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Diffusive scaling of the Kob–Andersen model in $\mathbb{Z}^d$

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Abstract. We consider the Kob–Andersen model, a cooperative lattice gas with kinetic constraints which has been widely analysed in the physics literature in connection with the study of the liquid/glass transition. We consider the model in a finite box of linear size L with sources at the boundary. Our result, which holds in any dimension and significantly improves upon previous ones, establishes for any positive vacancy density q a purely diffusive scaling of the relaxation time $T_{\mathrm{rel}}(q, L)$ of the system. Furthermore, as $q \downarrow 0$ we prove upper and lower bounds on $L^{-2}T_{\mathrm{rel}}(q, L)$ which agree with the physicists belief that the dominant equilibration mechanism is a cooperative motion of rare large droplets of vacancies. The main tools combine a recent set of ideas and techniques developed to establish universality results for kinetically constrained spin models, with methods from bootstrap percolation, oriented percolation and canonical flows for Markov chains.

Résumé. On considère le modèle de Kob–Andersen (KA), un modèle de particules sur réseau avec dynamique conservative et contraintes cinétiques. KA a été étudié en profondeur dans la literature physique en relation avec l'étude de la transition liquide/verre. On étudie le modèle dans une boite finie de taille L avec un réservoir de particules aux bords. Notre résultat, qui est valable en toute dimension et qui améliore d'une façon très significative les résultats précédents, établit que pour toute densité de sites vides q telle que $q > 0$, le temps de relaxation du système, $T_{\mathrm{rel}}(q; L)$, est purement diffusif. De plus, on établit des bornes supérieures et inférieures pour $L^{-2}T_{\mathrm{rel}}(q; L)$ qui sont en accord avec l'heuristique des physiciens selon laquelle, quand $q \downarrow 0$, le mécanisme dominant de relaxation est un mouvement coopératif de grandes et rares gouttelettes formées par des sites vides. Nos outils principaux mélangent : un ensemble de techniques et idées developpées pour établir les résultats d'universalité pour les modèles de spin avec contraintes cinétiques ; des méthodes developpeés en percolation bootstrap ; des méthodes developpées en percolation dirigée ; et la technique de chemins canoniques pour les chaînes de Markov.

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1. Introduction

Kinetically constrained lattice gases (KCLG) are interacting particle systems on the integer lattice $\mathbb{Z}^d$ with hard core exclusion, i.e. with the constraint that on each site there is at most one particle. A configuration is therefore defined by giving for each site $x \in \mathbb{Z}^d$ the occupation variable $\eta_x \in \{0, 1\}$, which represents an empty or occupied site respectively. The evolution is given by a continuous time Markov process of Kawasaki type, which allows the exchange of the occupation variables across a bond $e = (x, y)$ of neighbouring sites x and y with rate $c_e(\eta)$ (bonds are non oriented, namely $(x, y) \equiv (y, x)$ and $c_{yx}(\omega) = c_{xy}(\omega)$). This exchange rate equals one if the current configuration satisfies an a priori specified local constraint and zero otherwise. In the former case we say that the exchange is *legal*. A key feature of the constraint is that it does not depend on the occupation variables η_x, η_y and therefore for any $q \in [0, 1]$ detailed balance w.r.t. $(1-q)$-Bernoulli product measure μ is verified. Thus, μ is an invariant reversible measure for the process. However, at variance with the simple symmetric exclusion process (SSEP), that corresponds to the case in which the constraint is always verified, KCLG have several other invariant measures. This is related to the fact that due to the constraints there exist *blocked configurations*, namely configurations for which some exchange rates remain zero forever.

KCLG have been introduced in physics literature (see [11,21] for a review) to model the liquid/glass transition that occurs when a liquid is suddenly cooled. In particular they were devised to mimic the fact that the motion of a molecule in a low temperature (dense) liquid can be inhibited by the geometrical constraints created by the surrounding molecules. Thus, to encode this local caging mechanism, the exchange rates of KCLG require a minimal number of empty sites in a certain neighbourhood of $e = (x, y)$ in order for the exchange at e to be allowed. There exists also a non-conservative version of KCLG, the so called Kinetically Constrained Spin Models, which feature a Glauber type dynamics and have been recently studied in several works (see e.g. [4,17,18] and references therein).

In this paper we focus on the class of KCLG which has been most studied in physics literature, the so-called Kob–Andersen (KA) models [13]. Each KA model leaves on $\mathbb{Z}^d$, with $d \geq 2$, and is characterised by an integer parameter k with $k \in [2, d]$. The nearest neighbour exchange rates are defined as follows: $c_{xy}(\eta) = 1$ if at least $k - 1$ neighbours of x different from y are empty and at least $k - 1$ neighbours of y different from x are empty too, $c_{xy} = 0$ otherwise. The name *KA-kf model* is used in the literature to refer to the model with parameter k.[1] The choices $k = 1$ and $k > d$ are discarded: $k = 1$ would correspond to SSEP; $k > d$ would yield the existence of finite clusters of particles which are blocked, and therefore for this choice at any $q < 1$ the infinite volume process would not be ergodic.[2] For example for $k = 3, d = 2$ a 2×2 square fully occupied by particles is blocked: none of these particles can ever jump to their neighboring empty positions.

In [23] it has been proven that for all $k \in [2, d]$ the infinite volume KA-kf models are ergodic for all $q \in (0, 1]$, thus disproving previous conjectures [10,13,14] on the occurrence of an ergodicity breaking transition at $q_c > 0$ based on numerical simulations. In [2] it has been proved that for all $q \in (0, 1]$ the rescaled position of a marked particle at time $\varepsilon^{-2}t$ converges as $\varepsilon \to 0$, to a d-dimensional Brownian motion with non-degenerate diffusion matrix. This again disproves a conjecture that had been put forward in physics literature on the occurrence of a diffusive/non-diffusive transition at a finite critical density $q_c > 0$ [13,14]. Motivated by the fact that numerical simulations [13,16] suggest the possibility of an anomalous slowing down at high density, in [5] the relaxation time T_{rel} (namely the inverse of the spectral spectral gap) has been studied. For KA-2f in dimension $d = 2$ it has been proved that in a box of linear size L with boundary sources, T_{rel} is upper bounded by $L^2 \log L$ at any $q \in (0, 1]$. The same technique can be extended to establish an analogous upper bound for all choices of d and $k \in [2, d]$. By using this result in [5] it is also proved that the infinite volume time auto-correlation of local functions decays at least as $1/t$ modulo logarithmic corrections [5]. A lower bound as $1/t^{d/2}$ follows by comparison with SSEP.

The description of the state of the art for KCLG would not be complete without mentioning that a purely diffusive scaling L^2 for the inverse of the spectral gap has been established for some KCLG [1,12,20], with and without boundary sources. However, all the models considered in these papers belong to the so called class of *non-cooperative* KCLG, namely models for which the constraints are such that it is possible to construct a finite group of vacancies, the mobile cluster, with two key properties. (i) For any configuration it is possible to move the mobile cluster to any other position in the lattice by a sequence of allowed exchanges; (ii) any nearest neighbour exchange is allowed if the mobile cluster is in a proper position in its vicinity. The existence of finite mobile clusters is a key tool in the analysis of non-cooperative KCLG and allows the application of some techniques (e.g. paths arguments) developed for SSEP. It is immediate to verify that instead, for all $k \in [2, d]$, KA models belong to the *cooperative* class, which contain all models that are not non-cooperative. For example for $k = d = 2$, one can easily check that there cannot exist a finite mobile cluster by noticing that any a fully occupied double column which spans the lattice can never be destroyed.

Besides being a challenging mathematical issue, developing a new set of techniques to prove a purely diffusive scaling for KA and for cooperative models in general is important from the point of view of the modelization of the liquid/glass transition, since in this context cooperative models are undoubtedly the most relevant ones. Indeed, very roughly speaking, non cooperative models behave like a rescaled SSEP with the mobile cluster playing the role of a single vacancy and are less suitable to describe the rich behavior of glassy dynamics.

Here we significantly improve upon the existing results, by establishing $T_{\mathrm{rel}}(L) \approx L^2$ for all KA-kf models in a finite box of side L of $\mathbb{Z}^d$, $d \geq 2$, with sources at the boundary (Theorem 1). This is the first result establishing a pure diffusive scaling for a cooperative KCLG. The technique that we develop, which is completely different from the one in [5], combines a set of ideas and techniques recently developed by two of the authors to establish universality results for kinetically constrained spin models [18], with methods from oriented percolation and canonical flows for Markov chains. Although we have applied our technique for KA models, we expect our tools to be robust enough to be extended to analyse other KCLG in the ergodic regime.

Our main result (cf. Theorem 1) establishes upper and lower bounds on T_{rel} of the form $C_-(q) \times L^2 \leq T_{\mathrm{rel}}(q, L) \leq C_+(q) \times L^2$ with two parameters C_+, C_- that diverge as $q \to 0$. Remarkably, the divergence of both $C_-(q)$ and $C_+(q)$

[1] Here f stands for "facilitation", since k denotes the minimal number of empty sites to allow motion.

[2] Here ergodic means that zero is a simple eigenvalue for the generator of the Markov process in $\mathbb{L}_2(\mu)$.

has the same leading behavior, and it is qualitatively in agreement with that conjectured by the physicists [23] and based on the assumption that the dominant mechanism driving the system to equilibrium is a complex cooperative motion of rare large droplets of vacancies.

The plan of the paper is the following. In Section 2 we introduce the notation and the results. In Section 3 we prove the upper bound on the relaxation time in several steps. We start by performing a coarse graining (Section 3.1), and proving a coarse-grained constrained Poincaré inequality (Section 3.3). A key ingredient for this proof is the probability that a certain good event has a large probability, a result that is proved in Section 3.2 by using tools from supercritical oriented percolation. In Sections 3.4 and 3.5 we use canonical flows techniques in order to bound from above the r.h.s. of the coarse-grained Poincaré inequality with the Dirichlet form of KA model, and we conclude by using the variational characterization of the spectral gap. Finally, in Section 4 we prove the lower bound on the relaxation time, finding an appropriate test function and using again the variational characterization of the gap.

2. The Kob–Andersen model and the main result

Given an integer L, and a parameter $q \in (0,1)$, we let $\Lambda = [L]^d$

$$\partial\Lambda = \big\{x \in \Lambda : \exists y \notin \Lambda \text{ with } \|x-y\|_1 = 1\big\}$$

and consider the probability space $(\Omega_\Lambda, \mu_\Lambda)$ where

$$\Omega_\Lambda = \big\{\eta \in \{0,1\}^{\mathbb{Z}^d} : \eta_x = 0 \text{ for all } x \notin \Lambda\big\}$$

and μ_Λ is the product Bernoulli$(1-q)$ measure. Given $\eta \in \Omega_\Lambda$ and $V \subset \Lambda$, we shall say that V is empty (for η) if $\eta_x = 0$ $\forall x \in V$.

Fix an integer $k \in [2,d]$ and, for any given a pair of nearest neighbour sites x, y in Λ, write $c_{xy}(\cdot)$ for the indicator of the event that both x and y have at least $k-1$ empty neighbours among their nearest neighbours in Λ without counting x, y

$$c_{xy}(\eta) = \begin{cases} 1 & \text{if } \sum_{z:\|x-z\|_1=1, z\neq y}(1-\eta_z) \geq k-1 \\ & \text{and } \sum_{z:\|y-z\|_1=1, z\neq x}(1-\eta_z) \geq k-1, \\ 0 & \text{otherwise,} \end{cases} \tag{2.1}$$

and set

$$\eta_z^{xy} := \begin{cases} \eta_z & \text{if } z \notin \{x,y\}, \\ \eta_x & \text{if } z = y, \\ \eta_y & \text{if } z = x, \end{cases}$$

$$\eta_z^{x} := \begin{cases} \eta_z & \text{if } z \neq x, \\ 1-\eta_x & \text{if } z = x. \end{cases}$$

The Kob–Andersen model in Λ with parameter k, for short the KA-kf model, with *constrained exchanges* in Λ and *unconstrained sources* at the boundary $\partial\Lambda$ is the continuous time Markov process defined through the generator which acts on local functions $f : \Omega_\Lambda \to \mathbb{R}$ as

$$\mathcal{L}f(\eta) = \sum_{\substack{x,y\in\Lambda \\ \|x-y\|_1=1}} c_{xy}(\eta)\big[f\big(\eta^{xy}\big) - f(\eta)\big] + \sum_{x\in\partial\Lambda}\big[(1-\eta_x)(1-q) + \eta_x q\big]\big[f\big(\eta^x\big) - f(\eta)\big]. \tag{2.2}$$

In words, every pair of nearest neighbours sites x, y such that $c_{xy}(\eta) = 1$, with rate one and independently across the lattice, exchange their states η_x, η_y. In the sequel we will sometimes refer to such a move as a *legal exchange*. Furthermore every boundary site, with rate one and independently from anything else, updates its state by sampling it from the Bernoulli$(1-q)$ measure. Notice that these latter moves are unconstrained and that for $k=1$ the KA-1f chain coincides with the symmetric simple exclusion in Λ with sources at $\partial\Lambda$. It is easy to check that the KA-kf chain is reversible w.r.t. μ_Λ and irreducible thanks to the boundary sources. Let $T_{\mathrm{rel}}(q, L)$ be its *relaxation time* i.e. the inverse of the spectral gap in the spectrum of its generator $\mathcal{L}_\Lambda$ (see e.g. [15]).

Theorem 1. *For any $q \in (0,1)$ there exist two constants $C_+(q), C_-(q)$ such that*

$$C_-(q)L^2 \le T_{\mathrm{rel}}(q,L) \le C_+(q)L^2.$$

Moreover, as $q \to 0$ the constants $C_\pm(q)$ can be taken equal to

$$C_+(q) = \begin{cases} \exp_{(k-1)}(c/q^{1/(d-k+1)}) & \text{if } 3 \le k \le d, \\ \exp(c\log(q)^2/q^{1/(d-1)}) & \text{if } k = 2 \le d, \end{cases} \tag{2.3}$$

$$C_-(q) = \begin{cases} \exp_{(k-1)}(c'/q^{1/(d-k+1)}) & \text{if } 3 \le k \le d, \\ \exp(c'/q^{1/(d-1)}) & \text{if } k = 2 \le d, \end{cases} \tag{2.4}$$

where $\exp_{(r)}$ *denotes the r-times iterated exponential and c, c' are a numerical constants.*

3. Proof of the upper bound in Theorem 1

The standard variational characterisation of the spectral gap of $\mathcal{L}$ (see e.g. [15]) implies immediately that the *upper bound* on $T_{\mathrm{rel}}(q,L)$ of Theorem 1 is equivalent to the Poincaré inequality

$$\mathrm{Var}(f) \le C(q)L^2\mathcal{D}(f) \quad \forall f : \Omega_\Lambda \mapsto \mathbb{R}, \tag{3.1}$$

where $C(q)$ is as (2.3). Above $\mathrm{Var}(f)$ denotes the variance of f w.r.t. the reversible measure μ and $\mathcal{D}(f)$ is the Dirichlet form associated to the generator (2.2)

$$\mathcal{D}(f) = \sum_{\substack{x,y\in\Lambda \\ \|x-y\|_1=1}} \mu\big(c_{xy}(\nabla_{xy}f)^2\big) + \sum_{x\in\partial\Lambda} \mu\big(\mathrm{Var}_x(f)\big), \tag{3.2}$$

where $(\nabla_{xy}f)(\eta) := f(\eta^{xy}) - f(\eta)$ and $\mathrm{Var}_x(f)$ is the local variance w.r.t. η_x, i.e. the variance conditioned on $\{\eta_y\}_{y\neq x}$.

Remark 3.1. Consider two systems with sizes $L < L'$, and let γ, γ' be the spectral gaps associated with the two generators. Then

$$\gamma' \le (2d+1)\gamma. \tag{3.3}$$

To see that, let $\Lambda = \Lambda_L$, $\Lambda' = \Lambda_{L'}$, and let $\mathcal{D}_\Lambda, \mathcal{D}_{\Lambda'}$ be the Dirichlet forms of the dynamics in Λ, Λ'. Take a function $f : \Omega_{\Lambda'} \mapsto \mathbb{R}$ depending only on the variables in Λ and observe that $\mathrm{Var}_{\Lambda'}(f) = \mathrm{Var}_\Lambda(f)$. Next we bound $\mathcal{D}_{\Lambda'}(f)$ as:

$$\begin{aligned}
\mathcal{D}_{\Lambda'}(f) &= \sum_{\substack{x,y\in\Lambda' \\ \|x-y\|_1=1}} \mu\big(c_{xy}(\nabla_{xy}f)^2\big) + \sum_{x\in\partial\Lambda'\cap\partial\Lambda} \mu\big(\mathrm{Var}_x(f)\big) \\
&\le \mathcal{D}_\Lambda(f) + \sum_{\substack{x\in\partial\Lambda, y\in\Lambda'\setminus\Lambda \\ \|x-y\|_1=1}} \mu\big(c_{xy}(\nabla_{xy}f)^2\big) \\
&\le \mathcal{D}_\Lambda(f) + \sum_{\substack{x\in\partial\Lambda, y\in\Lambda'\setminus\Lambda \\ \|x-y\|_1=1}} \mu\big(2\,\mathrm{Var}_x(f)\big) \le (2d+1)\mathcal{D}_\Lambda(f).
\end{aligned}$$

The last line follows because f does not depend on η_y for $y \notin \Lambda$ and therefore an exchange for the pair xy, $x \in \Lambda$ and $y \in \Lambda' \setminus \Lambda$, is equivalent to a spin flip at x. Thus,

$$\mathrm{Var}_\Lambda(f) = \mathrm{Var}_{\Lambda'}(f) \le \frac{1}{\gamma'}\mathcal{D}_{\Lambda'}(f) \le \frac{2d+1}{\gamma'}\mathcal{D}_\Lambda(f)$$

implying equation (3.3).

We will prove (3.1) in several steps. The first step consists in proving a coarse-grained constrained Poincaré inequality with long range constraints (see Proposition 3.15) under the assumption that the probability $\pi_\ell(k,d)$ of a certain *good* event (see Definition 3.9) is sufficiently large. Here ℓ is the mesoscopic scale characterising the coarse-grained construction and $2 \le k \le d$ is the parameter of the KA-model. The necessary tools for this part are developed in Sections 3.1 and 3.2.

The second step (see Section 3.4) consists developing canonical flows techniques (see e.g. [15, Chapter 13.5]) for the KA model in order to bound from above the r.h.s. of the coarse-grained Poincaré inequality by $C(\ell,q)(L/\ell)^2\mathcal{D}(f)$, with $C(\ell,q) \le e^{O(\ell^{d-1}(|\log(q)|+\log(\ell)))}$ (see Proposition 3.24 and Corollary 3.25).

The final step (see Section 3.6) proves that it is possible to choose $\ell = \ell(q,k,d)$ in such a way that $\pi_\ell(d,k)$ is large enough and $C(\ell,q) \le C(q)$ as $q \to 0$, where $C(q)$ is as in (2.3).

3.1. *Coarse graining*

Let $\ell \in \mathbb{N}$ to be fixed later on. By Remark 3.1, we may assume that $N := L/\ell$ satisfies $N = 100^n, n \in \mathbb{N}$, so that, in particular, $\frac{1}{2}\sqrt{N} \in \mathbb{N}$. Later on (see Section 3.6) we will choose ℓ as a function of q and suitably diverging as $q \to 0$. We will then consider the coarse grained lattice of boxes with side ℓ. These boxes will be of the form $B_i = \ell i + [\ell]^d$ for $i \in \mathbb{Z}^d$. In order to distinguish between the standard lattice and the coarse-grained lattice we denote the latter by $\mathbb{Z}^d_\ell$. Vertices of the original lattice will always be called sites and they will be denoted using the letters $x, y, \dots$ while the vertices of $\mathbb{Z}^d_\ell$ will represent boxes and they will always be denoted using the letters $i, j, \dots$.

For $x \in \mathbb{Z}^d$ we let $B(x) := B_{i(x)}$, where $i(x) \in \mathbb{Z}^d_\ell$ is such that $B_{i(x)} \ni x$. We also define $\Lambda_\ell = [N]^d \subset \mathbb{Z}^d_\ell$ so that, in particular, $\Lambda = \bigcup_{i\in\Lambda_\ell} B_i$. Sometimes we shall simply write "the box i" meaning the box B_i and whenever we shall refer to a "box" it will be a generic box i.

Definition 3.2 (Slice). Let E be a subset of the standard basis with size $|E| \le d-1$, $V \subset \mathbb{Z}^d$ a set of sites, and fix a site $x \in V$. Then the $|E|$-dimensional *slice* of V passing through x in the directions of E is defined as $V \cap (x + \operatorname{span} E)$, where $\operatorname{span} E$ is the linear span of E.

Definition 3.3 (Frameable configurations). Given the d-dimensional cube $\mathcal{C}_n = [n]^d \subset \mathbb{Z}^d$ and an integer $j \le d$ we define the jth-frame of $\mathcal{C}_n$ as the union of all $(j-1)$-dimensional slices of $\mathcal{C}_n$ passing through $(1,\dots,1)$. Next we introduce the set of (d,j)-*frameable* configurations of $\{0,1\}^{\mathcal{C}_n}$ as those configurations which are connected by legal KA-jf exchanges inside $\mathcal{C}_n$ to a configuration for which the jth-frame of $\mathcal{C}_n$ is empty.

We are finally ready for our definition of a box B_i being good for a given configuration.

Definition 3.4 (Good boxes). Given $\eta \in \Omega_\Lambda$, we say that the box B is (d,k)-good for η if all $(d-1)$-dimensional slices of B are $(d-1,k-1)$-frameable for all configurations $\eta' \in \Omega_\Lambda$ that differ from η in at most one site. The probability that the d-dimensional box B is (d,k)-good will be denoted by $\pi_\ell(d,k)$.

Remark 3.5. For $d=2$, $k=2$ a box is $(2,2)$-good if it contains at least two empty sites in every row and every column.

Notation warning. Whenever the value of d,k is clear from the context we shall simply write that a box is good if it is (d,k)-good. We shall also say that a vertex $i \in \mathbb{Z}^d_\ell$ is (d,k)-good if the box B_i is (d,k)-good.

3.2. *Tools from oriented percolation*

In this section we collect and prove certain technical results from oriented percolation which will be crucial to prove the aforementioned coarse-grained constrained Poincaré inequality. We shall work on the coarse-grained lattice $\mathbb{Z}^d_\ell$ so that any vertex $i \in \mathbb{Z}^d_\ell$ is representative of the mesoscopic box B_i in the original lattice $\mathbb{Z}^d$. Given a configuration $\eta \in \Omega$ we shall consider the induced subgraph of $\mathbb{Z}^d_\ell$ whose vertices are the representatives of the good boxes for η. In other words, under the measure μ, we declare each vertex of $\mathbb{Z}^d_\ell$ *good* with probability $\pi_\ell(d,k)$ and *bad* with probability $1-\pi_\ell(d,k)$, independently of all the other vertices. We shall study certain oriented percolation features of the random subgraph of $\mathbb{Z}^d_\ell$ consisting of the good vertices when π_ℓ is sufficiently large and the main result here is Proposition 3.11. Throughout this section the parameters d,k will be kept fixed and we shall write $\pi_\ell := \pi_\ell(d,k)$. In the sequel, and up to Proposition 3.11, we shall assume that a partition of the vertices of $\mathbb{Z}^d_\ell$ into good and bad ones has been given.

Definition 3.6 (Paths). An *up-right* or *oriented* path γ in $\mathbb{Z}_\ell^d$ starting at i and of length $n \in \mathbb{N}$ is a sequence $(\gamma^{(1)}, \dots, \gamma^{(n)}) \subset \mathbb{Z}_\ell^d$ such that $\gamma^{(1)} = i$ and $\gamma^{(t+1)} \in \{\gamma^{(t)} + \vec{e}_1, \gamma^{(t)} + \vec{e}_2\}$ for all $t \in [n-1]$. γ is *focused* if $d_\gamma(t) := d(\gamma^{(t)}, \{j : j = i + s(\vec{e}_1 + \vec{e}_2), s \in \mathbb{N}\})$ satisfies $\max_{t\in[n]} d_\gamma(t) \le \sqrt{n}$. Two consecutive elements of γ form an edge of γ and we say that γ, γ' are *edge-disjoint* if they do not share an edge. We say that γ is *good* if $\gamma^{(t)}$ is good for all $t \in [n]$.

Definition 3.7 (Good family of paths). Fix $i \in \mathbb{Z}_\ell^d$. A family of paths $\mathcal{G}$ is said to form a good family for i if the following conditions hold:

(1) All paths in $\mathcal{G}$ are good up-right focused paths starting at i of length $2N$.
(2) The paths of $\mathcal{G}$ are *almost edge-disjoint* i.e. any common edge is at distance at most $\sqrt{N}$ from i.
(3) $|\mathcal{G}| \ge \frac{1}{2}\sqrt{N}$.

Remark 3.8. There are $\sqrt{N}$ sites at distance $\sqrt{N}$ from i that can be reached by an up-right path (recall that distances are in the ℓ_1-norm). Consider now the paths of $\mathcal{G}$ passing through a vertex j whose distance from i is $\sqrt{N}$. Such a path could go either up or to the right, and since these edges are at distance larger than $\sqrt{N}$ from i, (2) in the above definition implies that only two such paths could exist. Therefore, $|\mathcal{G}| \le 2\sqrt{N}$.

Given $i \in \mathbb{Z}_\ell^d$ let D_i be the segment

$$D_i = \left\{ i + \left(\frac{1}{2}\sqrt{N} - t\right)\vec{e}_1 + \left(\frac{1}{2}\sqrt{N} + t\right)\vec{e}_2 \;\Big|\; -\frac{1}{2}\sqrt{N} \le t \le \frac{1}{2}\sqrt{N} \right\}. \tag{3.4}$$

Let also H_n, V_n be the rectangular subsets of the form

$$H_n = \left\{ j \in \mathbb{Z}_\ell^d : j = i + a\vec{e}_1 + b\vec{e}_2, a \in [0, \ell_n], b \in [0, \ell_{n-1}] \right\},$$
$$V_n = \left\{ j \in \mathbb{Z}_\ell^d : j = i + a\vec{e}_1 + b\vec{e}_2, a \in [0, \ell_{n-1}], b \in [0, \ell_n] \right\},$$

where $\ell_n = 10^n$. We shall prove that the existence of a good family of paths for the vertex $i \in \mathbb{Z}_\ell^d$ is guaranteed by the simultaneous occurrence of certain events $\mathcal{A}, \mathcal{B}$ and $\{\mathcal{C}_n\}_{n=1}^{n_*}$, where, recalling the choice of L in the beginning of Section 3.1, n_* is such that $\ell_{n^*} = \sqrt{N}$ (cf. Figure 1).

Definition 3.9 (The events $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}_n$).

(i) Let R_i be the rectangle in $\mathbb{Z}_\ell^d$ whose short sides are D_i and $D_i + 2N(\vec{e}_1 + \vec{e}_2)$. Then $\mathcal{A}$ is the event that there are at least $1.9\sqrt{N}$ edge-disjoint good up-right paths contained in R_i and connecting D_i with $D_i + 2N(\vec{e}_1 + \vec{e}_2)$.
(ii) $\mathcal{B}$ is the event that the set $\bigcup_{t\in[0,\sqrt{N}]}\{i + t\vec{e}_1\} \cup \{i + t\vec{e}_2\}$ is connected to at least $0.7\sqrt{N}$ vertices of $D_i \setminus (H_{n_*} \cup V_{n_*})$ by a good up-right path,
(iii) $\mathcal{C}_n$ is the event that i is good and there exists a good up-right hard-crossing of both V_n and H_n, i.e. a good up-right path connecting the two short sides of $V_n(H_n)$ and which is contained in $V_n(H_n)$.

The next lemma guarantees the existence of a good family of paths for $i \in \mathbb{Z}_\ell^d$. We emphasise that these are paths whose vertices represent good boxes.

Lemma 3.10. *Assume that $\mathcal{A} \cap \mathcal{B} \cap \mathcal{C}_n$ occurs for all $n \in [n_*]$. Then there exists a good family of paths for $i \in \mathbb{Z}_\ell^d$.*

Proof. We show first that i is connected by a good up-right path to the set $D_i \cap H_{n_*}$ and to the set $D_i \cap V_{n_*}$. Let $n_1 = 1$, and define recursively $n_{k+1}, k \in [n_\star - 1]$ as the largest integer $n \in [n_*]$ such that there exists a crossing of H_n starting from the set $\{i + t\vec{e}_2, t \in [0, \ell_{n_k}]\}$. Since the events $\mathcal{C}_n$ all occur, the sequence $\{n_k\}_{k=1}^n$ is strictly increasing as long as $n_k \le n_*$.

Then, starting from $V_1 \equiv V_{n_1}$ we can first follows the lowest hard crossing of H_{n_2} until we reach a hard crossing of V_{n_2}. Then we follow the latter until meeting a hard crossing of H_{n_3} and so on. At the end of this procedure the set V_1, and a fortiori the box i, becomes connected by a good up-right path to the right short side of H_{n_*} and hence also to one of the vertices of $D_i \cap H_{n_*}$. The same construction can be repeated symmetrically by inverting the role of H and V. Therefore we conclude that there exists a good up-right path connecting i to $D_i \cap H_{n_*}$ and a good up-right path connecting i to $D_i \cap V_{n_*}$. See Figure 1.

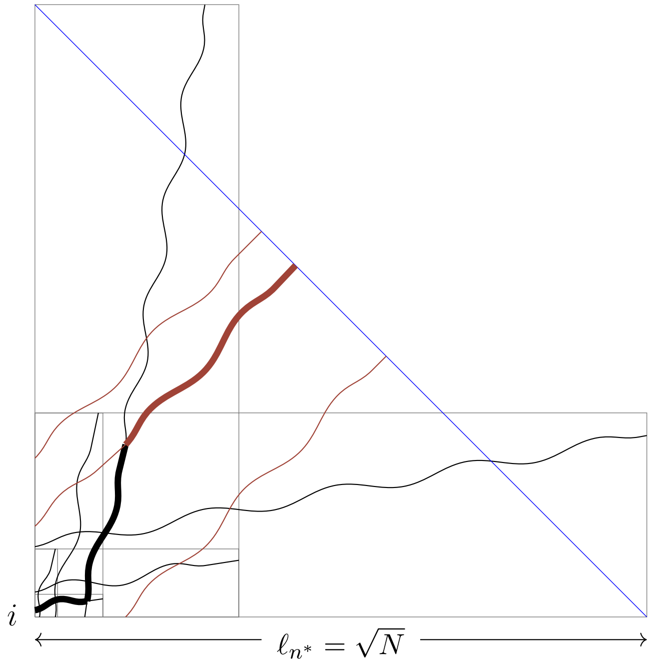

Fig. 1. A graphical illustration of the proof of Lemma 3.10. For better rendering the drawn paths are not perfectly oriented and the ratio among the sides of rectangles in the drawings is not $1/10$ as it should be. The blue segment corresponds to the set D_i. The red paths are the good up-right paths guaranteed by the event $\mathcal{B}$. The blacks paths are the good up-right hard crossings guaranteed by the events C_n.

Recall that $\ell_{n_*} = \sqrt{N}$. Then, since $|D_i \cap H_{n_*}| = |D_i \cap V_{n_*}| = \sqrt{N}/10$ and since each each vertex can be the starting point of at most two edge disjoint paths, there could be at most $2|(D_i \cap H_{n_*}) \cup (D_i \cap V_{n_*})| = 0.4\sqrt{N}$ edge-disjoint path starting from $(D_i \cap H_{n_*}) \cup (D_i \cap V_{n_*})$. Hence, the event $\mathcal{A}$ guarantees that there are at least $1.9\sqrt{N} - 0.4\sqrt{N} = 1.5\sqrt{N}$ edge-disjoint paths starting in $D_i \setminus (H_{n_*} \cup V_{n_*})$. Since each starting point for these paths could belong to at most two paths, at least $0.75\sqrt{N}$ boxes of $D_i \setminus (H_{n_*} \cup V_{n_*})$ are the starting point of an edge-disjoint up-right paths crossing R_i.

Using event $\mathcal{B}$ and noticing that $|D_i \setminus (H_{n_*} \cup V_{n_*})| = 0.8\sqrt{N}$, at most $0.1\sqrt{N}$ boxes in $D_i \setminus (H_{n_*} \cup V_{n_*})$ are not connected to $\bigcup_{t\in[0,\sqrt{N}]}\{i + t\vec{e}_1\} \cup \{i + t\vec{e}_2\}$. That is, the number of boxes in $D_i \setminus (H_{n_*} \cup V_{n_*})$ that are at the same time the starting point of an edge-disjoint up-right path crossing R_i and connected to $\bigcup_{t\in[0,\sqrt{N}]}\{i + t\vec{e}_1\} \cup \{i + t\vec{e}_2\}$ is at least $0.75\sqrt{N} - 0.1\sqrt{N} = 0.65\sqrt{N}$.

Using now the fact that i is connected by a good up-right path to the set $D_i \cap H_{n_*}$ and to the set $D_i \cap V_{n_*}$, we conclude that there exist at least $0.65\sqrt{N}$ good up-right paths from i to $D_i + 2N(\vec{e}_1 + \vec{e}_2)$ which, after crossing D_i become edge-disjoint and never leave R_i. The thick path of Figure 1 is one of these paths, drawn up to its crossing with D_i (R_i is not depicted in the figure due to lack of space). These paths form the sought good family as required. □

Our next task is to prove that if π_ℓ is sufficiently close to one then, uniformly in n, $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}_n$ are very likely. As proved in Section 3.6 that will be the case if the mesoscopic scale ℓ is suitably chosen as a function of q, d, k.

Proposition 3.11. *For any $\lambda > 0$ there exists $\pi_* < 1$ such that for $\pi_\ell \ge \pi_*$ and all $n, N \in \mathbb{N}$*

(a) $\mu(\mathcal{C}_n) \ge 1 - e^{-\lambda \ell_{n-1}}$,
(b) $\mu(\mathcal{B}) \ge 1 - e^{-\lambda\sqrt{N}}$,
(c) $\mu(\mathcal{A}) \ge 1 - e^{-\lambda\sqrt{N}}$.

In particular a family of good paths starting at i exists w.h.p. if π_ℓ is sufficiently close to one.

Proof. (a) This can be proven by a contour argument. Consider the rectangle V_n, and assume that it does not contain a good hard crossing. Then consider the path on the dual lattice that forms the upper contour of the set of sites that are connected to the bottom of the rectangle via an up-right good path. Since there is no vertical crossing, this path necessarily takes ℓ_n steps to the right and ends somewhere on the right boundary of V_n. By using the fact that each time this dual path makes a step to the right or downwards, this implies the presence of a bad vertex, it is not difficult to prove that for

π_ℓ sufficiently large depending on λ

$$\mu(\text{there is not a good hard crossing}) = \mu\big(\mathcal{C}_n^c\big) \le e^{-\lambda \ell_n}.$$

(b) Consider the down-left good oriented paths starting from sites of $D_i \setminus (H_{n_*} \cup V_{n_*})$. The event $\mathcal{B}$ certainly occurs if at least $7/8$ of the points in this set are the starting point of an infinite down-left good oriented path. The upper bound on the probability of $\mathcal{B}$ then follows directly from [8, Theorem 1].[3]

(c) The main tool here is the *max-flow min-cut theorem* (see e.g. [3]). Given a *directed* graph (V, E) a *flow* f is a non-negative function defined on the edges; we write $f(u, v)$ instead of $f(\overrightarrow{uv})$ for the value of the flow on the directed edge $\overrightarrow{uv}$. Given two disjoint sets of vertices $s = \{s_1, \dots, s_k\}$, $t = \{t_1, \dots, t_m\}$ called the *sources* and the *sinks* respectively we say that $f : E \to \mathbb{R}^+$ is a flow from s to t if for all $v \notin \{s_1, \dots, s_k\} \cup \{t_1, \dots, t_m\}$

$$\sum_{u:(u,v)\in E} f(u, v) = \sum_{w:(v,w)\in E} f(v, w).$$

In other words, for all vertices outside $s \cup t$ the incoming flow equals the outgoing flow. Finally, given a *capacity* function $c : E \to \mathbb{R}^+$ we say that a flow f satisfies the capacity constraint if $f(e) \le c(e)$ for all $e \in E$. Given a flow from s to t the *value of the flow* $v(f)$ is defined as the total flow going in the sinks (which is the same as the flow leaving the sources), namely

$$v(f) := \sum_{j=1}^{m} \sum_{v:(v,t_j)\in E} f(v, t_j).$$

A *cut* (S, T) is a partition of V in two subsets S and T, such that all the sources belong to S and all the sinks belong to T. The capacity of a cut (S, T) is the sum of capacities of the edges pointing from S to T.

Theorem (Max-Flow Min-Cut theorem). *Given a set of sources s and a set of sinks t and a capacity function c, the maximum value of a flow from s to t satisfying the capacity constraint is equal the minimum capacity of a cut. Moreover, if all capacities are integers, there is a flow f satisfying the above requirements such that $f(e) \in \mathbb{N}$ for all $e \in E$ and its value $v(f)$ is maximal.*

In order to use this theorem for the proof of part (c) of the proposition we first define our graph $G = (V, E)$. The vertex set is

$$V = \big\{i + a\vec{e}_1 + b\vec{e}_2 : a, b \in [N], a + b \ge \sqrt{N}, |a - b| \le \sqrt{N}\big\} \cap \Lambda_\ell,$$

and the directed edges are

$$E = \big\{\big(j, j'\big) : j \text{ is good and } j' \in \{j + \vec{e}_1, j + \vec{e}_2\}\big\}.$$

Remark 3.12. Notice that the requirement that edges have their starting point only at the good vertices of V is the only place where randomness enters.

We then choose as source set the set:

$$s = \big\{j \in V : \|i - j\|_1 = \sqrt{N}\big\},$$

and as sink set the set:

$$t = V \cap \big\{j \mid (\vec{e}_1, j) = N \text{ or } (\vec{e}_2, j) = N\big\}.$$

Finally we assign unitary capacity to all edges of E. With this choice, the maximal value of a flow f from s to t satisfying $f(e) \in \{0, 1\}$ will be exactly the number of edge-disjoint good up-right paths contained in R_i and connecting D_i with $D_i + 2N(\vec{e}_1 + \vec{e}_2)$. We have thus reduced the proof of part (c) to the following claim:

Claim 3.13. *If π_ℓ is large enough, with probability greater than $1 - e^{-\lambda\sqrt{N}}$ the graph constructed above is such that the capacity of any cut is at least $1.9\sqrt{N}$.*

[3] Though the Theorem is stated for the contact process, it also holds for oriented percolation as stated in [8].

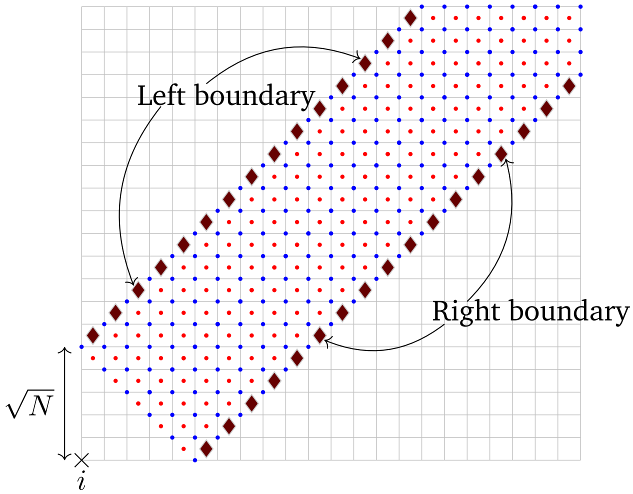

Fig. 2. Black dots are the vertices of V, grey dots are the vertices of V^*, diamonds are the left and right boundary of V^*.

Proof. In order to prove the claim, for every cut (S,T) we will construct a dual path $\gamma := \gamma^*_{S,T}$ that will separate S from T satisfying the following property. If the capacity of the cut is smaller than $1.9\sqrt{N}$ then there are at least $|\gamma|/2 - 0.9\sqrt{N}$ vertices in V which are bad and which neighbor γ. Here $|\gamma| \geq 2\sqrt{N}$ is the length of γ. A simple Peierls argument then proves that the latter event has probability at most $e^{-\lambda\sqrt{N}}$ for any π_ℓ large enough.

First, let us define a dual graph V^* for some fixed (S,T). Its vertices will be the faces of Λ_ℓ that have at least three neighbors in V. That is,

$$V^* = \left\{ i^* \in \Lambda_\ell + \frac{1}{2}\vec{e}_1 + \frac{1}{2}\vec{e}_2 : \#\left\{ i \in V : \left\| i^* - i \right\|_1 = 1 \right\} \geq 3 \right\}.$$

Its (directed) edges will depend on the cut (S,T). For $i^*, j^* \in V^*$, (i^*, j^*) is an edge if $\|i^* - j^*\|_1 = 1$, and if it has a site of S to its left and a site of T to its right. We will separate the vertices of V^* in three parts (see Figure 2):

(1) The right boundary

$$\left\{ i + \left(\sqrt{N} + \frac{1}{2} + a \right)\vec{e}_1 + \left(\frac{1}{2} + a \right)\vec{e}_2 : a \in [N] \right\} \cap V^*,$$

(2) the left boundary

$$\left\{ i + \left(\sqrt{N} + \frac{1}{2} + a \right)\vec{e}_1 + \left(\frac{1}{2} + a \right)\vec{e}_2 : a \in [N] \right\} \cap V^*,$$

(3) the interior, which will include all vertices that are neither in the right nor in the left boundary.

Focusing on a fixed vertex $j^* \in V^*$, we can count the edges going into j^* and the edges going out of j^* if we know which of the neighbouring vertices of V (namely $\{j \in V : \|j^* - i\|_1 = 1\}$) are in S and which are in T. By checking all possibilities, one can verify that the incoming degree of a vertex in the interior of V^* equals its outgoing degree (see right part of Figure 3).

At the boundaries, however, there could be vertices that have an outgoing degree different from the incoming degree. Consider a site on the right boundary (see left part of Figure 3) $j^*_a = i + (\sqrt{N} + \frac{1}{2} + a)\vec{e}_1 + (\frac{1}{2} + a)\vec{e}_2$. Let $j^+_a = j^*_a + \frac{1}{2}\vec{e}_1 + \frac{1}{2}\vec{e}_2$ and $j^-_a = j^*_a - \frac{1}{2}\vec{e}_1 - \frac{1}{2}\vec{e}_2$. If both j^+_a and j^-_a are in S, or if both are in T, then the incoming degree of j^*_a is the same as its outgoing degree. However, if $j^+_a \in S$ and $j^-_a \in T$ then the incoming degree is 1 and the outgoing degree is 0. For the case $j^+_a \in T$ and $j^-_a \in S$, we have an outgoing degree 1 and incoming degree 0. $j^+_a = j^-_{a+1}$, therefore the total outgoing degree of sites on the right boundary is

$$\#\left\{ a : j^-_a \in S, j^-_{a+1} \in T \right\}$$

and the total incoming degree is

$$\#\left\{ a : j^-_a \in T, j^-_{a+1} \in S \right\}.$$

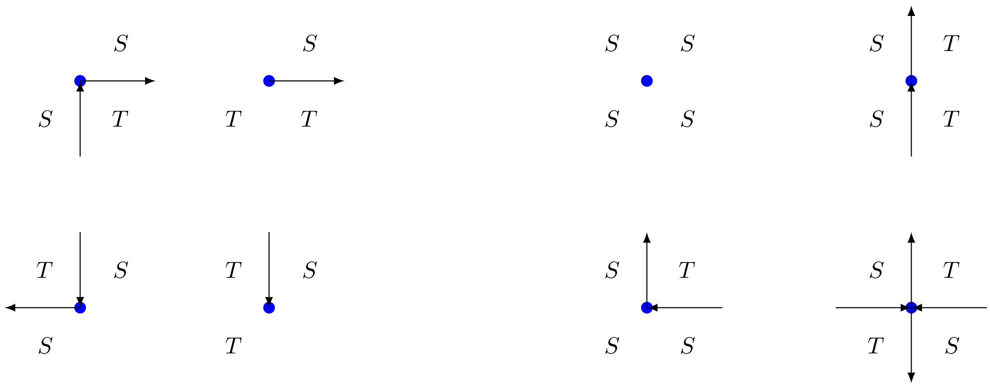

Fig. 3. Incoming and outgoing degrees of vertices on the boundary of V^* (left) and its interior (right).

But since the first site (i.e., j_0^-) is in s and the last site is in t, the incoming degree must be smaller by 1 than the outgoing degree.

By the exact same argument, we can find that the incoming degree of the left boundary is larger by 1 than its outgoing degree. This implies that there exists a dual path $\gamma_* = (j_*^{(1)}, \dots, j_*^{(n)})$, where $j_*^{(1)}$ in on the right boundary and $j_*^{(n)}$ is on the left boundary. In particular, $n \geq 2\sqrt{N}$.

The capacity of the cut (S, T) is at least the number of edges in E pointing from S to T and crossing γ_*. Thanks to the choice of the direction of the edges in V^*, this could be written as

$$\#\bigl\{t : j_*^{(t+1)} - j_*^{(t)} \in \{-\vec{e}_1, \vec{e}_2\} \text{ and } \bigl(j_*^{(t+1)}, j_*^{(t)}\bigr) \text{ crosses an edge in } E\bigr\}.$$

We therefore consider the number of steps that γ_* takes in each direction:

$$\begin{aligned}
R &= \#\bigl\{t : j_*^{(t+1)} - j_*^{(t)} = \vec{e}_1\bigr\}, \\
L &= \#\bigl\{t : j_*^{(t+1)} - j_*^{(t)} = -\vec{e}_1\bigr\}, \\
U &= \#\bigl\{t : j_*^{(t+1)} - j_*^{(t)} = \vec{e}_2\bigr\}, \\
D &= \#\bigl\{t : j_*^{(t+1)} - j_*^{(t)} = -\vec{e}_2\bigr\}.
\end{aligned}$$

Observe that $i_*^{(n)} - i_*^{(1)} = (R - L)\vec{e}_1 + (U - D)\vec{e}_2$, and since

$$\begin{aligned}
\bigl(\vec{e}_1 - \vec{e}_2, j_*^{(1)}\bigr) &= (\vec{e}_1 - \vec{e}_2, i) + \sqrt{N}, \\
\bigl(\vec{e}_1 - \vec{e}_2, j_*^{(n)}\bigr) &= (\vec{e}_1 - \vec{e}_2, i) - \sqrt{N},
\end{aligned}$$

$U + L - D - R = 2\sqrt{N}$. Therefore, since $U + L + R + D = n$ we get $U + L = \frac{n}{2} + \sqrt{N}$.

Definition 3.14. We will say that a pair (j, j') of vertices of V form a *erased edge* if the vertex j is bad and $j' \in \{j + \vec{e}_1, j + \vec{e}_2\}$. In other words, these are the edges of the original Λ_ℓ that do not belong to our graph $G = (V, E)$.

Assume now that the capacity of the cut is less than $1.9\sqrt{N}$. From the previous observations it follows that γ_* must cross at least $U + L - 1.9\sqrt{N} = n/2 - 0.9\sqrt{N}$ erased edges. Since every such erased edge comes from a bad vertex, and since at most two erased edges could share the same bad vertex, at least $n/4 - 0.45\sqrt{N}$ of the vertices to the left of γ_* are bad. Therefore, the probability that there exists a cut with capacity less than $1.9\sqrt{N}$ is upper bounded by the probability that there exists a dual path of length $n \geq 2\sqrt{N}$ with at least $n/4 - 0.45\sqrt{N}$ bad vertices on its left. Since there are at most 2^n dual paths of length n, if π_ℓ was taken large enough depending on λ, we get

$$\mu(\text{capacity of any cut} \geq 1.9\sqrt{N}) \geq 1 - \sum_{n=2\sqrt{N}}^{\infty} 2^n \sum_{k=n/4-0.45\sqrt{N}}^{n} \binom{n}{k} (1 - \pi)^k \geq 1 - e^{-\lambda\sqrt{N}}.$$

□

The proof of the proposition is complete. □

3.3. *A long range Poincaré inequality*

Recall the setting of Sections 3.1, 3.2 and in particular Definition 3.7 of a good family of paths for a vertex $i \in \mathbb{Z}^d_\ell$. Let $Q_i = i + \{0,1\}^d \setminus \{0\}^d \subset \mathbb{Z}^d_\ell$ and define

$$\hat{c}_i = \begin{cases} 1 & \text{if any } j \in Q_i \text{ is good and there exists a good family of paths for } i + \vec{e}_1, \\ 0 & \text{otherwise.} \end{cases} \tag{3.5}$$

In this section we shall prove the following result. Recall that $\pi_\ell := \pi_\ell(d,k)$ is the probability that any given $i \in \mathbb{Z}^d_\ell$ is (d,k)-good.

Proposition 3.15. *There exists $\pi_* < 1$ such that for any $\pi_\ell \geq \pi_*$ and any local function $f : \Omega_\Lambda \to \mathbb{R}$*

$$\operatorname{Var}(f) \leq 4 \sum_{i \in \Lambda_\ell} \mu\big(\hat{c}_i \operatorname{Var}_{B_i}(f)\big). \tag{3.6}$$

Proof. We will closely follow the proof of [18, Theorem 2.6]. Let $\tilde{c}_i$ be the indicator of the event $\mathcal{A} \cap \mathcal{B} \bigcap_{n=1}^{n_*} \mathcal{C}_n$ (see Definition 3.9), together with the requirement that Q_i is good. By Lemma 3.10 $\tilde{c}_i \leq \hat{c}_i$ for all $i \in \mathbb{Z}^d_\ell$. Hence, in order to prove (3.6), it is enough to prove the stronger constrained Poincaré inequality in which in the r.h.s. of (3.6) the constraint $\hat{c}_i$ is replaced by $\tilde{c}_i$. Using Proposition 3.11 together with the obvious bound $\mu(Q_i \text{ is good}) \geq 1 - (2^d - 1)(1 - \pi_\ell)$, the proof of the latter is now identical to the one given in [18, Theorem 2.6]. $\square$

3.4. *Constructing the canonical path on the coarse-grained lattice*

In this section we will construct a set of T-step moves – sequences of at most $T \in \mathbb{N}$ legal moves for the KA dynamics (i.e. legal exchanges or resampling of boundary sites) that could be chained together in order to flip the state of an arbitrary point $x \in \mathbb{Z}^d$. The construction of the move is quite cumbersome, so we will only give here the required definitions and the statement of the result. For details see [22, Chapter 5] and the supplementary file to the arXiv version of this paper.

The next definition describes how to move from one configuration to the other using only legal KA exchanges. It will provide a way to construct, for a given initial configuration, a certain path in configuration space. We emphasise that, unlike the paths introduced earlier, this *is not a geometric path* in Λ_ℓ, but a path in the much larger configuration space Ω_Λ.

Definition 3.16 (**T-step move**). Fix an integer $k \leq d$. Given a finite connected subset V of Λ and $\mathcal{M} \subset \Omega$, a T-step move for KA-kf dynamics $M = (M_0, \dots, M_T)$ taking place in V and with domain $\operatorname{Dom}(M) = \mathcal{M}$ is a function from $\mathcal{M}$ to Ω^{T+1} such that the sequence $M(\eta) = (M_0(\eta), \dots, M_T(\eta))$, $\eta \in \mathcal{M}$, satisfies:

(i) $M_0(\eta) = \eta$,
(ii) for any $t \in [T]$, the configurations $M_{t-1}(\eta)$ and $M_t(\eta)$ are either identical or linked by a legal move contained in V, i.e. either a legal KA-kf exchange among sites $x, y \in V$ or a resampling at a boundary site $z \in \partial V$.

Definition 3.17 (Information loss and energy barrier). Given a T-step move M its *information loss* $\operatorname{Loss}_t(M)$ at time $t \in [T]$ is defined as

$$2^{\operatorname{Loss}_t(M)} = \sup_{\eta' \in \operatorname{Dom}(M)} \#\big\{\eta \in \operatorname{Dom}(M) \mid M_t(\eta) = M_t\big(\eta'\big), M_{t+1}(\eta) = M_{t+1}\big(\eta'\big)\big\}.$$

In other words, knowing $M_t(\eta)$ and $M_{t+1}(\eta)$, we are guaranteed that η is one of at most $2^{\operatorname{Loss}_t(M)}$ configurations. We also set $\operatorname{Loss}(M) := \sup_t \operatorname{Loss}_t(M)$. The energy barrier of M is defined as

$$E(M) = \sup_{\eta \in \operatorname{Dom}(M)} \sup_{t \in [T]} \big|\#\big\{\text{empty sites in } M_t(\eta)\big\} - \#\{\text{empty sites in } \eta\}\big|.$$

The main result is the following proposition that guarantees the existence of a T-step move with a bounded information loss and energy barrier that allows to flip the configuration at x (namely to go from $\eta \to \eta^x$) provided η has a certain up-right good path. Recall that $Q_i = i + \{0,1\}^d \setminus \{0\}^d \subset \mathbb{Z}^d_\ell$ and that $N = L/\ell$.

Proposition 3.18. *Fix an integer $k \le d$. Fix $i \in \Lambda_\ell$ and $x \in B_i$. If $i + \vec{e}_1 \in \Lambda_\ell$ fix also an up-right path γ connecting $i + \vec{e}_1$ to $\partial\Lambda_\ell$. Then there exists a T-step move M with*

$$\mathrm{Dom}(M) = \{\eta \mid \gamma \text{ is good and all } j \in Q_i \cap \Lambda_\ell \text{ are good}\},$$

taking place in $\bigcup_{j\in\gamma} B_j \cup (Q_i \cap \Lambda_\ell)$ and such that, for all $\eta \in \mathrm{Dom}(M)$ and all $t \in [T]$, $M_t(\eta) \in \mathrm{Dom}(M)$ and $M_T(\eta)$ is the configuration η flipped at x. Moreover, $E(M) \le C\ell^{k-1}$, and for all $j \in \Lambda_\ell$

$$\mathrm{Loss}(M) \le C \log_2(\ell)\ell, \qquad T \le CN\ell^\lambda, \qquad \left|\mathcal{T}_M^{(j)}\right| \le C\ell^\lambda \quad \text{for } k = 2,$$

$$\mathrm{Loss}(M) \le C\ell^d, \qquad T \le CN2^{\ell d}, \qquad \left|\mathcal{T}_M^{(j)}\right| \le C2^{\ell d} \quad \text{for } k \ge 3,$$

where $\mathcal{T}_M^{(j)}$ is the set of indices $t \in [T]$ such that for some $\eta \in \mathrm{Dom}(M)$ the configurations $(M_t(\eta), M_{t+1}\eta)$ are linked together by a legal KA-transition inside B_j. The constants C, λ may depend only on k and d.

In order to flip the state of a site x we must perform a legal KA exchange touching x which in turn requires having enough empty sites in the vicinity of x. Patches of empty sites (e.g. a super-good box i.e. a good box with an empty row for $d = k = 2$) are certainly present inside the percolation structure of the good boxes. However, typically they will be quite far from x because $q \ll 1$. The main idea behind the proof of the proposition is to prove that such super-good boxes can be moved at will inside the good percolation network and brought near x by concatenating suitable "elementary" T-steps moves. This concatenation will form the sought global T-step move M.

Unfortunately the general construction of the elementary moves is a bit cumbersome and technical. For the interested reader we refer to [22, Chapter 5] and to the supplementary file attached to the arXiv version of this paper. Still, we will present a sketch of the construction for the particular case $k = d = 2$ that will give a flavour of the type of arguments used there.

3.4.1. *Sketch of the proof of Proposition 3.18 for $k = 2, d = 2$*

We will first introduce the notion of almost good – a box is said to be *almost good* if it contains at least one empty site in every line and every column. Recall that a good box is a box that remains almost good even after filling one of its sites.

Claim 3.19 (Exchanging rows). *Fix $y \in \mathbb{Z}^d$, and consider the configurations in which the row $y + [\ell] \times \{0\}$ is empty, and the row above it contains at least one empty site. Then there exists a T-step move M whose domain consists of these configurations, and in the final state $M_T(\eta)$ the rows $y + [\ell] \times \{0\}$ and $y + [\ell] \times \{1\}$ are exchanged. Moreover,* $\mathrm{Loss}(M) = O(\log_2 \ell)$ *and $T = O(\ell)$. See Figure 4.*

Note that even though the claim is formulated for exchanging rows, the same will hold for columns.

The path γ in Proposition 3.18 is a general up-right path, but imagine for the moment that it is a straight path going right all the way to the boundary of Λ_ℓ. In this case, we can create an empty column on the boundary of Λ, and use Claim 3.19 to propagate it until it reaches the right side of the box B_i.

Assume further that all of the boxes in Q_i are connected to the boundary by straight paths. The same construction as before will then allow us to empty all the sites in the outer up-right boundary of B_i. Figure 5 shows how in this state one can permute sites in B_i, so in particular we are able to change the occupation at x.

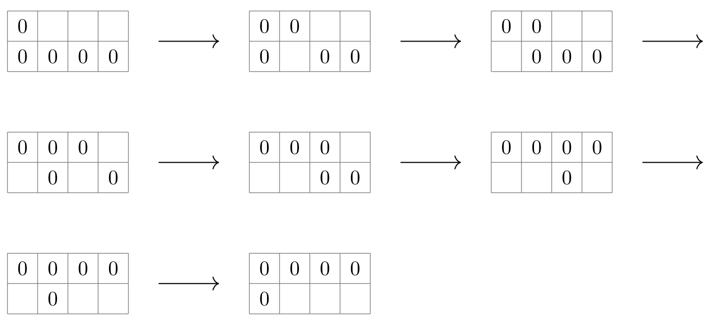

Fig. 4. This figure shows how to exchange an empty row with a neighbouring row containing at least one empty site (see Claim 3.19). The loss comes from the fact that there are ℓ options for the position of the empty site in the upper row.

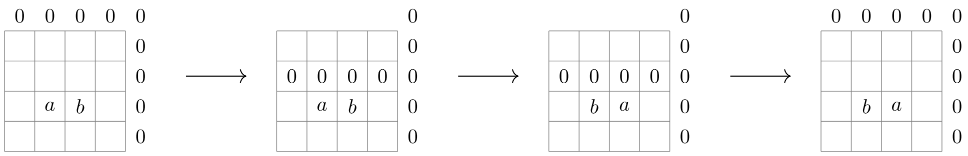

Fig. 5. This figure shows how to exchange two arbitrary neighbouring sites of a box if the external top row and right column are empty. By a concatenation of such moves it is possible to exchange any two internal sites.

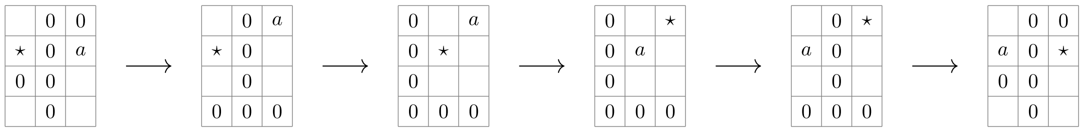

Fig. 6. This figure shows how to make a marked site jump beyond an empty column. See Claim 3.20.

This, however, only allows us to exchange x with another site in the same box; in order to flip its state without changing the other sites we must exchange it with a boundary site, which is connected to the reservoir (namely, it is being resampled from equilibrium, and in particular the number of particles is not conserved). In order to move the site to the other side of the empty column we use the following claim:

Claim 3.20 (Moving a marked site). *Fix $y \in \mathbb{Z}^d$ and some marked site $\star \in y + \{-1\} \times [\ell]$. Consider the configurations in which the column $y + \{0\} \times [\ell]$ is empty, and each of the columns $y + \{\pm 1\} \times [\ell]$ contains at least one empty site, not counting the site $\star$. Then there exists a T-step move M whose domain consists of these configurations, and in the final state $M_T(\eta)$ the sites $\star$ and $\star + (2,0)$ change their occupation values. Moreover,* $\mathrm{Loss}(M) = O(\log_2 \ell)$ *and $T = O(\ell)$. See Figure 6.*

For this (very untypical) case, when the paths have no turns, the last claim will finish the construction – after emptying the outer up-right boundary of B_i and bringing the site x to the rightmost column of B_i, apply consecutively Claim 3.19 and Claim 3.20. The energy barrier is at most 3ℓ since we empty three rows/columns, and the time is $O(L\ell^2)$. The loss of information is also $O(\ell)$, since at each step we only need to reconstruct the three empty row/columns. If we are in the course of a move described in Claim 3.19 we must also pay its loss, but this gives a lower order contribution.

When the paths turn, however, we cannot propagate the empty line like before, and a more complicated mechanism is required. The first step consists in framing a frameamble box (recall Definition 3.3). When $d = k = 2$, it means the following:

Claim 3.21 (Framing a box). *Fix a box, and consider the configurations for which the box is almost good, and, in addition, its bottom row is empty. Then there exists a T-step move M whose domain consists of these configurations, and in the final state $M_T(\eta)$ the left column is also empty. Moreover,* $\mathrm{Loss}(M) = O(\ell \log_2 \ell)$ *and $T = O(\ell^2)$. See Figure 7.*

Once a box is framed, we can use again Figure 5 in order to construct a general permutation inside it, and in particular we are able to reflect the configuration.

Claim 3.22 (Reflecting a framed configuration). *Fix a box, and consider the configurations for which the box is framed, i.e., its bottom row and left column are empty. Then there exists a T-step move M whose domain consists of these configurations, and in the final state $M_T(\eta)$ the configuration inside the box is reflected along the axis connecting its bottom left corner with its up right corner. Moreover,* $\mathrm{Loss}(M) = 0$ *and $T = O(\ell^4)$.*

Now we are allowed, when reaching a turn of the path, to frame the box, reflect the configuration, "unframe" the reflected box, and continue propagating the marked site to finish the construction as in the corner-less case. The dominating contribution to the total loss is coming from the framing move, $O(\ell \log_2(\ell))$, the energy barrier remains the one coming from the creation of empty sites on the boundary, $O(\ell)$, and the time is $O(L\ell^4)$.

3.5. *From the long range Poincaré inequality to the Kob–Andersen dynamics*

In this section we bound from above the Dirichlet form with the long range constraints appearing in the r.h.s. of (3.6) with that of the KA model in Λ (3.2). Given $i \in \Lambda_\ell$ our aim is to bound the quantity $\mu(\hat{c}_i \operatorname{Var}_{B_i}(f))$ appearing in the r.h.s.

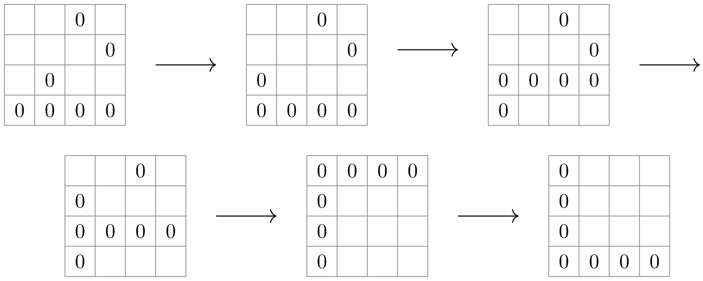

Fig. 7. Framing an almost good box. See Claim 3.21.

of (3.6) using the T-step moves that have been constructed in the previous section. In order to do that, it is convenient to first condition on the environment of the coarse-grained variables $\{\mathbb{1}_{\{B_j \text{ is good}\}}\}_{j\in\Lambda_\ell, j\neq i}$. The main advantage of the above conditioning is that the good family for the vertex $i+\vec{e}_1$, whose existence is guaranteed by the long range constraint $\hat{c}_i$, become deterministic. We will thus work first in a fixed realisation of the coarse-grained variables satisfying $\hat{c}_i = 1$ and only at the end we will take an average and we will sum over i. The main technical step of the above program is as follows.

Given $i \in \Lambda_\ell$ let γ be an up-right focused path γ of length $2N$ starting at $i+\vec{e}_1$ and let $G_{i,\gamma}$ be the event that γ is good and all $j \in Q_i \cap \Lambda_\ell$ are good. Let also $V_{i,\gamma} := B_i \bigcup_{j\in\gamma\cup Q_i} B_j$ and let $\mathcal{F}_i$ be the σ-algebra generated by the random variables $\{\mathbb{1}_{\{B_j \text{ is good}\}}\}_{j\in\Lambda_\ell, j\neq i}$. Notice that $G_{i,\gamma}$ is measurable w.r.t. $\mathcal{F}_i$. Finally write

$$\mathcal{D}_{i,\gamma}(f \mid \mathcal{F}_i) := \sum_{\substack{x,y\in V_{i,\gamma}\cap\Lambda \\ \|x-y\|_1=1}} \mu\big(c_{xy}(\nabla_{xy} f)^2 \mid \mathcal{F}_i\big) + \sum_{x\in V_{i,\gamma}\cap\partial\Lambda} \mu\big(\mathrm{Var}_x(f) \mid \mathcal{F}_i\big),$$

where $\mathrm{Var}_x(f)$, as before, is the variance conditioned on the occupation of the sites in $\Lambda \setminus \{x\}$. Clearly the average w.r.t. μ of $\mathcal{D}_{i,\gamma}(f \mid \mathcal{F}_i)$ represents the contribution coming from the set $V_{i,\gamma} \cap \Lambda$ to the total Dirichlet form $\mathcal{D}(f)$.

For simplicity in the sequel we shall write $C(\ell, q)$ for any positive function such that, as $\ell \uparrow +\infty$, $q \downarrow 0$,

$$C(\ell,q) = e^{O(\ell|\log(q)|+\ell\log(\ell))} \quad \text{for } k=2, \tag{3.7}$$

$$C(\ell,q) = e^{O(\ell^{k-1}|\log(q)|+\ell^d\log(\ell))} \quad \text{for } k\geq 3. \tag{3.8}$$

Of course the constant in the $O(\cdot)$ notation may change from line to line.

Lemma 3.23. *On the event* $G_{i,\gamma}$

$$\mu\big(\mathrm{Var}_{B_i}(f) \mid \mathcal{F}_i\big) \leq O(N)C(\ell,q)\mathcal{D}_{i,\gamma}(f \mid \mathcal{F}_i) \quad \forall f : \Omega_\Lambda \mapsto \mathbb{R}.$$

Proof. Assume $\mathbb{1}_{G_{i,\gamma}} = 1$. Since the marginal of $\mu(\cdot \mid \mathcal{F}_i)$ on $\{0,1\}^{B_i}$ is a product measure we have immediately that

$$\mu\big(\mathrm{Var}_{B_i}(f) \mid \mathcal{F}_i\big) \leq \sum_{x\in B_i} \mu\big(\mathrm{Var}_x(f) \mid \mathcal{F}_i\big),$$

and it is sufficient to prove that

$$\max_{x\in B_i} \mu\big(\mathrm{Var}_x(f) \mid \mathcal{F}_i\big) \leq O(N)C(\ell,q)\mathcal{D}_{i,\gamma}(f \mid \mathcal{F}_i).$$

Given $x \in B_i$, Proposition 3.18 and the assumption $\mathbb{1}_{G_{i,\gamma}} = 1$ imply that there exists a T-step move M with $\mathrm{Dom}(M) = G_{i,\gamma}$, taking place in $V_{i,\gamma} \cap \Lambda$ and such that for all $\eta \in \mathrm{Dom}(M)$ $M_T\eta$ is the configuration η flipped at x. Notice that M does not change the variables $\{\mathbb{1}_{\{B_j \text{ is good}\}}\}_{j\in\Lambda_\ell, j\neq i}$. Hence, on the event $G_{i,\gamma}$,

$$\mathrm{Var}_x(f) = pq\big(f(\eta) - f(\eta^x)\big)^2 \leq \left(\sum_{t=0}^{T-1}\big(f\big(M_t(\eta)\big) - f\big(M_{t+1}(\eta)\big)\big)\right)^2 \leq T\sum_{t=0}^{T-1}\big(f\big(M_t(\eta)\big) - f\big(M_{t+1}(\eta)\big)\big)^2. \tag{3.9}$$

In order to proceed it is convenient to introduce the following notation.

A pair of configurations $e = (\eta, \eta') \in \Omega^2$ is called a *KA-edge* if $\eta \neq \eta'$ and η' is obtained from η by applying to η either a legal exchange at some bond b_e of Λ or a spin flip at some site $z_e \in \partial\Lambda$. If b_e or z_e belong to a given $V \subset \Lambda$ we say that *the edge e occurs in V*. Given a KA-edge $e = (\eta, \eta')$ we write $\nabla_e f := f(\eta') - f(\eta)$. Finally the collection of all KA-edges in Ω^2 is denoted Ω_{KA}.

By construction, if $M_{t+1}(\eta) \neq M_t(\eta)$ then $e_t := (M_t(\eta), M_{t+1}(\eta))$ is a KA-edge and the r.h.s. of (3.9) can be written as

$$T \sum_{t=0}^{T-1} c_{e_t} (\nabla_{e_t} f)^2,$$

where c_{e_t} is the kinetic constraint associated to the KA-edge e_t. Taking the expectation over η w.r.t. $\mu(\cdot \mid \mathcal{F}_i)$ yields

$$\mu\big(\mathrm{Var}_x(f) \mid \mathcal{F}_i\big) \leq T \sum_{e \in \Omega_{\mathrm{KA}}} \sum_{t=0}^{T} \mu\big(c_e (\nabla_e f)^2 \mathbb{1}_{\{e=(M_t(\eta), M_{t+1}(\eta))\}} \mid \mathcal{F}_i\big). \tag{3.10}$$

Next we use the following chain of observations (recall Proposition 3.18 and the relevant definitions therein).

(i) For any KA-edge e and any η such that $e = (M_t(\eta), M_{t+1}(\eta))$ for some $t \leq T$ it holds that (for $q < 1/2$)

$$\mu(\eta) \leq q^{-E(M)} \mu\big(M_t(\eta)\big).$$

(ii) Since the T-move M takes place in the set $V_{i,\gamma} \cap \Lambda$, in the r.h.s. of (3.10) we can replace $\sum_{e \in \Omega_{\mathrm{KA}}}$ by

$$\sum_{\substack{e \in \Omega_{\mathrm{KA}} \\ e \text{ occurs in } V_{i,\gamma} \cap \Lambda}} .$$

(iii) Given a KA-edge e occurring in some $B_j \subset V_{i,\gamma} \cap \Lambda$,

$$\sum_{\eta \in \Omega} \sum_{t=1}^{T} \mathbb{1}_{\{e=(M_t(\eta), M_{t+1}(\eta))\}} \leq 2^{\mathrm{Loss}(M)} \big|\mathcal{T}_M^{(j)}\big|.$$

Using the above remarks, on the event $G_{i,\gamma}$,

$$\mu\big(\mathrm{Var}_x(f) \mid \mathcal{F}_i\big) \leq T 2^{\mathrm{Loss}(M)} \big|\mathcal{T}_{\mathrm{M}}^{(j)}\big| q^{E(\mathrm{M})} \sum_{\substack{e=(\eta,\eta') \in \Omega_{\mathrm{KA}} \\ e \text{ occurs in } V_{i,\gamma} \cap \Lambda}} \mu(\eta \mid \mathcal{F}_i) c_e(\eta) (\nabla_e f)^2.$$

This expression, by Proposition 3.18, satisfies the required bound. □

We are now ready to state the main result of this section.

Proposition 3.24. *Let $\mathcal{D}^\ell(f) = \mu(\sum_{i \in \Lambda_\ell} \hat{c}_i \mathrm{Var}_{B_i}(f))$ and let $\mathcal{D}(f)$ be the Dirichlet form of the KA model. Then*

$$\mathcal{D}^\ell(f) \leq O\big(N^2\big) C(\ell, q) \mathcal{D}(f).$$

Corollary 3.25. *Fix $2 \leq k \leq d$ together with $q \in (0,1)$. Assume that it is possible to choose the mesoscopic scale ℓ depending only on k, d, q in such a way that $\pi_\ell(d,k) \geq \pi_*$, where π_* is the constant appearing in Proposition 3.15. Then*

$$\mathrm{Var}(f) \leq O\big(N^2\big) C(\ell, q) \mathcal{D}(f).$$

Equivalently

$$T_{\mathrm{rel}}(q, L) \leq O\big(L^2\big) C(\ell, q).$$

Proof. The first part of the corollary follows at once from Propositions 3.15 and 3.24. The second part is an immediate consequence of the first one and of the variational characterisation of the relaxation time (see the beginning of Section 3). □

Proof of Proposition 3.24. Recall definition (3.5) of the long range constraints $\hat{c}_i$ and let us consider one term $\mu(\hat{c}_i \operatorname{Var}_{B_i}(f))$ appearing in the definition of $\mathcal{D}^\ell(f)$. Observe that $\hat{c}_i$ is measurable w.r.t. the σ-algebra $\mathcal{F}_i$. Conditionally on $\mathcal{F}_i$ and assuming that $\hat{c}_i = 1$, let $\mathcal{G}$ be a family of good paths for the vertex $i + \vec{e}_1 + \vec{e}_2 \in \mathbb{Z}_\ell^d$. Clearly $\hat{c}_i = 1$ implies that $G_{i,\gamma}$ occurs for each path $\gamma \in \mathcal{G}$. Hence, by applying Lemma 3.23 to each path in $\mathcal{G}$ we get

$$\begin{aligned}
\mu\big(\operatorname{Var}_{B_i}(f) \mid \mathcal{F}_i\big) &\le O(N)C(\ell,q)\frac{1}{|\mathcal{G}|}\sum_{\gamma\in\mathcal{G}}\mathcal{D}_{i,\gamma} \\
&= O(N)C(\ell,q)\Bigg[\sum_{\substack{x,y\in\Lambda\\ \|x-y\|_1=1}} \mu\big(c_{xy}(\nabla_{xy}f)^2 \mid \mathcal{F}_i\big)\frac{1}{|\mathcal{G}|}\sum_{\gamma\in\mathcal{G}}\mathbb{1}_{\{(x,y)\subset V_{i,\gamma}\}} \\
&\quad + \sum_{x\in\partial\Lambda}\mu\big(\operatorname{Var}_x(f)\mid\mathcal{F}_i\big)\frac{1}{|\mathcal{G}|}\sum_{\gamma\in\mathcal{G}}\mathbb{1}_{\{x\in V_{i,\gamma}\}}\Bigg].
\end{aligned}\tag{3.11}$$

For a given bond $(x,y)\subset\Lambda$ (respectively $x\in\partial\Lambda$) let $j=j(x)$ be such that $B_j \ni x$ and let Π_j denote the $(\vec{e}_1,\vec{e}_2)$-plane in $\mathbb{Z}_\ell^d$ containing j. Since all the paths forming the family $\mathcal{G}$ belong to the plane Π_i, and are focused we immediately get that

$$\frac{1}{|\mathcal{G}|}\sum_{\gamma\in\mathcal{G}}\mathbb{1}_{\{(x,y)\subset V_{i,\gamma}\}} = \mathbb{1}_{\{j\in\Pi_i\}}\mathbb{1}_{\{j\in\mathcal{R}_i\}}\frac{1}{|\mathcal{G}|}\sum_{\gamma\in\mathcal{G}}\mathbb{1}_{\{(x,y)\subset V_{i,\gamma}\}},$$

$$\frac{1}{|\mathcal{G}|}\sum_{\gamma\in\mathcal{G}}\mathbb{1}_{\{x\in\partial V_{i,\gamma}\}} = \mathbb{1}_{\{j\in\Pi_i\}}\mathbb{1}_{\{j\in\mathcal{R}_i\}}\frac{1}{|\mathcal{G}|}\sum_{\gamma\in\mathcal{G}}\mathbb{1}_{\{x\in\partial V_{i,\gamma}\}},$$

where $\mathcal{R}_i$ is the set of points at distance at most $\sqrt{N}$ from the set $\{k : k = i + s(\vec{e}_1+\vec{e}_2), s\in\mathbb{N}\}$.

Next, for $(x,y)\subset\Lambda$ (respectively $x\in\partial\Lambda$) such that $\|i-j\|_1\le\sqrt{N}$ we bound $\frac{1}{|\mathcal{G}|}\sum_{\gamma\in\mathcal{G}}\mathbb{1}_{\{(x,y)\subset V_{i,\gamma}\}}$ (respectively $\frac{1}{|\mathcal{G}|}\sum_{\gamma\in\mathcal{G}}\mathbb{1}_{\{x\in\partial V_{i,\gamma}\}}$) by one. If instead $\|i-j\|_1>\sqrt{N}$ then we use the fact that the paths of $\mathcal{G}$ are almost edge-disjoint to bound from above both sums by $2/|\mathcal{G}|\le 2/\sqrt{N}$.

In conclusion, the first and second term inside the square bracket in the r.h.s. of (3.11) are bounded from above by

$$\sum_{\substack{x,y\in\Lambda\\ \|x-y\|_1=1}} \mu\big(c_{xy}(\nabla_{xy}f)^2\mid\mathcal{F}_i\big)\mathbb{1}_{\{j\in\Pi_i\}}\mathbb{1}_{\{j\in\mathcal{R}_i\}}\bigg[\mathbb{1}_{\{\|i-j(x)\|_1\le\sqrt{N}\}}+\frac{2}{\sqrt{N}}\mathbb{1}_{\{\|i-j(x)\|_1>\sqrt{N}\}}\bigg]$$

and

$$\sum_{x\in\partial\Lambda}\mu\big(\operatorname{Var}_x(f)\mid\mathcal{F}_i\big)\mathbb{1}_{\{j\in\Pi_i\}}\mathbb{1}_{\{j\in\mathcal{R}_i\}}\bigg[\mathbb{1}_{\{\|i-j(x)\|_1\le\sqrt{N}\}}+\frac{2}{\sqrt{N}}\mathbb{1}_{\{\|i-j(x)\|_1>\sqrt{N}\}}\bigg]$$

respectively. Clearly the same bounds hold for their average w.r.t. μ.

In order to conclude the proof it is enough to sum over i the above expressions and use the fact that, uniformly in $x\in\Lambda$,

$$\sum_{i\in\Lambda_\ell}\mathbb{1}_{\{j\in\Pi_i\}}\mathbb{1}_{\{j\in\mathcal{R}_i\}}\bigg[\mathbb{1}_{\{\|i-j(x)\|_1\le\sqrt{N}\}}+\frac{2}{\sqrt{N}}\mathbb{1}_{\{\|i-j(x)\|_1>\sqrt{N}\}}\bigg]\le O(N).$$

□

3.6. *Completing the proof of the upper bound*

Using Corollary 3.25, the proof of the upper bound is complete if we can prove that for all $\pi^*<1$, for any given $q\in(0,1)$ and $2\le k\le d$ it is possible to choose $\ell=\ell(q,k,d)$ in such a way that

(i) the probability that any given $i\in\mathbb{Z}_\ell^d$ is (d,k)-good satisfies $\pi_\ell(d,k)\ge\pi_*$;
(ii) $C(\ell,q)\le C(q)$ as $q\to 0$, where $C(q)$ is as in (2.3) and $C(\ell,q)$ satisfies (3.7).

Let us start by stating a key result on the probability of the set of frameable configurations

Proposition 3.26 (Probability of frameable configurations [23]). *Fix q and let $\mathcal{F}_q(\ell, d, j)$ be the probability that the cube $C_\ell = [\ell]^d$ is (d, j)-frameable. Then there exists $C > 0$ s.t. for $q \to 0$*

$$\mathcal{F}_q(\ell, d, j) \geq 1 - Ce^{-\ell_q/\Xi_{d,j}} \quad \forall \ell_q \text{ s.t. } \Xi_{d,k}(q) = O(\ell_q)$$

with

$$\Xi_{d,1}(q) := \left(\frac{1}{q}\right)^{1/d}$$

and

$$\Xi_{d,j}(q) := \exp_{(j-1)}\left(\frac{1}{q^{\frac{1}{d-j+1}}}\right) \quad \forall j \in [2, d].$$

Proof. The case $j = 1$ follows immediately from the definition of frameable configurations (see Definition 3.3). The cases $j \in [2, d]$ are proven in Section 2 of [23], see formula (34)[4] and (36), where the results are stated in terms of the parameter $s = j - 1$. Actually, the definition of frameable in [23] is more restrictive than our Definition 3.3. Indeed in [23] the frame that should be emptiable is composed by all the faces of dimension $j - 1$ containing one corner of $\mathcal{C}_\ell$ (and not only those that contain the vertex $(1, \ldots, 1)$). However, since we only need a lower bound on the probability of being frameable we can directly use the results of [23]. □

Then, by using Proposition 3.26 and the Definition 3.4, we get that there exists $c > 0$ s.t. by choosing

$$\ell(q, k, d) = \exp_{(k-2)}\left(c/q^{\frac{1}{d-j+1}}\right) \quad \forall k \in [3, d]$$

and

$$\ell(q, 2, d) = |\log q|/q^{\frac{1}{d-1}}$$

we get

$$\pi_\ell(d, k) \geq \left(1 - C\exp^{-\ell/\Xi_{d-1,k-1}}\right)^{\ell d}$$

which goes to 1 as $q \to 0$, and thus implies that condition (i) is satisfied for all $q \in (0, 1)$ (since $\pi_\ell(d, k)$ is non decreasing with q). Finally, it is immediate to verify that the above choice of ℓ satisfies also condition (ii) for all $k \in [2, d]$.

4. Proof of the lower bound in Theorem 1

In this section we will prove the lower bound on the relaxation time by finding suitable positive constants c, λ depending on d, k and a function f such that

$$\mathrm{Var}(f) \geq e^{\lambda m} L^2 \mathcal{D}(f), \tag{4.1}$$

where $m := m(q)$ satisfies

$$m(q) = \begin{cases} \lfloor cq^{-\frac{1}{d-1}} \rfloor, & k = 2, \\ \lfloor \exp_{(k-2)}(cq^{-\frac{1}{d-k+1}}) \rfloor, & k \geq 3. \end{cases} \tag{4.2}$$

Note that m defined here, up to logarithmic corrections, describe a length scale similar to ℓ of the previous section. Roughly speaking, this is the scale at which large structures that contain many empty sites could propagate and influence their neighborhood.

[4]There is a misprint in formula (34) of [23]: in the exponential a minus sign is missing

4.1. *Bootstrap percolation*

In order to define f we first need to introduce the *k-neighbor bootstrap percolation* (see e.g. [19] and references therein). Fix $V \subseteq \Lambda$, and consider a set $A \subseteq \Lambda$. The k-neighbor bootstrap percolation in V starting at A is the deterministic growth process in discrete time, defined as

$$A_0 = A \cap V,$$
$$A_{t+1} = A_t \cup \{x \in V : |\{y \in A_t \text{ such that } y \sim x\}| \geq k\}, \quad t \in \mathbb{N}.$$

That is, at each step the set A_{t+1} is obtained by adding to the set A_t the sites that have at least k neighbors already in A_t. The set $\bigcup_{t\geq 0} A_t$ will be denoted by $[A]^V$ and it forms a subgraph of Λ whose connected components will be referred to as *clusters*. Given $x \in V$ we shall write $\mathcal{C}_x^V$ for either the cluster of $[A]^V$ containing x if $x \in [A]^V$ or for the set $\{x\}$ otherwise. Given $\eta \in \Omega_V$ we shall define the bootstrap percolation process started from η as the above process with initial set $A = A_\eta := \{x \in \Lambda : \eta_x = 0\}$.

4.2. *Construction of the test function*

We start with a few geometric definitions:

Definition 4.1. Denote the box $x + [-m, m]^d$ by B_x. Its inner boundary is denoted by ∂B_x. We say that an edge $y \sim z$ *crosses* ∂B_x, and write $yz \in \bar{\partial} B_x$, if one of its endpoints is in B_x and the other is not.

Definition 4.2. Fix a configuration $\eta \in \Omega$ and a site $x \in \Lambda$, and consider the cluster of x in $[A_\eta]^{B_x}$. We define $r_x(\eta)$ to be the maximal site in this cluster, according to the lexicographic order.

We are now ready to define the function f. Let $g : [0, 1]^d \to \mathbb{R}$ a positive smooth function supported in $[0.1, 0.9]^d$. Then

$$f(\eta) = \sum_{x\in\Lambda} g\big(r_x(\eta)/L\big)\eta_x. \tag{4.3}$$

Remark 4.3. The above choice is inspired by the test function $\varphi = \sum_{x\in\Lambda} g(x/L)\eta_x$ for the symmetric simple exclusion process, with g related to the lowest eigenfunction of the discrete Laplacian in Λ (see e.g. [6, Section 4.1]). The only crucial difference between f and φ is the choice of $r_x(\eta)$ instead of x inside the slowly varying function $g(r_x(\eta)/L)$ as a proxy for the *effective position* of the particle at x. Actually any other quantity depending only the cluster $\mathcal{C}_x$ (e.g. its center of mass) would work as well. Evaluating g at this effective position is the cause of the prefactor $e^{\lambda m}$ in front of the diffusive term L^2 in (4.1). In fact, as proved below, the cluster $\mathcal{C}_x$ is influenced by an exchange of the KA dynamics in the box B_x only if $\mathcal{C}_x \cap \partial B_x \neq \varnothing$. Since the latter event has probability $e^{-O(m)}$ if the constant c appearing in (4.2) is small enough, the sought prefactor emerges.

We shall now bound separately the variance and Dirichlet form of f. In the sequel, c and λ will denote generic positive constants that may depend only on k, d, and g.

4.3. *Bounding the variance*

Proposition 4.4. *For q small enough and L large enough*,

$$\operatorname{Var}(f) \geq cqL^d.$$

Proof. Let $H = (2m+1)\mathbb{Z}^d$, and for $\xi \in B_0$ let $H_\xi = (\xi + H) \cap \Lambda$. Clearly $H_\xi \cap H_{\xi'} = \varnothing$ iff $\xi \neq \xi'$ and $\bigcup_{\xi\in B_0} H_\xi = \Lambda$. Note also that for any $x, x' \in H_\xi$, $x \neq x'$, $B_x \cap B_{x'} = \varnothing$.

For $\xi \in B_0$ denote by $f_\xi(\eta)$ the part of the sum in equation (4.3) that corresponds to H_ξ:

$$f_\xi(\eta) = \sum_{x\in H_\xi} g(r_x/L)\eta_x.$$

By the previous observation this is a sum of independent variables so that

$$\begin{aligned}\operatorname{Var}(f_\xi) &= \sum_{x\in H_\xi} \operatorname{Var}\left[g\left(\frac{1}{L} r_x(\eta)\right)\eta_x\right] = \sum_{x\in H_\xi} \operatorname{Var}\left[\big(1+O(m/L)\big)g(x/L)\eta_x\right] \\ &= \sum_{x\in H_\xi} g(x/L)^2 pq\big(1+O(m/L)\big) \geq \frac{1}{2} pq \sum_{x\in H_\xi} g(x/L)^2.\end{aligned}$$

The notation $O(\cdot)$ stands for a random variable deterministically bounded by the expression inside the parentheses times a constant. Above we used $|r_x(\eta)| = O(m)$ to write

$$g\big(r_x(\eta)/L\big) = g(x/L)\big(1+O(m/L)\big), \tag{4.4}$$

recalling that g is smooth.

Next, for $\xi \neq \xi'$,

$$\begin{aligned}\operatorname{Cov}(f_\xi, f_{\xi'}) &= \sum_{x\in H_\xi}\sum_{x'\in H_{\xi'}} \operatorname{Cov}\big(g(r_x/L)\eta_x, g(r_{x'}/L)\eta_{x'}\big) \\ &= \sum_{x\in H_\xi}\sum_{x\in H_{\xi'}} \mathbb{1}_{\|x-x'\|\leq 2m+1} \operatorname{Cov}\big(g(r_x/L)\eta_x, g(r_{x'}/L)\eta_{x'}\big).\end{aligned}$$

Considering one of these terms, using equation (4.4) and $\operatorname{Cov}(\eta_x, \eta_{x'}) = 0$, we find that

$$\operatorname{Cov}\big(g(r_x/L)\eta_x, g(r_{x'}/L)\eta_{x'}\big) = O(m/L),$$

yielding

$$\big|\operatorname{Cov}(f_\xi, f_{\xi'})\big| = \sum_{x\in H_\xi}\sum_{x\in H_{\xi'}} \mathbb{1}_{\|x-x'\|\leq 2m+1} O(m/L) \leq |H_\xi|(4m+3)^d O(m/L).$$

Putting everything together,

$$\begin{aligned}\operatorname{Var}(f) &= \operatorname{Var}\left(\sum_{\xi\in B_0} f_\xi\right) = \sum_\xi \operatorname{Var} f_\xi + \sum_{\xi\neq\xi'} \operatorname{Cov}(f_\xi, f_{\xi'}) \\ &\geq \sum_\xi \frac{1}{2} pq \sum_{x\in H_\xi} g(x/L)^2 - \sum_{\xi\neq\xi'} |H_\xi| O\big(m^{d+1}/L\big) \\ &= \frac{1}{2} pq \sum_x g(x/L)^2 - O\big(m^{2d+1}L^{d-1}\big) \geq \frac{1}{4} pq L^d \int g(s)^2 \,\mathrm{d}s\end{aligned}$$

for L large enough. □

4.4. *Bounding the Dirichlet form*

In order to bound $\mathcal{D}(f)$, we will use [7, Lemma 5.1]. Plugging our choice of m for small enough c in their result yields the following lemma:[5]

Lemma 4.5. *Consider the bootstrap percolation in the box B_0 starting at A_η, where η is a configuration distributed according to μ. Then the probability that the cluster of the origin in $[A_\eta]^{B_0}$ contains a site in ∂B_0 is bounded from above by $e^{-\lambda m}$. The same bound holds replacing B_0 by the box $[-m, m+1] \times [-m, m]^d$ (or any of its rotations).*

We will now make a few combinatorial observations.

[5] Note that in [7] the parameter k is called ℓ, and that the length scale m of equation (4.2) corresponds (up to a constant) to m_- of [7].

Observation 1. Fix a configuration η, and two sets $U, V \subseteq \Lambda$. Then $[A_\eta]^U \subset [A_\eta]^{U \cup V}$.

Observation 2. Fix a configuration η, two sets $U \subseteq V \subseteq \Lambda$, and a site $x \in V \setminus U$. Assume that $x \in [A_\eta]^V$, but $x \notin [A_\eta]^{V \setminus U}$. Then $\mathcal{C}_x^V \cap U \neq \varnothing$.

Observation 3. Fix a configuration η, a set $V \subseteq \Lambda$, and two sites $y \sim z \in V$. Assume the constraint c_{yz} is satisfied in V (i.e., when fixing all sites outside V to be occupied), and that $\eta_y \neq \eta_z$. Then both y and z are contained in $[A_\eta]^V$. In particular, $\mathcal{C}_y^V = \mathcal{C}_z^V$.

Observation 4. Fix a configuration η, a set $V \subseteq \Lambda$, and two sites $y \sim z \in V$. Assume the constraint c_{yz} is satisfied in V. Then $[A_\eta]^V = [A_{\eta^{yz}}]^V$.

Claim 4.6. *Fix a configuration η and an edge $y \sim z$, such that $c_{yz} = 1$ and $\eta_y \neq \eta_z$. Assume that $\mathcal{C}_y^{B_y \cup B_z} \cap \partial(B_y \cup B_z) = \varnothing$. Then $r_y(\eta) = r_z(\eta^{yz})$.*

Proof. Using Observation 3 and the fact that $y \sim z$ we have that $\mathcal{C}_y^B = \mathcal{C}_z^B$ for $B \in \{B_y, B_z, B_y \cup B_z\}$. Moreover, using Observation 4 these clusters are the same for η and η^{yz}. We will show that $\mathcal{C}_y^B$ is the same for all three boxes, which will imply the result. We start by showing that $\mathcal{C}_y^{B_y} = \mathcal{C}_y^{B_y \cup B_z}$. By Observation 1 $\mathcal{C}_y^{B_y} \subseteq \mathcal{C}_y^{B_y \cup B_z}$. In the other direction, let, by contradiction, $w \in \mathcal{C}_y^{B_z \cup B_y} \setminus [A_\eta]^{B_y}$. Then, setting $V = B_z \cup B_y$ and $U = (B_z \cup B_y) \setminus B_y$, Observation 2 implies that $\mathcal{C}_y^{B_z \cup B_y} \cap U \neq \varnothing$. Noticing that $U \subset \partial(B_y \cup B_z)$, this contradicts the assumption of the claim. We conclude that $\mathcal{C}_y^{B_z \cup B_y} = \mathcal{C}_y^{B_y}$. Similarly one proves that $\mathcal{C}_y^{B_z \cup B_y} = \mathcal{C}_y^{B_z}$, and the result follows. □

Claim 4.7. *Fix $x \in \Lambda$ and $y \sim z$. Then*:

(1) *If $yz \in \bar{\partial} B_x$ (so in particular $x \notin \{y, z\}$),*

$$\mu\left(c_{yz}\left(\nabla_{yz}\left[g(r_x/L)\eta_x\right]\right)^2\right) \le e^{-\lambda m} m^2/L^2.$$

(2) *If $x \in \{y, z\}$,*

$$\mu\left(c_{zy}\left(\nabla_{yz}\left[g(r_y/L)\eta_y + g(r_z/L)\eta_z\right]\right)^2\right) \le e^{-\lambda m} m^2/L^2.$$

(3) *Otherwise,*

$$c_{yz}\left(\nabla_{yz}\left[g(r_x/L)\eta_x\right]\right) = 0.$$

Proof. Since $\nabla_{xy}(\cdot) = 0$ whenever $\eta_y = \eta_z$, we may assume throughout the proof of this claim that $\eta_y \neq \eta_z$, using freely Observation 3 and Claim 4.6.

For the first part, we note that η_x does not change when exchanging the sites y and z, so the only contribution to $\nabla_{yz}[g(r_x/L)\eta_x]$ comes from the change of r_x. Assume without loss of generality that $y \in B_x$ and $z \notin B_x$. By Observations 1 and 2, $[A_\eta]^{B_x}$ (and therefore r_x) cannot change when changing η_y, unless $y \in [A_\eta]^{B_x} \cup [A_{\eta^{yz}}]^{B_x}$. Hence, by equation (4.4), Lemma 4.5 and defining $\mathcal{C}_x$ to be $\mathcal{C}_x^{B_x}(\eta)$ when $\eta_y = 0$ and $\mathcal{C}_x^{B_x}(\eta^{yz})$ when $\eta_z = 0$,

$$\begin{aligned}\mu\left(c_{yz}\left(\nabla_{yz}\left[g(r_x/L)\eta_x\right]\right)^2\right) &= \mu\left(c_{yz}\left(\nabla_{yz}\left[g(r_x/L)\eta_x\right]\right)^2 \mathbb{1}_{y \in \mathcal{C}_x}\right) \\ &\le O\left(m^2/L^2\right)\mu(y \in \mathcal{C}_x) \le e^{-\lambda m} m^2/L^2.\end{aligned}$$

In order to prove the second part, note first that $y, z \in B_y \cap B_z$ so that Observation 4 together with $c_{yz}(\eta) = 1$ implies that $\mathcal{C}_y^{B_y}(\eta) = \mathcal{C}_y^{B_y}(\eta^{yz})$ and in particular $r_y(\eta) = r_y(\eta^{yz})$. In the same manner $r_z(\eta) = r_z(\eta^{yz})$. Suppose now that $r_y(\eta) = r_z(\eta^{yz})$, so also $r_z(\eta) = r_y(\eta^{yz})$. Then

$$g\left(r_y(\eta)/L\right)\eta_y + g\left(r_z(\eta)/L\right)\eta_z = g\left(r_z\left(\eta^{yz}\right)/L\right)\eta_z^{yz} + g\left(r_y\left(\eta^{yz}\right)/L\right)\eta_y^{yz}.$$

Therefore, $c_{yz}\mathbb{1}_{\{r_y(\eta) = r_z(\eta^{yz})\}}\nabla_{yz}[g(r_y(\eta)/L)\eta_y + g(r_z(\eta)/L)\eta_z] = 0$. We are thus left with estimating

$$\mu\left(c_{xy}\mathbb{1}_{\{r_y(\eta) \neq r_z(\eta^{yz})\}}\left(\nabla_{yz}\left[g\left(r_y(\eta)/L\right)\eta_y + g\left(r_z(\eta)/L\right)\eta_z\right]\right)^2\right).$$

Using (4.4)

$$\left|\nabla_{yz}\left[g\left(r_y(\eta)/L\right)\eta_y + g\left(r_z(\eta)/L\right)\eta_z\right]\right| = O(m/L).$$

Moreover, by Claim 4.6 and Lemma 4.5, $\mu(c_{xy}\mathbb{1}_{\{r_y(\eta)\neq r_z(\eta^{yz})\}}) \le e^{-\lambda m}$, and this concludes the proof.

The third part is a direct consequence of Observation 4. □

We are now ready to bound from above the Dirichlet form $\mathcal{D}(f)$.

Proposition 4.8. *For any small enough $c > 0$ in (4.2) there exists $\lambda > 0$ such that $\mathcal{D}(f) \le e^{-\lambda m} L^{d-2}$.*

Proof. First, note that, since g is supported in $[0.1, 0.9]$, the term $\sum_{x\in\partial\Lambda} \mathrm{Var}_x(f)$ in $\mathcal{D}(f)$ equals 0. Consider then one of the exchange terms $c_{yz}(\nabla_{yz} f)^2$, and split the sum over x according to the different cases in Claim 4.7:

$$\begin{aligned}
c_{yz}(\nabla_{yz} f)^2 &= c_{yz}\left(\sum_{x\in\Lambda} \nabla_{yz}\left[g\left(r_x(\eta)/L\right)\eta_x\right]\right)^2 \\
&= c_{yz}\left(\nabla_{yz}\left[g\left(r_y(\eta)/L\right)\eta_y + g\left(r_z(\eta)/L\right)\eta_z\right] + \sum_{x:yz\in\bar{\partial} B_x} \nabla_{yz}\left[g\left(r_x(\eta)/L\right)\eta_x\right]\right)^2 \\
&\le cm^{d-1} c_{yz}\left(\left(\nabla_{yz}\left[g\left(r_y(\eta)/L\right)\eta_y + g\left(r_z(\eta)/L\right)\eta_z\right]\right)^2 + \sum_{x:yz\in\bar{\partial} B_x} \left(\nabla_{yz}\left[g\left(r_x(\eta)/L\right)\eta_x\right]\right)^2\right),
\end{aligned}$$

where we have used the Cauchy–Schwarz inequality and the fact that the sum $\sum_{x:yz\in\partial B_x}$ contains $2(2m+1)^{d-1}$ terms corresponding to the $(2m+1)^{d-1}$ possible translations of the face crossing the edge, doubled by the reflection along it.

By Claim 4.7, this inequality implies

$$\mu\left(c_{yz}(\nabla_{yz} f)^2\right) \le cm^{d-1}\left(e^{-\lambda m} m^2/L^2 + m^{d-1} e^{-\lambda m} m^2/L^2\right) \le e^{-\lambda m}/L^2.$$

In conclusion,

$$\mathcal{D}(f) = \mu\left[\sum_{x\in\partial\Lambda} \mathrm{Var}_x(f) + \sum_{z\sim y} c_{yz}(\nabla_{yz} f)^2\right] \le e^{-\lambda m} L^{d-2}.$$

□

Propositions 4.4 and 4.8 show that indeed that the relaxation time is greater than $e^{\lambda m} L^2$, which by the choice of m coincides with the lower bound of Theorem 1. □

5. Concluding remarks and further questions

The general scheme of the proof has already been proven effective in the study of kinetically constrained spin models (and specifically in obtaining universality results [17]). It consists in analysing the microscopic dynamics up to some mesoscopic scale ℓ; and then understanding the long range dynamics, which depends on connectivity properties of a percolation process on the lattice of mesoscopic boxes. The long range dynamics depends very weakly on the details of the model. For example, we were able to restrict this dynamics to paths in two dimensions rather than d, since already in two dimensions percolation with large enough parameter is supercritical, and satisfies strong enough connectivity properties. We believe that applying these methods to other cooperative kinetically constrained lattice gases could yield new interesting results.

In the context of the Kob–Andersen model, the techniques presented in this paper can be used in order to find the diffusion coefficient of a marked particle ([9]). We believe that they may also help understanding further properties of the this model, e.g., improving the bound of [4] on the loss of correlation for local functions, or understanding its hydrodynamic limit. See also [22, Chapter 6].

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