\mathcal{I}$ the set of invariant measures. As a consequence, for any $\mu \in \mathcal{I}$ and for any measurable set A, it holds

$$\mathbb{P}^{\mu}(\eta_s \in A) = \mathbb{P}^{\mu}(\eta_{s+t} \in A), \quad \forall s, t \geq 0.$$

The invariant measures satisfy the following properties:

Theorem 3.2.2 (Properties of $\mathcal{I}$)**.**

(i) $\mathcal{I}$ is a compact and convex subset of $\mathcal{P}(X)$;

(ii) Given an initial measure π, if the weak limit $\lim_{t\to\infty} \pi P_t$ exists, i.e. if exists μ t.q.

$$\lim_{t\to\infty} \int P_t f d\pi = \mu(f), \ \ \forall f \in \mathcal{C}$$

then $\mu \in \mathcal{I}$;

(iii) $\mathcal{I}$ is non empty;

(iv) $\mu \in \mathcal{I}$ iff $\mu(\mathcal{L}f) = 0$ for any $f \in \mathcal{D}(\mathcal{L})$ with $\mathcal{D}(\mathcal{L})$ the domain [2] of the generator $\mathcal{L}$.

Note that property (ii) means that any measure which is obtained as a limit distribution under the evolution is necessarily invariant. Before proving the theorem, we state a result that plays a crucial role in the proof of point (iv).

[2]See [Lig85] for a formal definition of the domain. For practical purposes you can think of the domain as being the sets of local functions, i.e. continuous functions that depend on finitely many coordinates.

Theorem 3.2.3 (Hille Yoshida)**.** *There is a one to one correspondence between Markov semigroups and generators given as follows*

$$\mathcal{L}f = \lim_{t\to 0} \frac{P_t f - f}{t} \quad \forall f \in \mathcal{D}(\mathcal{L})$$

$$P_t f = \lim_{n\to\infty} (\mathbb{1} - \frac{t}{n}\mathcal{L})^{-n} f \quad \forall\, f \in \mathcal{C}(X),\ t \geqslant 0$$

where $\mathcal{D}(\mathcal{L}) \subset C(X)$ *is the set of functions for which* $\lim_{t\to 0} \frac{P_t f - f}{t}$ *exists.*

Furthermore

- *for* $f \in D(\mathcal{L})$ *it holds* $P_t f \in \mathcal{D}(\mathcal{L})$
- *the following backward forward equation holds*

$$\frac{d}{dt} P_t f = P_t(\mathcal{L}f) = \mathcal{L}(P_t f) \quad \forall\, f \in \mathcal{D}(X)$$

We will not provide a prove of the Hille Yoshida theorem, the interested reader can find it Chapter 1 of [EK85].

Proof. (i) Since $\mathcal{I}$ is a subset of the compact set $\mathcal{P}(X)$ we only have to show that it is closed to prove the claimed property. Let $\{\mu_n\}_{n\in\mathbb{N}}$ be such that

1. $\mu_n \in \mathcal{I}$ for all n;
2. there exists $\mu \in \mathcal{P}$ such that $\lim_{n\to\infty} \mu_n = \mu$.

By definition it holds

(a) $\mu_n = \mu_n P_t \ \ \forall n$;

(b) $\lim_{n\to\infty} \int f d\mu_n = \int f d\mu$ for any $f \in \mathcal{C}$;

(c) if $f \in \mathcal{C}$, it holds $P_t f \in \mathcal{C}$ for any $t \geqslant 0$

Therefore

$$\int P_t f d\mu = \lim_{n\to\infty} \int P_t f d\mu_n = \lim_{n\to\infty} \int f d\mu_n = \int f d\mu,$$

which implies that $\mu \in \mathcal{I}$. We used: (b) and (c) to obtain the first equality; (a) to obtain the second equality; and (b) to obtain the third equality. Convexity of $\mathcal{I}$ follows by definition;

(ii) Recall that the semigroup verifies the following property

(d) for $f \in \mathcal{C}$ and any $s, t \geqslant 0$ it holds $P_t P_s f = P_{t+s} f$.

Therefore

$$\int P_s f d\mu = \lim_{t\to\infty} \int P_t(P_s f) d\pi = \lim_{t\to\infty} \int P_{t+s} f d\pi = \int f d\mu$$

where we used point (c) and (d).

(iii) Fix a measure $\mu \in \mathcal{P}$ and a sequence $\{T_n\}_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} T_n = \infty$. Define a sequence of measures $\{\mu_n\}_{n\in\mathbb{N}}$ by letting for all $f \in \mathcal{C}(X)$

$$\int f d\mu_n := \frac{1}{T_n} \int_0^{T_n} dt \int f d(\mu P_t)$$

The compactness of $\mathcal{P}$ yields the existence of a converging subsequence, namely the existence of a measure $\tilde{\mu} \in \mathcal{P}$ and an increasing sequence $\{a_i\}_{i\in\mathbb{N}}$ with $a_i \in \mathbb{N}, \forall i$ s.t. $\lim_{n\to\infty} \tilde{\mu}_n = \tilde{\mu}$ where $\tilde{\mu}_n := \mu_{a_n}$. Therefore $\forall s > 0$ it holds

$$\int P_s f d\tilde{\mu} = \lim_{n\to\infty} \int P_s f d\tilde{\mu}_n = \lim_{n\to\infty} \int P_s f d\left(\frac{1}{T_{a_n}} \int_0^{T_{a_n}} \mu P_t dt\right) =$$

$$= \lim_{n\to\infty} T_{a_n}^{-1} \int_0^{T_{a_n}} \left(\int P_t P_s f d\mu\right) dt = \lim_{n\to\infty} T_{a_n}^{-1} \int_0^{T_{a_n}} \left(\int P_{t+s} f d\mu\right) dt =$$

$$= \lim_{n\to\infty} T_{a_n}^{-1} \int_s^{T_{a_n}+s} \left(\int P_\tau f d\mu\right) d\tau = \lim_{n\to\infty} \int f d\left(T_{a_n}^{-1} \int_0^{T_{a_n}} \mu P_\tau d\tau\right) = \int f d\tilde{\mu}$$

which implies $\tilde{\mu} \in \mathcal{I}$. In order to obtain the second-last equality we used the fact that

$$\Big| \int_s^{T_{a_n}+s} \left(\int P_t f d\mu\right) - \int_0^{T_{a_n}} \left(\int P_t f d\mu\right) \Big| \leqslant 2s \sup_{\eta\in X} f(\eta)$$

and therefore, since $f \in \mathcal{C}(X)$, $\lim_{n\to\infty} T_{a_n}^{-1} = 0$ and s is fixed, it holds

$$\lim_{n\to\infty} T_{a_n}^{-1} \Big| \int_s^{T_{a_n}+s} \left(\int P_t f d\mu\right) - \int_0^{T_{a_n}} \left(\int P_t f d\mu\right) \Big| = 0$$

For the other equalities we used=: $P_t P_s = P_{t+s}$, the definition of μP_t, the definition of μ_n and the fact that $\lim_{n\to\infty} \tilde{\mu}_n \to \tilde{\mu}$.

(iv) Let $f \in \mathcal{D}(\mathcal{L})$. Then

$$\int \mathcal{L} f d\mu = \lim_{t\to 0} \frac{\int P_t f d\mu - \int f d\mu}{t} = 0$$

and it follows immediately that $\mu \in \mathcal{I}$ implies $\int \mathcal{L} f d\mu = 0$.
Suppose that for any function $f \in \mathcal{D}(\mathcal{L})$ it holds $\mu(\mathcal{L}f) = 0$. In order to prove that $\mu \in \mathcal{I}$ we proceed as follows. Fix $g \in \mathcal{C}$ and $\lambda \geqslant 0$, and let $f_1 = f_1(g, \lambda) \in \mathcal{D}$ be defined via

$$(\mathbb{1} - \lambda \mathcal{L}) f_1 = g,$$

which implies by integrating

$$\int f_1 d\mu = \int g d\mu.$$

We extend the above definition letting for $n \geqslant 1$

$$f_n = (\mathbb{1} - \lambda \mathcal{L})^{-n} g.$$

By iterating the above argument we get

$$\int f_n d\mu = \int g d\mu. \tag{3.2.1}$$

Therefore by using Hille Yoshida theorem (Theorem 3.2.3) to get

$$\int P_t g d\mu = \lim_{n \to \infty} \int (\mathbb{1} - \frac{t}{n} \mathcal{L})^{-n} g d\mu = \int g d\mu$$

et which implies $\mu \in \mathcal{I}$. In order to establish the above result we use the definition of the semigroup for the first equality and (3.2.1) with the choice

$$\lambda := \frac{t}{n}$$

to obtain the second equality.

□

3.3 Ergodicity

Definition 3.3.1 (Ergodicity and phase transitions). *We say that an IPS is ergodic*[3] *if the following two conditions hold:*

(i) there is only one invariant measure, $\mathcal{I} = \{\mu\}$

[3]The term *ergodic* (which is the current jargon in IPS) can be misleading. Indeed the term ergodic usually denotes a process for which all events that are invariant under time shifts have probability either zero or one. Actually the stationary process of an ergodic (in the sense of definition 3.3.1) IPS is ergodic in this sense. However, the converse is not true.

(ii) $\lim_{t\to\infty} \pi P_t = \mu$ *for any* π

We say that a phase transition occurs for an IPS if: (i) the definition of the IPS contains a parameter (temperature, density, ...) that can vary in a (finite or infinite) real interval $R = [a, b]$ *(ii) if we denote by* λ *the parameter, there exists a value* $\lambda_c \in R$ *such that the IPS is ergodic for* $\lambda < \lambda_c$ *and it is not ergodic for* $\lambda > \lambda_c$ *or vice versa. In this case, we call* λ_c *the critical value.*

Remark 3.3.2 (Irreducibility)**.** *Given a Markov processes on a finite state* X*, we say that it is irreducible if for any couple* $(\eta, \eta') \in X \times X$ *it holds* $\mathbb{P}^\eta(\eta_t = \eta') > 0$ *for some* $t \geqslant 0$*. A finite state irreducible Markov process has a unique stationary measure.*

Instead, for processes with infinite state space X, it is not enough to exhibit for each couple $(\eta, \eta') \in X \times X$ a chain of moves with positive transition rate connecting η to η' to deduce the uniqueness of the stationary measure. For example, for SIM in the low temperature regime there is more than one invariant measure, and yet the existence of a chain of moves that connect any two configurations is guaranteed by the fact that the rate at which we can change the value of the spin at a given site is strictly positive in any configuration.

We will conclude by stating two sufficient conditions for ergodicity.

Theorem 3.3.3. *A sufficient condition for ergodicity is that* $M < \inf_x \inf_\eta \{r(x, \eta) + r(x, \eta^x)\}$ *with* $M = \sup_x \sum_{y\neq x} \sup_\eta |r(x, \eta^y) - r(x, \eta)|$*. Furthermore under this condition the convergence to the unique invariant measure is exponentially fast.*

The interested reader may find the proof of Theorem 3.3.3 on page 31 of [Lig85]. Let's just note that the conditions of the above theorem go in the right direction: they essentially require that the individual spin flip rate exceeds the dependence of the rate on the environment.

Exercise 10. *Use Theorem 3.3.3 to prove that there is an ergodic regime for CP (at small enough* λ*) and for SIM (at small enough* β*).*

A different sufficient condition for ergodicity is the following

Theorem 3.3.4. *Let* K *be defined as in* (2.2.6)*. If* $K < 0$ *the IPS is ergodic.*

The proof of the above result follows easily from Lemma 2.2.2. The fact that this condition is sufficient but not necessary can be easily seen by noticing that CP (for all values of λ) does not satisfy the condition $K < 0$, and yet it is ergodic when λ is small enough as proven in Exercise 10.

3.4 Reversibility

We will now introduce the of reversibility, which as we shall see is stronger than the notion of invariance.

Definition 3.4.1 (Reversible measure)**.** *We say that $\mu \in \mathcal{P}$ is reversible for the process if*

$$\int f P_t g \, d\mu = \int g \, P_t f d\mu, \quad \forall t \geqslant 0, \forall f, g \in \mathcal{C}(X)$$

Remark 3.4.2 (Reversibility vs stationarity)**.** *Letting $g := 1$ in the above definition it is easily seen that reversibility implies stationarity .*

Theorem 3.4.3. *A measure μ is reversible iff for all $f, g \in \mathcal{D}$ it holds*

$$\int f \mathcal{L} g \, d\mu = \int g \mathcal{L} f d\mu$$

namely iff $\mathcal{L}$ is self-adjoint w.r.t. μ.

Exercise 11. *Prove Theorem 3.4.3 along analogous lines as Theorem 3.2.2 (iv).*

Remark 3.4.4 (Invariance and reversibility in finite volume)**.** *For a continuous time Markov chain on a countable set X with transition rate $c(\eta, \eta')$ the generator $\mathcal{L}$ acts on continuous functions as*

$$\mathcal{L} f(\eta) = \sum_{\eta' \in X} c(\eta, \eta')(f(\eta') - f(\eta))$$

Thus

- *$\mu \in \mathcal{P}(X)$ is invariant iff $\mu(\mathcal{L}\mathbb{1}_\xi) = 0$ for any $\xi \in X$ which* [4] *yields the condition*

$$\sum_\eta [\mu(\eta)c(\eta, \xi) - \mu(\xi)c(\xi, \eta)] = 0 \ \ \forall \xi$$

- *$\mu \in \mathcal{P}(X)$ is reversible iff $\mu(\mathbb{1}_\xi \mathcal{L} \mathbb{1}_{\xi'}) = \mu(\mathbb{1}_{\xi'} \mathcal{L} \mathbb{1}_\xi)$ for any two configuraitions ξ, ξ'. This corresponds to the so called detailed balance condition*

$$\mu(\xi')c(\xi', \xi) = \mu(\xi)c(\xi, \xi') \ \ \forall \xi, \xi'$$

  which corresponds to requiring that each term is zero in the sum appearing in the stationarity condition.

[4] Use $\mathcal{L}\mathbb{1}_\xi(\eta) = 1_{\eta \neq \xi} c(\eta, \xi) - 1_{\eta = \xi} \sum_{\eta'} c(\xi, \eta')$

Exercise 12. *Prove the necessary and sufficient conditions for stationarity and for reversibility on finite volume stated in Remark 3.4.4.*

Exercise 13. *Prove that for SIM on a finite volume at inverse temperature β the Gibbs measure are reversible (and thus stationary) for the process.*

Exercise 14. *Prove that, at any $q \in (0,1]$ and for any $d \geqslant 1$, FA-1f is not ergodic on $\mathbb{Z}^d$.* [**Hint.** *Consider FA-1f on a finite interval $[a,b]$ with empty boundary condition on $b+1$ and $a-1$. Search for a measure that satisfies the detailed balance condition stated in Remark 3.4.4. Since for any couple η, η' the transition rate $c(\eta,\eta')$ is zero unless the two configuration differ on a single site, you have to search for a measure that satisfies $\mu(\eta)r(x,\eta) = \mu(\eta^x)r(x,\eta^x)$ with $r(x,\eta)$ defined by* (1.2.8) *and* (1.2.9).
From the knowledge of the finite volume reversible measure in this case try to guess which is a reversible invariant measure for the model on $\mathbb{Z}^d$ (besides δ_1).]

3.5 Monotonicity or attractivness

For $X = \{0,1\}^{\mathbb{Z}^d}$ we define the following *partial order*

$$\eta \leqslant \xi \quad \text{iff} \quad \eta(x) \leqslant \xi(x) \quad \forall x \in \mathbb{Z}^d$$

We say that a function $f : X \to \mathbb{R}$ is *increasing* if

$$\eta \leqslant \xi \qquad \text{implies} \quad f(\eta) \leqslant f(\xi)$$

and we let $\mathcal{M} \subset \mathcal{C}$ be the set of continuous increasing functions.

Exercise 15. *(very easy one!) Let $A \subset \mathbb{Z}^d$ with $|A| < \infty$ and set $f_A(\eta) := \prod_{x \in A} \eta(x)$. Show that $f_A \in \mathcal{M}$.*

Given μ_1, μ_2 two probability measures on S we say that μ_1 is *stochastically dominated* by (or stochastically smaller than) μ_2 and we write $\mu_1 \leqslant \mu_2$ if the following holds:

$$\int f d\mu_1 \leqslant \int f d\mu_2 \quad \forall f \in \mathcal{M}$$

Definition 3.5.1. *We say that an IPS is monotone (or attractive) if $\mu_1 \leqslant \mu_2$ implies $\mu_1 P_t \leqslant \mu_2 P_t$ for all $t \geqslant 0$.*

Exercise 16. *Prove that Definition is equivalent to the following: "We say that an IPS is monotone if for any* $f \in \mathcal{M}$ *and any* $t \geqslant 0$*, the function* $P_t f$ *also belongs to* $\mathcal{M}$*."*

Theorem 3.5.2. *A spin IPS is monotone iff the following holds: for any couple of configurations* η, ξ *that satisfy* $\eta \leqslant \xi$ *it holds*

(i) $r(x,\eta) \leqslant r(x,\xi)$ *if* $\eta(x) = \xi(x) = 0$

(ii) $r(x,\eta) \geqslant r(x,\xi)$ *if* $\eta(x) = \xi(x) = 1$

Exercise 17. *Use Theorem 3.5.2 to prove that*

- *CP, VM are monotone*
- *FA-1f model is* not *monotone.*

Exercise 18. *Consider SIM under the change of variables* $\eta \in \{\pm 1\}^\Lambda \to \tilde{\eta} \in \{0,1\}^\Lambda$ *with* $\tilde{\eta}(x) = \frac{1-\eta(x)}{2}$ *for all* x *and prove using Theorem 3.5.2 that it is monotone.*

Before giving the proof of Theorem 3.5.2 we should understand better the notion of stochastic domination among measure by introducing the notion of *coupling*.

Definition 3.5.3 (Coupling)**.** *A coupling of two random variables is a joint construction of the variables on a common probability space. More precisely, given* μ_1*,* μ_2 *on* X*, a coupling is a measure* μ *on* $X \times X$ *whose marginals are* μ_1 *and* μ_2 *, i.e. such that for* $i \in [1,2]$ *and any* $A \subset X$ *it holds* $\mu(\{\eta : \eta^i \in A\}) = \mu_i(A)$*, where for* $\eta \in X \times X$ *we denote by* η^1 *(resp.* η^2 *the first (resp. second) coordinate of* η*.*

Theorem 3.5.4 (Strassen)**.** *Given* μ_1*,* μ_2 *on* X*, it holds*

$$\mu_1 \leqslant \mu_2$$

iff $\exists$ *a coupling* μ *on* $X \times X$ *s.t.*

$$\mu\{\eta = (\eta^1, \eta^2) : \eta^1 \leqslant \eta^2\} = 1$$

Proof. A direction of the proof is easy. Fix f an increasing function. If a coupling μ with the property $\mu\{\eta = (\eta^1,\eta^2) : \eta^1 \leqslant \eta^2\} = 1$ exist, with probability 1 w.r.t. μ it holds $f(\eta^1) \leqslant f(\eta^2)$. Therefore

$$\mu_1(f) = E_\mu f(\eta^1) \leqslant E_\mu f(\eta^2) \leqslant \mu_2(f)$$

The other direction is more tricky, full proof on [Lig85] (Theorem 2.4, pag 72). □

Exercise 19. *Fix* $p_1, p_2 \in [0,1]$ *with* $p_1 < p_2$. *Let* $X = \{0,1\}^\Lambda$ *with* $|\Lambda| < \infty$ *and*

$$\mu_i = \prod_{x\in\Lambda} p_i^{\eta(x)}(1-p_i)^{1-\eta(x)} \quad \text{for } i \in \{1,2\}.$$

Construct a coupling μ *of* μ_1 *and* μ_2 *such that* $\mu\{\eta = (\eta^1,\eta^2) : \eta^1 \leqslant \eta^2\} = 1$. [**Hint.** *Case* $|\Lambda| = 1$. *Let* z *be a uniform random variable on the interval* $[0,1]$. *If you set* $\eta^1 = \mathbb{1}_{z<p_1}$ *and* $\eta^2 = \mathbb{1}_{z<p_2}$ *it follows that* $\eta^1 \leqslant \eta^2$ *and it is easily checked that* η^1 *is distributed with* μ_1 *and* η^2 *is distributed with* μ_2. *Therefore we have provided the coupling. It is now very easy to extending the coupling to the case* $|\Lambda| > 1$.]

Neither Definition 3.5.3 nor Theorem 3.5.4 give an efficient way to check whether, given $\mu_1, \mu_2 \in \mathcal{P}$, one of the two measures is stochastically dominated by the other. A precious result is the following sufficient condition. Let $\eta \vee \xi$ and $\eta \wedge \xi$ be the configurations defined by

$$\eta \vee \xi(x) = \max(\eta(x),\xi(x)), \quad \eta \wedge \xi(x) = \min(\eta(x),\xi(x))$$

Theorem 3.5.5 (Holley theorem)**.** *Given* μ_1, μ_2 *that assign a strictly positive probability to any point in* X*, if it holds*

$$\mu_1(\eta\wedge\xi)\mu_2(\eta\vee\xi) \geqslant \mu_1(\eta)\mu_2(\xi) \quad \forall \eta,\xi \in X$$

then it holds

$$\mu_1 \leqslant \mu_2.$$

We refer the reader to [Lig85] pag. 75 for a detailed proof. The strategy of the proof is to construct a Markov chain (η_t, ξ_t) on $X \times X$ with starting point η, ξ s.t. $\eta \leqslant \xi$ and preserving this property during the evolution and such that the first (resp. second) marginal is a Markov chain with stationary measure μ_1 (resp. μ_2).

We are now ready to prove the necessary and sufficient condition for an IPS to be monotone.

Proof of Theorem 3.5.2. We should prove that

(a) any IPS satisfying conditions (i) and (ii) is necessarily monotone

(b) any monotone IPS satisfies conditions (i) and (ii)

Proof of (a). Fix $\mu_1, \mu_2 \in \mathcal{P}$ s.t. $\mu_1 \leqslant \mu_2$. We must show that (i) and (ii) imply that for any $t > 0$ it holds $\mu_1 P_t \leqslant \mu_2 P_t$. To this aim we construct a coupling of $\mathbb{P}^{\mu_1}, \mathbb{P}^{\mu_2}$ that preserves the partial order at any fixed time, namely a probability $P(\{\eta_t^1\}_{t\geqslant 0}, \{\eta_t^2\}_{t\geqslant 0})$ with marginals $\mathbb{P}^{\mu_1}$ and $\mathbb{P}^{\mu_2}$ and such that $P(\{\eta_t^1 \leqslant \eta_t^2 \quad \forall t \geqslant 0\}) = 1$. If we exhibit such a coupling, then the result follows by Theorem 3.5.4.

In order to construct the coupling with the desired properties notice that, since $\mu_1 \leqslant \mu_2$, there is a distribution μ on $X \times X$ that is a coupling for μ_1, μ_2 and that satisfies $\mu(\eta^1 \leqslant \eta^2) = 1$. Consider the Markov process on $X \times X$ that evolves from μ and has the following rates:

1) if $\eta(x) = \xi(x) = 0$ then
   - $(\eta, \xi) \to (\eta^x, \xi^x)$ at rate $r(x, \eta)$
   - $(\eta, \xi) \to (\eta, \xi^x)$ at rate $r(x, \xi) - r(x, \eta)$
2) if $\eta(x) = \xi(x) = 1$ then
   - $(\eta, \xi) \to (\eta^x, \xi^x)$ at rate $r(x, \xi)$
   - $(\eta, \xi) \to (\eta^x, \xi)$ at rate $r(x, \eta) - r(x, \xi)$
3) if $\eta(x) = 0$ and $\xi(x) = 1$ then
   - $(\eta, \xi) \to (\eta^x, \xi)$ at rate $r(x, \eta)$
   - $(\eta, \xi) \to (\eta, \xi^x)$ at rate $r(x, \xi)$.

It is immediate to verify that

- (i) and (ii) guarantee that all the transition rates of the constructed process are non negative, thus it is well defined
- each transition preserves the partial order
- each marginal process evolves according to the correct transition rates

Proof of (b). Consider a monotone spin IPS, fix a site $x \in \Lambda$ and define $f(\eta) = \eta(x)$. Since f is increasing, by monotonicity of the process also $P_t f$ is increasing. Choose two configurations η, ξ s.t.

$$\eta(x) = \xi(x) \quad \text{and } \eta \leqslant \xi.$$

Then it holds

$$\mathcal{L}f(\eta) - \mathcal{L}f(\xi) = \lim_{t \to 0} \frac{P_t f(\eta) - P_t f(\xi) + \eta(x) - \xi(x)}{t} \leqslant 0 \tag{3.5.1}$$

where we used the fact that: $\eta(x) = \xi(x)$, $\eta \leqslant \xi$ and $P_t f$ increasing. Furthermore it holds

$$\mathcal{L}f(\eta) = \sum_y r(y, \eta)(f(\eta^y) - f(\eta)) = r(x, \eta)(1 - 2\eta(x)) \tag{3.5.2}$$

Thus it can be easily seen that (3.6.5) together with (3.5.2) imply that (i) and (ii) necessarily hold.

□

Theorem 3.5.6 (Invariant measures for monotone spin IPS)**.** *For a monotone spin IPS it holds*

(a) $\delta_0 P_s \leqslant \delta_0 P_t$ *for all* $s \in [0, t]$

(b) $\delta_1 P_s \geqslant \delta_1 P_t$ *for all* $s \in [0, t]$

(c) $\delta_0 P_t \leqslant \mu P_t \leqslant \delta_1 P_t$ *for all* $t \geqslant 0$ *and any* μ

(d) $\lim_{t\to\infty} \delta_0 P_t$ *and* $\lim_{t\to\infty} \delta_1 P_t$ *exist. We let* $\underline{\nu} := \lim_{t\to\infty} \delta_0 P_t$ *and* $\bar{\nu} := \lim_{t\to\infty} \delta_1 P_t$

(e) let $\mu \in \mathcal{P}$. *If* $\nu := \lim_{t\to\infty} \mu P_t$ *exists, it holds* $\underline{\nu} \leqslant \nu \leqslant \bar{\nu}$

(f) $\underline{\nu}$ *et* $\bar{\nu}$ *are extremal on* $\mathcal{I}$

Proof.

(a) Fix s, t with $s \leqslant t$. Since δ_0 is concentrated on the smallest element of X, it necessarily holds $\delta_0 \leqslant \delta_0 P_{t-s}$. Due to monotonicity this order is preserved at any later time, thus $\delta_0 P_s \leqslant \delta_0 P_{t-s} P_s = \delta_0 P_t$.

(b) analogous to (a)

(c) use $\delta_0 \leqslant \mu \leqslant \delta_1$ and use monotonicity to get the claim.

(d) Fix any increasing sequence of times $\{t_n\}_{n\in\mathbb{N}}$ and let $\mu_n := \delta_0 P_{t_n}$. By point (a) we have that $\mu_n \leqslant \mu_m$ for any $n < m$, thus $\lim_{n\to\infty} \mu_n$ exists and belongs to $\mathcal{P}$ due to compactness. Suppose that we fix two increasing sequences of times $\{t_n^1\}_{n\in\mathbb{N}}$ and $\{t_n^2\}_{n\in\mathbb{N}}$ and call μ_1 and μ_2 the corresponding limit measures. Then it follows that for any $f \in \mathcal{M}$ it holds $\mu_1(f) = \mu_2(f)$, which implies $\mu_1 = \mu_2$. We proceed analogously to prove the existence of $\lim_{t\to\infty} \delta_1 P_t$

(e) it follows from (c), (d) and monotonicity

(f) $\underline{\nu}$ et $\bar{\nu}$ are invariant thanks to Theorem 3.2.2. To prove extremality we proceed by contradiction. Suppose that $\bar{\nu}$ is not extremal, namely suppose that $\exists \mu_1, \mu_2 \in \mathcal{I}$ with μ_1 and μ_2 different from $\bar{\nu}$ and $\alpha \in (0, 1)$ s.t. $\bar{\nu} = \alpha\mu_1 + (1 - \alpha)\mu_2$. Since μ_1, μ_2 are invariant measures they can be obtained as infinite limit of a process started with themselves, thus (e) implies $\mu_1, \mu_2 \leqslant \bar{\nu}$. Therefore for any $f \in \mathcal{M}$ it holds $\mu_i(f) \leqslant \bar{\nu}(f)$ and $\bar{\nu}(f) = \alpha\mu_1(f) + (1-\alpha)\mu_2(f)$ which implies $\mu_1(f) = \mu_2(f) = \bar{\nu}(f)$. Thus we deduce that for any $f \in \mathcal{M}$ it holds $\mu_1(f) = \mu_2(f) = \bar{\nu}(f)$ which implis $\mu_1 = \mu_2 = \bar{\nu}$ and contradicts the hypothesis.

□

The interested reader might have a look at Theorem 3.13 p.152 in [Lig85] which provides a a sufficient condition for an IPS in $d = 1$ to have that only extremal invariant measures are $\underline{\nu}$ and $\bar{\nu}\}$. This condition is satisfied by CP, SIM, and VM.

Corollary 3.5.7. *For a monotone spin IPS the following three conditions are equivalent*

1. *the process is ergodic*
2. *$\mathcal{I}$ is a singleton*
3. *$\bar{\nu} = \underline{\nu}$*

3.6 Duality

Duality is a very useful tool that allows sometimes to connect two different IPS expressing the law of one process in term of the other and vice versa.
A first example: consider VM in d=1 and focus on the evolution of the position of the boundaries separating islands of 0's and 1's. It is not difficult to realise that these boundaries evolve as simple symmetric annihilating random walks on $\mathbb{Z}$: when two boundaries meet they annihilate and otherwise each boundary moves as a random walks jumping at rate 1/2 to each of its 2 nearest neighbours. So one can translate the probability law of one-dimensional VM in terms of the law for one-dimensional simple symmetric annihilating random walks. These two systems are dual one to the other [5].

Let us start by giving an abstract definition of duality. We will later provide specific examples.

[5] Actually duality here holds configuration wise, namely in a stronger sense than the one given by Definition 3.6.1.

Definition 3.6.1 (Duality and Self-duality). *Given two Markov processes $(\eta_t)_{t \geqslant 0}$ and $(\xi_t)_{t \geqslant 0}$ on space states X and Y, and given $H(\eta, \xi)$ a bounded measurable function on $X \times Y$, we say that $(\eta_t)_{t \geqslant 0}$ and $(\xi_t)_{t \geqslant 0}$ are dual to each other w.r.t. H if*

$$\mathbb{E}^\eta H(\eta_t, \xi) = \mathbb{E}^\xi H(\eta, \xi_t), \;\; \forall \eta \in X, \xi \in Y, t \geqslant 0$$

We say that $(\eta_t)_{t \geqslant 0}$ is self dual if it is dual w.r.t. H to a process $(\xi_t)_{t \geqslant 0}$ that has the same law as $(\eta_t)_{t \geqslant 0}$.

Theorem 3.6.2. *[A class of duality relations for spin IPS]*
Let

$$X = \{0,1\}^{\mathbb{Z}^d}, \quad Y := \{A : A \subset \mathbb{Z}^d, |A| < \infty\}$$

and

$$H(\eta, A) := \prod_{x \in A} (1 - \eta(x)) \;\; \text{for} \;\; A \neq \emptyset, \quad H(\eta, \emptyset) := 1 \tag{3.6.1}$$

Fix $c : \mathbb{Z}^d \to \mathbb{R}$ and $p : \mathbb{Z}^d \times Y \to [0,1]$ satisfying

(A1) $\sup_x c(x) < \infty$

(A2) $c(x) \geqslant 0 \; \forall x \in \mathbb{Z}^d$,

(A3) $p(x, A) \geqslant 0 \;\; \forall x \in \mathbb{Z}^d, \forall A \in Y$

(A4) $\sum_{B \in Y} p(x, B) = 1 \;\; \forall x \in \mathbb{Z}^d$

(A5) $\sup_x c(x) \sum_A |A| p(x, A) < \infty$.

Fix $\eta \in X$ and $A \in Y$ we define two Markov processes $(\eta_t)_{t \geqslant 0}$ on X and $(A_t)_{t \geqslant 0}$ on Y, as follows

- *$(\eta_t)_{t \geqslant 0}$ is the spin IPS with $\eta_0 = \eta$ and rates*

$$r(x, \eta) := c(x) \left[\eta(x) \sum_A p(x, A) H(\eta, A) + (1 - \eta(x)) \sum_A p(x, A)(1 - H(\eta, A)) \right] \tag{3.6.2}$$

- *$(A_t)_{t \geqslant 0}$ is the Markov process on Y with $A_0 = A$ and rates*

$$q(A, B) := \sum_{x \in A} c(x) \sum_{F : (A \setminus \{x\}) \cup F = B} p(x, F) \tag{3.6.3}$$

and we call $\mathbb{P}^\eta$ and $\mathbb{P}^A$ the laws of the process $(\eta_t)_{t \geqslant 0}$ and $(A_t)_{t \geqslant 0}$.
Then $(\eta_t)_{t \geqslant 0}$ and $(A_t)_{t \geqslant 0}$ are dual to each other w.r.t. the function H defined in (3.6.1).

In words we can describe the process $(A_t)_{t \geqslant 0}$ by saying that each $x \in A$ is removed from A at rate $c(x)$ and replaced by the set F with probability $p(x, F)$. The assumption in (A5) implies that $|A_t| < \infty$ for any $t > 0$. We are left with proving Theorem 3.6.2.

Proof of Theorem 3.6.2. Let $\mathcal{L}$ and $\bar{\mathcal{L}}$ be the generator of $(\eta_t)_{t \geqslant 0}$ and $(A_t)_{t \geqslant 0}$, so that

$$\mathcal{L}H(\eta, A) = \sum r(x, \eta)[H(\eta^x, A) - H(\eta, A)]$$
$$\bar{\mathcal{L}}H(\eta, A) = \sum_B q(A, B)[H(\eta, B) - H(\eta, A)]$$

Using the form of the rates and the function H, you can directly verify that

$$\mathcal{L}H(\eta, A) = \bar{\mathcal{L}}H(\eta, A) \;\; \forall \eta \in X, \, \forall A \in Y$$

Now we can use this result, plus Hille Yoshida theorem (Theorem 3.2.3) and the definition of semigroup to obtain

$$\frac{d}{dt}\mathbb{E}^\eta H(\eta_t, A) = \frac{d}{dt}P_t H(\eta, A) = \mathcal{L}(P_t H(\eta, A)) = P_t(\mathcal{L}H(\eta, A)) =$$
$$= P_t(\bar{\mathcal{L}}H(\eta, A)) = \bar{\mathcal{L}}(P_t H(\eta, A)) = \bar{\mathcal{L}}(\mathbb{E}^\eta H(\eta_t, A))$$

On the other hand it holds

$$\frac{d}{dt}\mathbb{E}^A H(\eta, A_t) = \bar{\mathcal{L}}(\mathbb{E}^A H(\eta, A_t))$$

Therefore we obtain that $\mathbb{E}^\eta H(\eta_t, A)$ and $\mathbb{E}^A H(\eta, A_t)$ solve the same differential equation. Since the two quantities are equal at time 0, they are necessarily equal at any later time, and the duality relation

$$\mathbb{E}^\eta H(\eta_t, A) = \mathbb{E}^A H(\eta, A_t)$$

is proven. □

Corollary 3.6.3. *Fix $\eta \in X$ and $A \in Y$, and let $\mathbb{P}^\eta$ and $\mathbb{P}^A$ be the laws defined in Theorem 3.6.2. It holds*

$$\mathbb{P}^\eta(\eta_t(x) = 0 \; \forall x \in A) = \mathbb{P}^A(\eta(x) = 0 \, \forall x \in A_t) \quad \forall \, t \geqslant 0$$

Proof. The proof follows easily by noticing that

$$\mathbb{E}^\eta \left(\prod_{x \in A} (1 - \eta_t(x) \right) = \mathbb{P}^\eta \left(\eta_t(x) = 0 \; \forall x \in A \right)$$

and

$$\mathbb{E}^A \left(\prod_{x \in A_t} (1 - \eta(x) \right) = \mathbb{P}^A \left(\eta(x) = 0 \; \forall x \in A_t \right)$$

□

Exercise 20. *Prove that if we let*

$$c(x) := 1 + 2d\lambda, \quad p(x,A) := \frac{1}{1+2d\lambda}\delta_{A=\emptyset} + \frac{\lambda}{1+2d\lambda}\sum_{y:y\sim x} \mathbb{1}_{A=\{x,y\}}$$

the two functions satisfy (A1)–(A5) of Theorem 3.6.2. Furthermore using (3.6.2) *and* (3.6.3) *it follows that* $(\eta_t)_{t\geqslant 0}$ *is CP(d,* λ*), i.e. the contact process with infection parameter* λ *on* $\mathbb{Z}^d$ *and* $(A_t)_{t\geqslant 0}$ *is the process evolving as the subset of* $\mathbb{Z}^d$ *containing all the infected sites of a CP(d,* λ*).*

Hint. *The present choice of c and p together with* (3.6.2) *and* (3.6.3) *yield*

$$r(x,\eta) = \eta(x) + (1-\eta(x))\lambda \sum_{y:y\sim x} \eta(y)$$

$$q(A,B) = |\{x : x \in A, A \setminus x = B\}| + \lambda|\{(x,y) : x \in A, A \cup y = B, y \sim x\}|$$

Remark 3.6.4. *As a by-product of Exercise 20 we have proven that CP is self-dual. For an alternative proof of self-duality for the contact process the interested reader might read section 2.1 of [Swaa] (see in particular Lemma 2.1 therein)* [6]*.*

Exercise 21. *Prove that if we let*

$$c(x) := 1, \quad p(x,A) := \frac{1}{2d}\sum_{y:y\sim x} \mathbb{1}_{A=\{y\}}$$

the two functions satisfy (A1)–(A5) of Theorem 3.6.2.

Prove that with this choice $(\eta_t)_{t\geqslant 0}$ *is the voter model (VM) and* $(A_t)_{t\geqslant 0}$ *evolve as independent random walks on* $\mathbb{Z}^d$ *that jump to a neighbouring site at rate* $1/2d$ *and that coalesce when they meet. Thus in any dimension VM is dual to coalescing RW.*

Hint. *The present choice of c and p together with* (3.6.2) *and* (3.6.3) *yield*

$$r(x,\eta) = \frac{1}{2d}\sum_{y,y\sim x} \mathbb{1}_{\eta(y)\neq\eta(x)}$$

$$q(A,B) = |\{(x,y) : x \in A, B = (A \setminus \{x\}) \cup \{y\}, y \sim x\}|$$

[6] The proof presented by Swart is completely graphical (and less abstract) and based on a simple observation. Draw the occurrences of the arrival times of the infection and healing maps as described in Section 2.1. For $A \subset \mathbb{Z}^d$ and $s,t \geqslant 0$, let $\eta_t^{A,s}$ be the set of points $i \in \mathbb{Z}^d$ s.t. $\exists y \in A$ with $(y,s) \to (i,A)$. Let also $\eta_t^{+,A,s}$ be the set of points $i \in \mathbb{Z}^d$ s.t. there exists $y \in A$ with $(i, s-t) \to (y,s)$. Then the law of $\eta_t^{+,A,s}$ and $\eta_t^{A,s}$ coincide. The proof of the above result can be done as follows: (1) take a piece of paper, (2) draw a realisation of the arrows and crosses corresponding to the arrival times of the infection and healing maps of CP (see Section 2.1) , (3) turn the paper upside down, (4) invert the direction of each infection arrow and put a $-$ sign in front of each time (so that e.g. an original horizontal line at time 10 is now at time -10), (5) notice that thanks to the fact that infections from $i \to i+1$ have the same rate as infections from $i \to i-1$, the crosses and arrows that you see now are still distributed as for a contact process, (6) notice that a path of influence occurs now from $j, -(s+t)$ iff in your original (non upside down) picture a path of influence was occurring from i,s to $j, s+t$.

The following very useful theorem is the key ingredient to prove that for the contact process the critical value of λ separating the regime in which CP started from a single infection dies out from the regime in which it survives coincides with the critical value separating the ergodic and non ergodic regimes (see Theorem **??**).

Theorem 3.6.5. *[Duality and Ergodicity] Let $(\eta_t)_{t\geqslant 0}$ and $(A_t)_{t\geqslant 0}$ be defined as in Theorem 3.6.2, then $(\eta_t)_{t\geqslant 0}$ is ergodic iff for all $A\in Y$ it holds $P^A(\tau<\infty)=1$ where $\tau:=\inf\{t\geqslant 0 : A_t=\emptyset\}$.*

Corollary 3.6.6. *VM is not ergodic, namely $\delta_0=\underline{\nu}\neq\bar{\nu}=\delta_1$.*

Proof. By definition coalescing random walk never die out, namely starting from any finite number of random walks there is at least one walker at any subsequent time, therefore it holds $P^A(\tau<\infty)=0$. This, together with Theorem 3.6.5, implies that VM is not ergodic. □

Proof of Theorem 3.6.5. The key idea in the proof is that the processes defined in Theorem 3.6.2 are such that the completely healthy configuration is a trap for $(\eta_t)_{t\geqslant 0}$, while $A=\emptyset$ is a trap for $(A_t)_{t\geqslant 0}$.

- Suppose that we know that the survival time for A_t is finite. Then, using Theorem 3.6.2, we write for any μ and any A

  $$\mathbb{P}^\mu(\eta_t(x)=0\,\forall x\in A)=\int\mathbb{E}^\eta H(\eta_t,A)\mu(d\eta)=\int\mathbb{E}^A H(\eta,A_t)\mu(d\eta)=$$

  $$=\mathbb{P}^A(\tau\leqslant t)+\int\mathbb{P}^A(\eta(x)=0\,\forall x\in A_t,\tau>t)\mu(d\eta)$$

  Now letting $t\to\infty$ and using the hypothesis we get

  $$\lim_{t\to\infty}\mathbb{P}^\mu(\eta_t(x)=0\,\forall x\in A)=1\ \ \forall\mu\in\mathcal{P}(X),A\in Y$$

  Thus $\lim_{t\to\infty}\mu P_t=\delta_0$ and so $(\eta_t)_{t\geqslant 0}$ is ergodic.

- Suppose that we know that $(\eta_t)_{t\geqslant 0}$ is ergodic, thus $\lim_{t\to\infty}\delta_1 P_t=\delta_0$. By using the same formulas as before with the choice $\mu=\delta_1$ and taking again the limit $t\to\infty$ we get

  $$\mathbb{P}^{\delta_1}(\eta_t(x)=0\ \ \forall x\in A)=\mathbb{P}^A(\tau\leqslant t)+\int\mathbb{P}^A(\eta(x)=0\,\forall x\in A_t,\tau=\infty)\delta_1(d\eta)=0$$

  If we now let $t\to\infty$ the l.h.s. goes to 1 for all A and the second term in the r.h.s. goes to 0 (since $A_t\neq\emptyset$ on the event $\tau=\infty$). Therefore we get $1=\mathbb{P}^A(\tau\leqslant\infty)$.

□

Chapter 4

CONTACT PROCESS

4.1 Main results

Fix a spatial dimension $d \in \mathbb{Z}_+$ and an infection rate $\lambda \geqslant 0$. We call CP(d, λ) the contact process with infection rate λ which has been constructed in Chapter 1. From Chapter 3 we already know that CP(d, λ) is a monotone spin IPS and the lower invariant measure is independent on λ and is concentrated in the completely empty configuration, $\underline{\nu} = \delta_0$. The upper invariant measure depends instead on λ and we denote it by $\bar{\nu}_\lambda$.

Exercise 22. *Fix $\lambda_1, \lambda_2 > 0$ with $\lambda_1 \leqslant \lambda_2$. Prove that $\bar{\nu}_{\lambda_1} \leqslant \bar{\nu}_{\lambda_1}$. As a consequence, $\rho(\lambda) := \bar{\nu}_\lambda(\eta(x))$ is non decreasing in λ.*

[**Hint.** *Use the graphical construction to couple the process with infection rate λ_1 and the process with infection rate λ_2. Both CP(d, λ_1) and CP(d,λ_2) have healing rate 1, so we can take the same realisation of the arrival times for the healing maps. Infection maps instead have different rates for the two process. Recall that for CP(d, λ), for any oriented couple of neighbouring sites, the arrival times of the infection maps form a Poisson point set of intensity λ on the time line and go back to Exercise 19 to have an idea on how to couple efficiently Poisson point set of intensity λ_1 and λ_2.*]

thm:CPerg

Let us introduce the notion of *survival*. We denote by $\mathbb{P}^\eta_{d,\lambda}$ the law of CP(d, λ) started by η and (with slight abuse of notation) for any $A \subset \mathbb{Z}^d$ we denote by $\mathbb{P}^A_{d,\lambda}$ the law of the CP(d, λ) started from a configuration η_A with $\eta_A(x) = 1$ iff $x \in A$. We also let

$$\theta_d(\lambda) := \mathbb{P}^{\{x\}}_{d,\lambda}(\eta_t \neq \emptyset \, \forall t \geqslant 0) = P((x, 0) \to \infty)$$

with P the measure on the Poisson point sets and $\{(x, 0) \to \infty\}$ the event that there a path of influence (as defined in Section 2.2) from $(x,0)$ to at least one point with time coordinate t for all $t \geqslant 0$ (i.e. a path following the direction of the arrows and never meeting crosses starting at $(x,0)$ reaching any possible time). Notice that by translation invariance the r.h.s. does not depend on x.

Definition 4.1.1. *If $\theta_d(\lambda) > 0$ we say that CP(d,λ) survives, otherwise we say that it dies out. We also define the critical infection rate for survival, $\bar{\lambda}_c(d)$, as*

$$\bar{\lambda}_c(d) := \sup\{\lambda \geqslant 0 : \ CP(d,\lambda) \text{ dies out}\}.$$

Exercise 23. *Use the graphical construction to prove that if $\lambda_1 < \lambda_2$ and if CP(d, λ_2) dies out, then CP(d, λ_1) also dies out. This implies in particular that for all $\lambda < \bar{\lambda}_c(d)$ CP dies out.*

Remark 4.1.2. *We could have defined survival starting from any finite sets of infections, it would have been an equivalent definition. Namely for any finite non-empty set $A \subset \mathbb{Z}^d$ the following holds*

- *CP dies out iff $\mathbb{P}^{\{A\}}_{d,\lambda}(\eta_t \neq \emptyset \ \forall t \geqslant 0) = 0$*
- *CP survives iff $\mathbb{P}^{\{A\}}_{d,\lambda}(\eta_t \neq \emptyset \ \forall t \geqslant 0) > 0$*

Indeed from the graphical construction it holds

$$\mathbb{P}^{\{A\}}_{d,\lambda}(\eta_t \neq \emptyset \ \forall t \geqslant 0) = P((A,0) \to \infty)$$

with

$$\{(A,0) \to \infty\} := \cup_{j \in A}\{(j,0) \to \infty\}$$

And therefore

$$\theta_d(\lambda) = P((x,0) \to \infty) \leqslant P((A,0) \to \infty) \leq \sum_{j \in A} P((j,0) \to \infty) = |A|\theta_d(\lambda)$$

Exercise 24. *Use the graphical construction to prove that the contact process is additive, namely that for any η, ξ the CP(d,λ) started at $\eta \vee \xi$ has the same law of $(\eta_t \vee \xi_t)_{t \geqslant 0}$, the max at each time among the CP(d,λ) started at η and ξ.*

Let $\lambda_c(d)$ be the critical threshold for ergodicity, namely

$$\lambda_c(d) := \sup\{\lambda \geqslant 0 : \underline{\nu} = \bar{\nu}\}$$

The main results for CP on $\mathbb{Z}^d$ are the two following theorems that we will prove in the remainder of this chapter:

Theorem 4.1.3. *For any $d \in \mathbb{Z}_+$, the critical threshold for survival coincides with the critical threshold for ergodicity, namely $\lambda_c(d) = \bar{\lambda}_c(d)$.*

Theorem 4.1.4. *For any $d \in \mathbb{Z}_+$ it holds $0 < \lambda_c(d) < \infty$.*

Other major results, that we won't prove since they require a heavier machinery are the followings:

Theorem 4.1.5 (Continuity)**.** *For any $d \in \mathbb{Z}_+$ it holds $\theta_d(\lambda_c) = 0$*

Theorem 4.1.6 (Complete convergence)**.** *For any $\pi \in \mathcal{P}$ it holds*

$$\lim_{t\to\infty} \pi P_t = \rho(A)\bar{\nu}_\lambda + (1 - \rho(A))\delta_0$$

where

$$\rho(A) := \int \mathbb{P}^\eta(\eta_t \neq \emptyset, \ \forall t \geqslant 0) d\pi(\eta).$$

This implies in particular that $\bar{\nu}$ and δ_0 are the only extremal invariant measures for CP.

We emphasise that complete convergence does not follow from monotonicity. A counterexample is the case of CP on regular trees, where despite monotonicity it has been proven that there exists $\tilde{\lambda_c}$ s.t. $\tilde{\lambda_c} > \lambda_c$ and for $\lambda \in (\lambda_c, \tilde{\lambda}_c)$ complete convergence does not hold (see [Lig85]).

Another issue which has been studied is the following: for $\lambda > \lambda_c$, how do infected areas look like at large time when we start from a single infection and we condition on survival? the rough answer is that the growth of the infected regions is linear. Let us conclude with a conjecture on the *behavior at criticality* that, despite very clear numerical confirmation and non rigorous analytical results in the physics community, still lacks a full rigorous proof:

Conjecture 1. *$\exists \beta = \beta(d) > 0$ s.t. for CP on $\mathbb{Z}^d$ it holds*

$$\theta(\lambda) \sim (\lambda - \lambda_c)^\beta \quad \textit{for } \lambda \downarrow \lambda_c,$$

namely $\lim_{\lambda\to\lambda_c} \log\theta(\lambda)/\log(\lambda - \lambda_c)^\beta = 1$. *Furthermore*

- *β is universal once the spatial dimension has been fixed, namely should not change by varying some details in the definition of CP (while λ_c is certainly not universal),*

- *$\beta(d) = 1$ for $d \geqslant 4$*

4.2 Survival vs ergodicity: proof of Theorem 4.1.3

The key ingredients of this proof are: (i) the self duality of the contact process (see Exercise 20) and (ii) Theorem 3.6.5 that connects survival and ergodicity for some special couples of dual processes, those defined in Theorem 3.6.2.

Proof. Fix $\lambda > 0$. Theorem 3.6.5 and Exercise 20 imply that CP(λ) is ergodic iff for any A a finite subset of $\mathbb{Z}^d$ it holds $\mathbb{P}^A(\tau < \infty)$, with $\mathbb{P}^A$ the evolution of the infected sets of the CP(λ) when at time 0 the set of infected sites coincides with A. On the other hand for CP survival from a single site is equivalent to survival from any finite set A (see Remark 4.1.2). Thus for CP the ergodic regime coincides with the regime in which the process dies out. □

4.3 $\lambda_c \in (0, 1)$: proof of Theorem 4.1.4

The proof follows immediately once the three lemmas below are proved.

Lemma 4.3.1. $\lambda_c(d) \geqslant \frac{1}{2d}$

Lemma 4.3.2. $\lambda_c(d) \leqslant \lambda_c(1)/d$

Lemma 4.3.3. $\lambda_c(1) < \infty$

Prof of Lemma 4.3.1. According to Theorem 3.6.5 CP(λ) dies out iff CP(λ) started from any finite set dies out. Consider an initial finite set of infections $A_0 = A$ with $|A| < \infty$. Then $|A_t|$ decreases by 1 at rate $|A_t|$ and increases by 1 at a rate which is upper bounded by $2d\lambda|A_t|$ (since a site can create a new infection only on an empty nearest neighbour). Thus if $2d\lambda < 1$, $|A_t|$ has a drift to decrease and will eventually hit 0. Therefore CP(λ) certainly dies out if $\lambda < 1/(2d)$, which yields $\lambda_c \geqslant 1/(2d)$. □

Proof of Lemma 4.3.2. The idea here is to couple versions of CP on different dimensions and with different parameters λ. More precisely we consider

- $(A_t)_{t \geqslant 0}$ the CP on $\mathbb{Z}^d$ with infection rate λ and started with $A_0 = \{(0, \dots, 0)\}$ (only the site with all the d coordinates equal to zero is infected)
- $(B_t)_{t \geqslant 0}$ the CP on $\mathbb{Z}$ with infection rate $d\lambda$ and started with $B_0 = \{0\}$.

Now we will prove that it holds

$$\mathbb{P}^{(0,\dots,0)}(A_t \neq \emptyset) \geqslant \mathbb{P}^0(B_t \neq \emptyset) \quad \forall t \geqslant 0. \tag{4.3.1}$$

This implies in particular that if B_t survives also A_t survives , namely

$$\text{if } d\lambda > \lambda_c(1) \text{ then necessarily } \lambda > \lambda_c(d)$$

which yields

$$\lambda_c(d) \leqslant \frac{\lambda_c(1)}{d}$$

We are left with proving inequality (4.3.1). Define the projection map $\pi_d : \mathbb{Z}^d \to \mathbb{Z}$ as

$$\pi_d(x_1, \dots, x_d) := \sum_{i=1}^{d} x_i$$

and let for $A \subset \mathbb{Z}^d$

$$\pi_d(A) := \{\pi_d(x) : x \in A\} \subset \mathbb{Z}.$$

Note that with this notation $B_0 \subset \pi_d(A_0)$ (actually $\pi(A_0) = B_0$ We will now construct a coupling of the processes $(A_t)_{t \geqslant 0}$ and $(B_t)_{t \geqslant 0}$ preserving this property at any later time, namely satisfying

$$B_t \subset \pi_d(A_t). \tag{4.3.2}$$

This implies that at any time if $B_t \neq \emptyset$ also $A_t \neq \emptyset$ and (4.3.1) follows. Start by associating to each $y \in \mathbb{Z}$, the point $\bar{x}(y) \in \mathbb{Z}^d$ s.t. $\bar{x}(y)\vec{e}_1 = y$ and $\bar{x}(y)\vec{e}_i = 0$ for $i \geqslant 2$. Then fix the Poisson point processes to construct $(A_t)_{t \geqslant 0}$ (namely the realisation of the arrival times of the healing maps for each site, and of the $2d$ infection maps pointing from each site to its nearest neighbours). Now we construct a coupled process $\tilde{B}_t$ as follows. We let $\tilde{B}_0 = B$ and let it evolve according to the following rules

- whenever an event occurs for the healing maps of a site $x \in \mathbb{Z}^d$ (i.e. in the $d+1$-dimensional drawing of the graphical representation for $(A_t)_{t \geqslant 0}$ we see a cross) and if $x = \bar{x}(y)$ for some $y \in \mathbb{Z}$, we heal at this time site y, namely we set it healthy if it was occupied;

- whenever an event occurs for the infection map from $x \in \mathbb{Z}^d$ to one of the d points $x - \vec{e}_i$ occurs (i.e. in the $d+1$-dimensional drawing of the graphical representation for $(A_t)_{t \geqslant 0}$ we see an arrow from x to $x - \vec{e}_i$), if $x = \bar{x}(y)$ for some $y \in \mathbb{Z}$, we infect at this time site $y - 1$ if y was infected;

- analogously, whenever an arrival time of the infection map from $x \in \mathbb{Z}^d$ to one of the d points $x + \vec{e}_i$ occurs, and if $x = \bar{x}(y)$ for some $y \in \mathbb{Z}$, we infect at this time site $y + 1$ if y was infected;

- on all the other times the process does not evolve.

It is not difficult to verify that the marginal under this coupling of $(\tilde{B}_t)_{t \geqslant 0}$, has the same evolution as the process B_t (use the fact that the union of the arrival times of d Poisson point sets of intensity λ is a Poisson point set of intensity $d\lambda$), namely the above construction provides a coupling of $(A_t)_{t \geqslant 0}$ and $(B_t)_{t \geqslant 0}$. Furthermore, it is also not difficult to verify that this coupling conserves the relation $B_t \subset \pi_d(A_t)$.

□

There are several alternative proves of Lemma 4.3.3. One of these proof uses comparison with *directed (or oriented) edge percolation on* $\mathbb{Z}^2$. Another proof, that we have decided to follow here, uses comparison with *directed (or oriented) site percolation on* $\mathbb{Z}^2$. The interested reader may find the first proof on Section 2.5 and 2.6 of [Swaa] or on Section 7.2, 7.3 and 7.4 of [Swab].

Definition 4.3.4 (Directed (or oriented) site percolation on $\mathbb{Z}^2$)**.** *Draw from each site of* $x \in \mathbb{Z}^2$ *two arrow directing towards its two neighbours in the positive direction, namely an arrow from* $x \to x + \vec{e}_1$ *and an arrow from* $x \to x + \vec{e}_2$*. Fix* $p \in [0,1]$ *and let* μ_p *be Bernoulli product measure at density* p *on* $\mathbb{Z}^2$*, namely each site, independently from all others, is empty with probability* $1 - p$ *and filled with probability* p*. We also say that filled sites are* open *and empty sites are* closed*. We let* $x \to y$ *for* $x, y \in \mathbb{Z}^2$ *iff there is a path that*

- *connects* x *to* y
- *traverses only edges along the orientation of the arrows*
- *visits only open sites.*

Let C_0 *be the set of all sites that are connected to the origin*

$$C_0 := \{x \in \mathbb{Z}^2 : (0,0) \to x\}$$

and let

$$p_c := \sup\{p \in [0,1] \ \ s.t. \ \ \mu_p(|C_0|) = \infty) = 0\}$$

Theorem 4.3.5.

$$p_c < \frac{80}{81} \tag{4.3.3}$$

Proof. The proof goes through a Peierls contour argument very similar to the one you met for the proof of the transition for the Ising model in the first part of the course. I did not have time to detail it here ((4.3.3)) but it must to be studied for the exam: please find all details on file Additional.pdf. □

Proof of Lemma 4.3.3. The proof uses a coupling argument that shows that for λ sufficiently large and δ sufficiently small then CP observed at times $t_i = i\delta$ dominates oriented percolation on $\mathbb{Z}^2$ with $\mu_p > 80/81$ [1] and therefore, thanks to Theorem 4.3.5, CP necessarily survives. I did not have time to detail the coupling in my notes but it must be studied for the exam: please find all details in the file Additional.pdf. □

[1] Actually this domination holds for an oriented lattice $\mathbb{Z}^2$ that has been tilted of 45, see the first page of file Additional.pdf for details

Bibliography

[EK85] S.N. Ethier and T.G. Kurtz. *Markov Processes: characterisation and convergence.* Wiley, New York, 1985.

[KL99] C. Kipnis and C. Landim. *Scaling limits of Interacting particle systems.* Springer-Verlag, 1999.

[Lig85] T.M. Liggett. *Interacting particle systems*, volume 276 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences].* Springer-Verlag, New York, 1985.

[Spo91] H. Spohn. *Large scale dynamics of interacting particle systems.* Springer-Verlag, 1991.

[Swaa] J. Swart. *Interacting particle systems.* Notes July 2011.

[Swab] J. Swart. *Interacting particle systems.* Notes January 2020, see https://arxiv.org/pdf/1703.10007.pdf.