

Coalescing and Branching Exclusion Process

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The model

Fix a connected graph $G = (V, E)$ and a parameter $0 < p < 1$

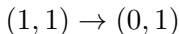
- **Configuration space:** $\Omega = \{0, 1\}^V$
- **Dynamics:** each edge $e = \{x, y\}$ containing at least one particle is resampled at rate 1 from

$$\pi_x \times \pi_y(\cdot \mid \exists \text{ at least one particle on } \{x, y\})$$

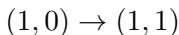
with $\pi_x = \pi_y = \text{Ber}(p)$

The model

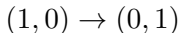
- If there are 2 particles they coalesce at rate $(1-p)/(2-p)$ to 1 particle on one of the two sites chosen uniformly



- If there is 1 particle it creates a new particle on the other site at rate $p/(2-p)$

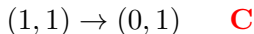


- if there is 1 particle it moves to the adjacent empty site at rate $(1-p)/(2-p)$

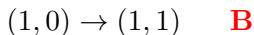


The model

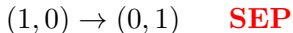
- If there are 2 particles they coalesce at rate $(1-p)/(2-p)$ to 1 particle on one of the two sites chosen uniformly



- If there is 1 particle it creates a new particle on the other site at rate $p/(2-p)$



- if there is 1 particle it moves to the adjacent empty site at rate $(1-p)/(2-p)$



\Rightarrow **CBSEP**

Coalescing random walks with neighbour births

System of particles on G that

- perform independent random walks jumping at rate 1
- branch at rate β creating a new particle on a neighbour empty vertex
- coalesce whenever they meet

Equivalent to CBSEP via a global time rescaling and setting $p = \beta/(1 + \beta)$

History

- introduced in '77 by Schwartz as dual of biased voter model (a.k.a. William-Bjerknes tumour growth model)
- for any $\beta > 0$ on \mathbb{Z}^d : weak convergence to the unique invariant measure starting from any configuration with at least one particle [Bramson, Griffeath '80, '81]
- for any $\beta > 0$ on \mathbb{Z}^d : shape theorem [Durrett, Griffeath '82]
→ mixing time cut-off on torus
- for $\beta \rightarrow 0$ on \mathbb{Z} : convergence to the Brownian net [Sun, Swart '08]

CBSEP: properties

- Attractive and additive
- reversible w.r.t. $\pi = \text{Ber}(\mathbf{p})^{\otimes V}$
- ergodic and reversible on $\Omega_+ := \Omega \setminus \{\text{empty configuration}\}$
w.r.t. $\mu := \pi(\cdot | \Omega_+)$

Mixing times and log Sobolev

- T_q is the ℓ^q -mixing time

$$h_\omega^t(\cdot) = P_\omega^t(\cdot)/\mu(\cdot), \quad \|f\|_q = (\mu(|f|^q))^{1/q}, \quad q \geq 1$$

$$T_q := \inf\{t > 0, \max_\omega \|h_\omega^t(\cdot) - 1\|_q \leq 1/e\}$$

$$T_{mix} = T_1$$

- T_{Sob} is the inverse of the Logarithmic Sobolev constant, i.e. the **inverse rate of decay of the entropy**

$$T_{\text{Sob}}^{-1} := \inf_f \frac{\mathcal{D}(f)}{\text{Ent}(f^2)} = \inf_f \frac{-\mu(f\mathcal{L}f)}{\mu(f^2 \log(f^2/\mu(f^2)))}$$

$$T_q \leq O\left(\log \log \left(\frac{1}{\min_\omega \mu(\omega)}\right)\right) T_{\text{Sob}} \quad \forall q \in [1, \infty]$$

CBSEP: results

Theorem [Hartarsky, Martinelli, C.T. '20]

Let $p_n = \Theta(1/n)$ and $G_n = (V_n, E_n)$ be a sequence of bounded degree graphs with $|V_n| = n$. Then $\exists c > 0$ s.t. $\forall n$

$$c T_{\text{meet}} \leq T_{\text{Sob}} \leq c^{-1} T_{\text{meet}} \log n$$

with T_{meet} the expected meeting time for two continuous time r.w. on G_n starting from two uniformly chosen sites

Corollary

If $G_n = \mathbb{T}_n^d = d$ -dimensional torus with n sites

$$\begin{aligned} cn^2 &\leq T_{\text{Sob}} \leq c^{-1} n^2 \log n & d = 1 \\ cn \log n &\leq T_{\text{Sob}} \leq c^{-1} n \log^2 n & d = 2 \\ cn &\leq T_{\text{Sob}} \leq c^{-1} n \log n & d \geq 3 \end{aligned}$$

CBSEP: ideas of the proof

$$\text{Ent}(f^2) = \mu(\text{Ent}(f^2|N)) + \text{Ent}(\mu(f^2|N)), \quad N = \# \text{ particles}$$

- first term
 - consider Bernoulli Laplace on complete graph K_n
 - $T_{\text{Sob}}^{\text{BL}} = \log n$ [Lee Yau]
 - $\mathcal{D}^{\text{BL}} \leq \frac{d_{\text{max}}^2 d_{\text{mean}}}{d_{\text{min}}^2} T_{\text{mix}}^{\text{lazy rw}} \mathcal{D}^{\text{SEP}}$ [Kozma Alon]
- second term
 - we construct an auxiliary birth death process with invariant measure the law of N
 - we determine $T_{\text{Sob}}^{\text{birth-death}}$ and we use path arguments to compare with CBSEP

A generalised version: g -CBSEP

- Fix a connected graph $G = (V, E)$ and a finite probability space (S, ρ) with $S = S_1 \cup S_0$ and set $p := \rho(S_1)$
- we say that *there is a particle at $v \in V$ iff $\omega_v \in S_1$*

g -CBSEP dynamics : any edge $e = (x, y)$ containing at least one particle is resampled at rate one from $\rho_x \times \rho_y(\cdot \mid \exists$ at least one particle on $e)$

Remark:

the projection $\phi : S^V \rightarrow \{0, 1\}^V$ with $\phi(\omega)_x = 1_{\omega_x \in S_1}$ is CBSEP.

g-CBSEP: results

$$T_{\text{cov}}^{\text{rw}} = \inf\{t > 0, \max_{x \in V} \mathbb{P}_x(\tau_{\text{cov}} > t) \leq 1/e\}$$

with τ_{cov} the cover time of the simple r.w. on G

Theorem [Hartarsky, Martinelli, C.T. '20]

$$T_{\text{mix}}^{\text{CBSEP}} \leq T_{\text{mix}}^{g\text{-CBSEP}} \leq c(T_{\text{mix}}^{\text{CBSEP}} + T_{\text{cov}}^{\text{rw}})$$

Idea:

wait for the projection (= CBSEP) to couple, then wait for one random walk to cover the graph (\rightarrow all sites are refreshed).

g-CBSEP: results

Corollary

On \mathbb{T}_n^d with $p = \Theta(1/n)$ we get

$$T_{\text{mix}}^{g\text{-CBSEP}} = n^2(\log n)^{\Theta(1)}, \quad d = 1$$

$$T_{\text{mix}}^{g\text{-CBSEP}} = n(\log n)^{\Theta(1)}, \quad d \geq 2$$

Remark

The result **does not** extend to the logarithmic Sobolev constant. Easy to find examples for which

$$T_{\text{Sob}}^{g\text{-CBSEP}} \gg T_{\text{Sob}}^{\text{CBSEP}} + T_{\text{cov}}^{\text{rw}}$$

1-neighbour KCM, a.k.a. FA-1f

Fix a connected graph $G = (V, E)$ and a parameter $0 < p < 1$

- **Configuration space:** $\Omega = \{0, 1\}^V$
- **Dynamics:** each site $v \in V$ that has at least 1 neighbouring particle is resampled to 1 with probability p and 0 with probability $1 - p$

→ As for CBSEP, the process is ergodic and reversible w.r.t. $\mu := \pi(\cdot | \exists \text{ at least one particle})$.

→ **BUT**

- **not attractive**
- we cannot embed a r.w.

1-neighbour KCM, a.k.a. FA-1f

- introduced [Friedrickson, Andersen '84] and extensively studied in physics as a model for the liquid glass transition
- scaling of the spectral gap on \mathbb{Z}^d as $p \downarrow 0$
[Cancrini, Martinelli, Roberto, C.T. '08, Shapira '20]
- convergence to equilibrium
[Blondel, Cancrini, Martinelli, Roberto, C.T. '13]
- Pillai and Smith '17 , '19: for $G = \mathbb{T}_n^d$ and $p = c/n$ it holds

$$C^{-1}n^2 \leq T_{\text{mix}} \leq Cn^2 \log^{14}(n) \quad d = 2$$

$$C^{-1}n^2 \leq T_{\text{mix}} \leq Cn^2 \log(n) \quad d \geq 3$$

FA-1f vs CBSEP

- branching and coalescing moves occur for FA-1f at the same rate as for CBSEP (when $p \rightarrow 0$)
- the SEP move $(1, 0) \rightarrow (0, 1)$ cannot occur on FA-1f, but it can be reconstructed via two consecutive FA-1f moves:

$$(1, 0) \xrightarrow{p} (1, 1) \xrightarrow{1-p} (0, 1)$$

$$\rightarrow c^{-1} \mathcal{D}^{\text{FA1f}}(f) \leq \mathcal{D}^{\text{CBSEP}}(f) \leq \frac{cd_{\max}}{p} \mathcal{D}^{\text{FA1f}}(f)$$

$$\rightarrow T_{\text{Sob}}^{\text{FA1f}} \leq O\left(\frac{d_{\max}}{p}\right) T_{\text{Sob}}^{\text{CBSEP}}$$

FA-1f: ℓ^q mixing

Corollary

On \mathbb{T}_n^d with $p = \Theta(1/n)$ it holds for all $q \geq 1$

$$T_q^{\text{FA1f}} \leq O(\log n) T_{\text{Sob}}^{\text{FA1f}} \leq \begin{cases} O(n^3 \log^2(n)) & d = 1 \\ O(n^2 \log^3(n)) & d = 2 \\ O(n^2 \log^2(n)) & d \geq 3 \end{cases}$$

- same results as Pillai, Smith '17+'19
- much simpler proof
- stronger : Pillai and Smith prove bounds on $T_{\text{mix}} = T_1$
- easy to generalise to different graphs, different scalings of p

FA-2f models on \mathbb{Z}^d

Constraint to update: at least 2 neighbouring particles

Theorem [Hartarsky, Martinelli, C.T. '20⁺]

Let τ_0 the first time at which the origin is zero. For FA-2f models on \mathbb{Z}^d it holds

$$\mathbb{E}(\tau_0) = \exp\left(\frac{\lambda_d + o(1)}{p^{1/(d-1)}}\right)$$

with $\lambda_d > 0$ an explicit constant. In particular $\lambda_2 = \pi^2/9$

How can we get this sharp threshold?

FA-2f: heuristics

- dominant relaxation : motion of large rare droplets
- droplets have density $q = e^{-\lambda_d/p^{1/(d-1)}}$
- a droplet can :
 - disappear near another droplet
 - create a new droplet nearby at rate q
 - move to a nearby position

→ droplets behave as CBSEP

FA-2f : some ideas of the proof

Key difficulties

- **droplets are not rigid objects**, they can be destroyed or change shape \rightarrow how do we identify them and follow their motion?
- **no monotonicity** \rightarrow no coupling or censoring arguments

Key ideas: upper bound

- translate heuristics into Poincaré inequalities
- renormalise to a g-CBSEP model
- use our results on g-CBSEP $\rightarrow T_{\text{rel}}^{\text{FA}2f} \leq 1/q \log(1/q)$

FA-2f : some ideas of the proof

Key ideas: lower bound

- $\mathbb{E}(\tau_0) \geq (\text{density of droplets})^{-1} = q^{-1}$
- the **deterministic version** of the dynamics in which sites are always filled is **2-neighbour bootstrap percolation**
- the dominant relaxation mechanism for BP is linear invasion of space by droplets $\rightarrow E^{BP}(\tau_0) \geq 1/q^{1/d}$
- sharp results on BP \rightarrow sharp results on q
[Holroyd '03, Balogh,Bollobas,Duminil-Copin,Morris '12]

The exponent for FA-2f is d times larger than for BP

Thanks!