Coalescing and Branching Exclusion Process

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The model

Fix a connected graph $G = (V, E)$ and a parameter $0 < p < 1$

- **Configuration space**: $\Omega = \{0, 1\}^V$
- **Dynamics**: each edge $e = \{x, y\}$ containing at least one particle is resampled at rate 1 from

$$\pi_x \times \pi_y(\cdot \,|\, \exists \text{ at least one particle on } \{x, y\})$$

with $\pi_x = \pi_y = \text{Ber}(p)$

The model

- If there are 2 particles they coalesce at rate $(1-p)/(2-p)$ to 1 particle on one of the two sites chosen uniformly

$$(1,1) \to (0,1)$$

- If there is 1 particle it creates a new particle on the other site at rate $p/(2-p)$

$$(1,0) \to (1,1)$$

- if there is 1 particle it moves to the adjacent empty site at rate $(1-p)/(2-p)$

$$(1,0) \to (0,1)$$

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$$(1,1) \to (0,1) \qquad \textbf{C}$$

- If there is 1 particle it creates a new particle on the other site at rate $p/(2-p)$

$$(1,0) \to (1,1) \qquad \textbf{B}$$

- if there is 1 particle it moves to the adjacent empty site at rate $(1-p)/(2-p)$

$$(1,0) \to (0,1) \qquad \textbf{SEP}$$

$\Rightarrow$ **CBSEP**

Coalescing random walks with neighbour births

System of particles on G that

- perform independent random walks jumping at rate 1
- branch at rate β creating a new particle on a neighbour empty vertex
- coalesce whenever they meet

Equivalent to CBSEP via a global time rescaling and setting $p = \beta/(1+\beta)$

History

- introduced in '77 by Schwartz as dual of biased voter model (a.k.a. William-Bjerknes tumour growth model)
- for any $\beta > 0$ on $\mathbb{Z}^d$: weak convergence to the unique invariant measure starting from any configuration with at least one particle [Bramson, Griffeath '80, '81]
- for any $\beta > 0$ on $\mathbb{Z}^d$: shape theorem [Durrett, Griffeath '82] $\rightarrow$ mixing time cut-off on torus
- for $\beta \rightarrow 0$ on $\mathbb{Z}$: convergence to the Brownian net [Sun, Swart '08]

CBSEP: properties

- Attractive and additive
- reversible w.r.t. $\pi = \text{Ber(p)}^{\otimes V}$
- ergodic and reversible on $\Omega_+ := \Omega \setminus \{\text{empty configuration}\}$ w.r.t. $\mu := \pi(\cdot|\Omega_+)$

Mixing times and log Sobolev

- T_q **is the ℓ^q-mixing time**

$$h_\omega^t(\cdot) = P_\omega^t(\cdot)/\mu(\cdot), \qquad ||f||_q = (\mu(|f|^q))^{1/q}, \quad q \geq 1$$

$$T_q := \inf\{t > 0, \max_\omega ||h_\omega^t(\cdot) - 1||_q \leq 1/e\}$$

$$T_{mix} = T_1$$

- $T_{\mathbf{Sob}}$ is the inverse of the Logarithmic Sobolev constant, i.e. the **inverse rate of decay of the entropy**

$$T_{\text{Sob}}^{-1} := \inf_f \frac{\mathcal{D}(f)}{\text{Ent}(f^2)} = \inf_f \frac{-\mu(f\mathcal{L}f)}{\mu(f^2 \log(f^2/\mu(f^2))}$$

$$T_q \leq O\left(\log\log\left(\frac{1}{\min_\omega \mu(\omega)}\right)\right) T_{\mathbf{Sob}} \qquad \forall q \in [1, \infty]$$

CBSEP: results

Theorem [Hartarsky,Martinelli, C.T. '20]

Let $p_n = \Theta(1/n)$ and $G_n = (V_n, E_n)$ be a sequence of bounded degree graphs with $|V_n| = n$. Then $\exists c > 0$ s.t. $\forall n$

$$c\,T_{\text{meet}} \leq T_{\text{Sob}} \leq c^{-1}\,T_{\text{meet}} \log n$$

with T_{meet} the expected meeting time for two continuous time r.w. on G_n starting from two uniformly chosen sites

Corollary

If $G_n = \mathbb{T}_n^d$= d-dimensional torus with n sites

$$\begin{aligned} cn^2 \leq T_{\text{Sob}} &\leq c^{-1} n^2 \log n \quad & d = 1 \\ cn \log n \leq T_{\text{Sob}} &\leq c^{-1} n \log^2 n \quad & d = 2 \\ cn \leq T_{\text{Sob}} &\leq c^{-1} n \log n \quad & d \geq 3 \end{aligned}$$

CBSEP: ideas of the proof

$$\mathrm{Ent}(f^2) = \mu(\mathrm{Ent}(f^2|N)) + \mathrm{Ent}(\mu(f^2|N)), \quad N = \#\ \text{particles}$$

- first term
  - consider Bernoulli Laplace on complete graph K_n
  - $T^{\mathrm{BL}}_{\mathrm{Sob}} = \log n$ [Lee Yau]
  - $\mathcal{D}^{\mathrm{BL}} \leq \frac{d^2_{max} d_{mean}}{d^2_{min}}\ T^{\text{lazy rw}}_{\mathrm{mix}}\ \mathcal{D}^{\mathrm{SEP}}$ [Kozma Alon]
- second term
  - we construct an auxiliary birth death process with invariant measure the law of N
  - we determine $T^{\text{birth-death}}_{\mathrm{Sob}}$ and we use path arguments to compare with CBSEP

A generalised version: g-CBSEP

- Fix a connected graph $G = (V, E)$ and a finite probability space (S, ρ) with $S = S_1 \cup S_0$ and set $p := \rho(S_1)$
- we say that *there is a particle at* $v \in V$ *iff* $\omega_v \in S_1$

g-CBSEP dynamics : any edge $e = (x, y)$ containing at least one particle is resampled at rate one from $\rho_x \times \rho_y(\cdot \,|\, \exists$ at least one particle on $e)$

Remark:
the projection $\phi : S^V \to \{0, 1\}^V$ with $\phi(\omega)_x = 1_{\omega_x \in S_1}$ is CBSEP.

g-CBSEP: results

$$T^{\mathrm{rw}}_{\mathrm{cov}} = \inf\{t > 0, \max_{x\in V} \mathbb{P}_x(\tau_{\mathrm{cov}} > t) \le 1/e\}$$

with τ_{cov} the cover time of the simple r.w. on G

Theorem [Hartarsky,Martinelli, C.T. '20]

$$T^{\mathrm{CBSEP}}_{\mathrm{mix}} \le T^{g\text{-CBSEP}}_{\mathrm{mix}} \le c(T^{\mathrm{CBSEP}}_{\mathrm{mix}} + T^{\mathrm{rw}}_{\mathrm{cov}})$$

Idea:
wait for the projection (= CBSEP) to couple, then wait for one random walk to cover the graph ($\rightarrow$ all sites are refreshed).

g-CBSEP: results

Corollary

On $\mathbb{T}_n^d$ with $p = \Theta(1/n)$ we get

$$T_{\text{mix}}^{g\text{-CBSEP}} = n^2 (\log n)^{\Theta(1)}, \quad d = 1$$

$$T_{\text{mix}}^{g\text{-CBSEP}} = n (\log n)^{\Theta(1)}, \quad d \geq 2$$

Remark

The result **does not** extend to the logarithmic Sobolev constant. Easy to find examples for which $T_{\text{Sob}}^{g\text{-CBSEP}} \gg T_{\text{Sob}}^{\text{CBSEP}} + T_{\text{cov}}^{\text{rw}}$

1-*neighbour KCM, a.k.a. FA*-1*f*

Fix a connected graph $G = (V, E)$ and a parameter $0 < p < 1$

- **Configuration space**: $\Omega = \{0, 1\}^V$
- **Dynamics**: each site $v \in V$ that has at least 1 neighbouring particle is resampled to 1 with probability p and 0 with probability $1 - p$

→ As for CBSEP, the process is ergodic and reversible w.r.t. $\mu := \pi(\cdot | \exists$ at least one particle).

→ **BUT**

- **not attractive**
- we cannot embed a r.w.

1-*neighbour KCM, a.k.a. FA*-1*f*

- introduced [Friedrickson, Andersen '84] and extensively studied in physics as a model for the liquid glass transition
- scaling of the spectral gap on $\mathbb{Z}^d$ as $p \downarrow 0$ [Cancrini, Martinelli, Roberto, C.T. '08, Shapira '20]
- convergence to equilibrium [Blondel, Cancrini, Martinelli, Roberto, C.T. '13]
- Pillai and Smith '17 ,'19: for $G = \mathbb{T}_n^d$ and $p = c/n$ it holds

$$C^{-1}n^2 \leq T_{\text{mix}} \leq Cn^2 \log^{14}(n) \quad d = 2$$

$$C^{-1}n^2 \leq T_{\text{mix}} \leq Cn^2 \log(n) \quad d \geq 3$$

FA-1f vs CBSEP

- branching and coalescing moves occur for FA-1f at the same rate as for CBSEP (when $p \to 0$)
- the SEP move $(1,0) \to (0,1)$ cannot occur on FA-1f, but it can be reconstructed via two consecutive FA-1f moves:

$$(1,0) \stackrel{p}{\to} (1,1) \stackrel{1-p}{\to} (0,1)$$

$$\to \quad c^{-1}\mathcal{D}^{\mathrm{FA1f}}(f) \le \mathcal{D}^{\mathrm{CBSEP}}(f) \le \frac{c d_{max}}{p}\,\mathcal{D}^{\mathrm{FA1f}}(f)$$

$$\to T^{\mathrm{FA1f}}_{\mathrm{Sob}} \le O\left(\frac{d_{max}}{p}\right) T^{\mathrm{CBSEP}}_{\mathrm{Sob}}$$

FA-1f: ℓ^q mixing

Corollary

On $\mathbb{T}_n^d$ with $p = \Theta(1/n)$ it holds for all $q \geq 1$

$$T_q^{\text{FA1f}} \leq O(\log n) T_{\text{Sob}}^{\text{FA1f}} \leq \begin{cases} O(n^3 \log^2(n)) & d = 1 \\ O(n^2 \log^3(n)) & d = 2 \\ O(n^2 \log^2(n)) & d \geq 3 \end{cases}$$

- same results as Pillai, Smith '17+'19
- much simpler proof
- stronger : Pillai and Smith prove bounds on $T_{mix} = T_1$
- easy to generalise to different graphs, different scalings of p

FA-2f models on $\mathbb{Z}^d$

Constraint to update: at least 2 neighbouring particles

Theorem [Hartarsky,Martinelli, C.T. '20^+]

Let τ_0 the first time at which the origin is zero. For FA-**2**f models on $\mathbb{Z}^d$ it holds

$$\mathbb{E}(\tau_0) = \exp\left(\frac{\lambda_d + o(1)}{p^{1/(d-1)}}\right)$$

with $\lambda_d > 0$ an explicit constant. In particular $\lambda_2 = \pi^2/9$

How can we get this sharp threshold?

FA-2f: heuristics

- dominant relaxation : motion of large rare droplets
- droplets have density $q = e^{-\lambda_d/p^{1/(d-1)}}$
- a droplet can :
  - disappear near another droplet
  - create a new droplet nearby at rate q
  - move to a nearby position

$\rightarrow$ droplets behave as CBSEP

FA-2f : some ideas of the proof

Key difficulties

- **droplets are not rigid objects**, they can be destroyed or change shape → how do we identify them and follow their motion?
- **no monotonicity** → no coupling or censoring arguments

Key ideas: upper bound

- translate heuristics into Poincaré inequalities
- renormalise to a g-CBSEP model
- use our results on g-CBSEP → $T_{\rm rel}^{\rm FA2f} \le 1/q \log(1/q)$

FA-2f : some ideas of the proof

Key ideas: lower bound

- $\mathbb{E}(\tau_0) \geq (\text{density of droplets})^{-1} = q^{-1}$
- the **deterministic version** of the dynamics in which sites are always filled is **2-neighbour bootstrap percolation**
- the dominant relaxation mechanism for BP is linear invasion of space by droplets $\rightarrow E^{BP}(\tau_0) \geq 1/q^{1/d}$
- sharp results on BP $\rightarrow$ sharp results on q [Holroyd '03, Balogh,Bollobas,Duminil-Copin,Morris '12]

The exponent for FA-2f is d times larger than for BP

Thanks!