Symmetric and asymmetric hydrodynamics for the facilitated exclusion process via mapping

Based on j.w. with O. Blondel, M. Sasada, M. Simon and L. Zhao

Clément Erignoux, INRIA Lille

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Symmetric Simple Exclusion Process (SSEP) on $\mathbb{Z}$

- Configuration $\eta \in \Omega := \{0,1\}^{\mathbb{Z}}$, with $\eta_x = 1$ for an occupied site, $\eta_x = 0$ for an empty site.
- **Stirring dynamics**: two neighboring sites are exchanged at rate 1.
- Initial profile $\rho_0 : \mathbb{R} \to [0,1]$ fixed, **initial configuration** e.g. $\eta_x(0) = 1$ w.p. $\rho_0(x/N)$.

Then, the **empirical measure** on a diffusive timescale

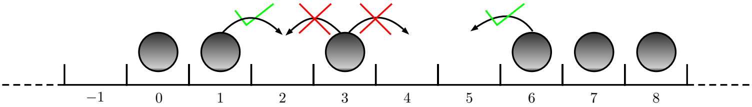
$$\pi^N_{tN^2} = \frac{1}{N} \sum_{x \in \mathbb{Z}} \eta_x(tN^2) \delta_{x/N}$$

converges in a weak sense to $\rho(t,u)du$, where ρ is the **solution to the heat equation**

$$\begin{cases} \partial_t \rho = \partial_{uu} \rho \\ \rho(0,\cdot) = \rho_0 \end{cases} .$$

Facilitated Exclusion Process (FEP)

Similar to *[Gonçalves, Landim, Toninelli '08]*, but with stronger kinetic constraint



Markov generator $\mathcal{L}f(\eta) = \sum_{x\in\mathbb{Z}} c_{x,x+1}(\eta)\{f(\eta^{x,x+1}) - f(\eta)\}$,

with

$$c_{x,x+1}(\eta) = p\eta_{x-1}\eta_x(1-\eta_{x+1}) + (1-p)\eta_{x+2}\eta_{x+1}(1-\eta_x).$$

The parameter $p \in [0,1]$ tunes the asymmetry, and $\eta^{x,x+1}$ is the configuration where sites x and $x+1$ have been exchanged.

- ▷ Bernoulli product measures are **not stationary**.
- ▷ **No mobile cluster** to mix the configuration (cooperative model).

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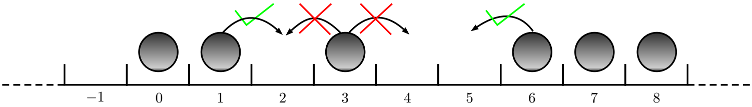

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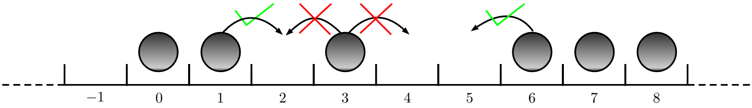

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Hydrodynamic limit for the symmetric FEP

Theorem (Blondel, E', Simon, Sasada 2018 & BES 2021)

*Given ρ_0, consider the **symmetric** ($p = 1 - p = 1/2$) process $\eta(t)$ on $\mathbb{T}_N := \{0, 1, \dots, N\}$ started from*

$$\mu^N = \mu_0^N := \bigotimes_{x \in \mathbb{T}_N} Ber(\rho_0(x/N)).$$

For any smooth compactly supported H

$$\frac{1}{N} \sum_{x \in \mathbb{T}_N} H(x/N) \eta_x(tN^2) \xrightarrow[N \to \infty]{\mathbb{P}} \int_{[0,1]} H(u) \rho(t,u) du$$

where ρ is solution to the parabolic Stefan problem $\rho(0,u) = \rho_0(u)$ and

$\boxed{\rho_0 > 1/2}$

$$\partial_t \rho = \frac{1}{2} \partial_{uu} \left\{ \frac{2\rho - 1}{\rho} \right\}$$

$\boxed{\rho_0 \in [0,1]}$

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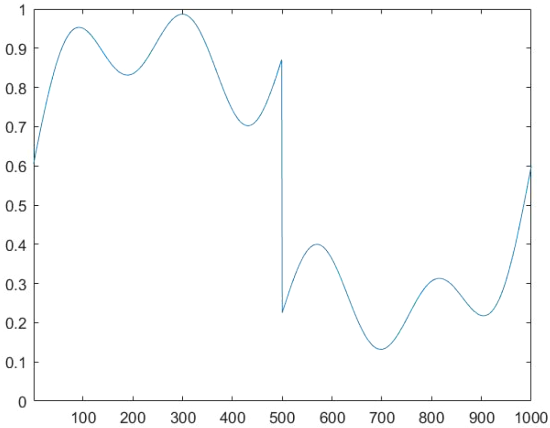
Stefan problem



$$\partial_t \rho = \frac{1}{2} \partial_{uu} \left\{ \frac{2\rho - 1}{\rho} \mathbf{1}_{\{\rho \geq 1/2\}} \right\}.$$

TYPES OF CONFIGURATIONS

Four types of configurations, depending on the **critical density** $\rho_c = 1/2$.

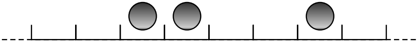
- ***Low density*** : *if* $\rho < 1/2$

Frozen configurations

$$\mathcal{F} = \{\eta \in \Omega \mid \eta_x \eta_{x+1} \equiv 0\}$$

Transient Bad configurations

$$\mathcal{TB} = \{\eta \in \Omega \mid \eta_x \eta_{x+1} \not\equiv 0\}.$$



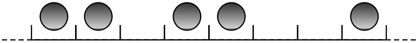
- ***Large density*** : *if* $\rho > 1/2$,

Ergodic configurations

$$\mathcal{E} = \{\eta \in \Omega \mid (1-\eta_x)(1-\eta_{x+1}) \equiv 0\}$$

Transient Good configurations

$$\mathcal{TG} = \{\eta \in \Omega \mid (1-\eta_x)(1-\eta_{x+1}) \not\equiv 0\}$$



Grand canonical measures

Because of kinetic constraint, Bernoulli product measures are not stationary for the dynamics. Canonical measures can be defined as **uniform measures on the ergodic components** with fixed number of particles.

The symmetric FEP is actually **reversible w.r.t. a family of explicit supercritical distributions** π_ρ, for $\rho \in (1/2, 1]$.

- π_ρ is supported on the **infinite ergodic component**.
- π_ρ is a Bernoulli product measure conditioned to having isolated empty sites (ergodic component)
- π_ρ exhibits **long-range correlations** as $\rho \searrow 1/2$.

Entropy tools and equilibrium distributions

The most classical techniques for hydrodynamic limits are based on **entropy bounds** between the measure μ_t^N of the process at time t and its reference measures π_α, namely

- ▷ Guo, Papanicolaou and Varadhan's **entropy method**,

$$H(\mu_t^N \mid \pi_\rho) \leq CN,$$

- ▷ Yau's **relative entropy method**

$$H(\mu_t^N \mid \pi_{\rho_t}) = o(N).$$

Supercritical case, in the transient regime, μ_t^N is not supported on ergodic configurations, whereas the grand canonical measures π_ρ are $\Rightarrow$ entropy estimate fails. In particular, we need to prove that the ergodic component is reached before the diffusive timescale $\tau = O(N^2)$.

General case, no real hope of using entropy methods : no reference measures because the two phase's stationary states have **disjoint supports**, and no smooth solutions to the hydrodynamic limit.

Strategy of proof

- **Supercritical case**, GPV's entropy method can be adapted, by proving that the ergodic component is reached in a subdiffusive time.
- **General case:**
  - entropy methods cannot be used, so we adapt Funaki's scheme for parabolic Stephan problems.
  - The **one-block estimate** is based on a De Finetti-type decomposition for translation invariant stationary states.
  - The two blocks estimate is bypassed by directly proving that the **Young measure is a dirac**.

Hydrodynamic limit for the asymmetric FEP

Theorem (E', Simon, Zhao 2022)

Given ρ_0, consider the ***asymmetric*** *($p \in (1/2, 1]$) process $\eta(t)$ started from*

$$\mu^N = \mu_0^N := \bigotimes_{x \in \mathbb{Z}} Ber(\rho_0(x/N)).$$

For any smooth compactly supported H

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where ρ is the ***unique entropy solution*** *to the hyperbolic Stefan problem*

$$\begin{cases} \partial_t \rho + (2p-1) \partial_u \left\{ \mathfrak{H}(\rho) \mathbf{1}_{\{\rho \geq 1/2\}} \right\} \\ \rho(0, u) = \rho_0(u) \end{cases}, \quad \textit{where} \quad \mathfrak{H}(\rho) = \frac{(1-\rho)(2\rho - 1)}{\rho}.$$

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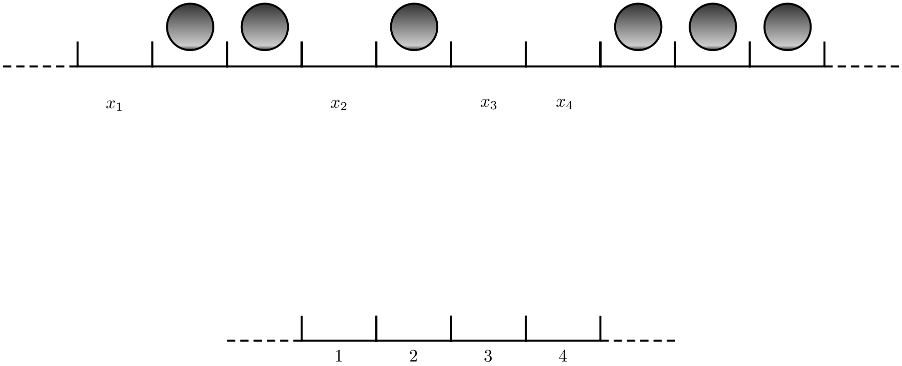
Possible strategies of proof

- ▷ GPV's **entropy method** for hyperbolic systems ? No two-blocks estimate in the asymmetric case.
- ▷ Yau's **relative entropy method** ? Only useful until the first shock, and even so, not at all straightforward for two-phased systems, and no smooth solution a priori even before the shock because of the Stefan problem.
- ▷ Fritz's **compensated compactness** arguments ? Blackbox tools, very technical, and requires adding up some lower-order stirring dynamics.
- ▷ **Attractiveness** ? A priori not available here.

Mapping with a Facilitated ZR process



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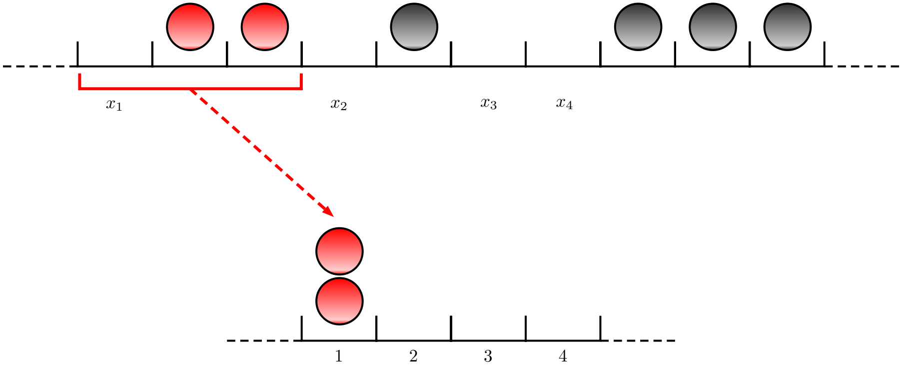

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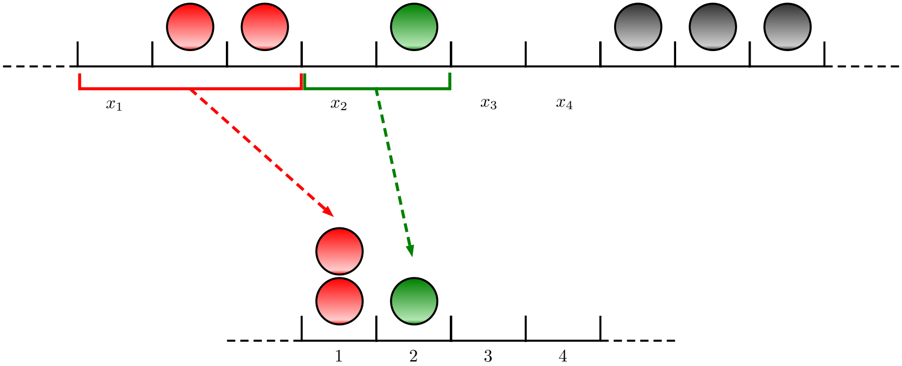

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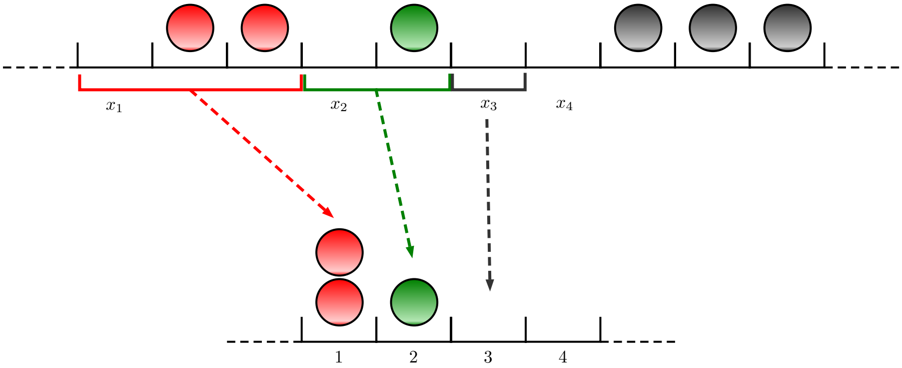

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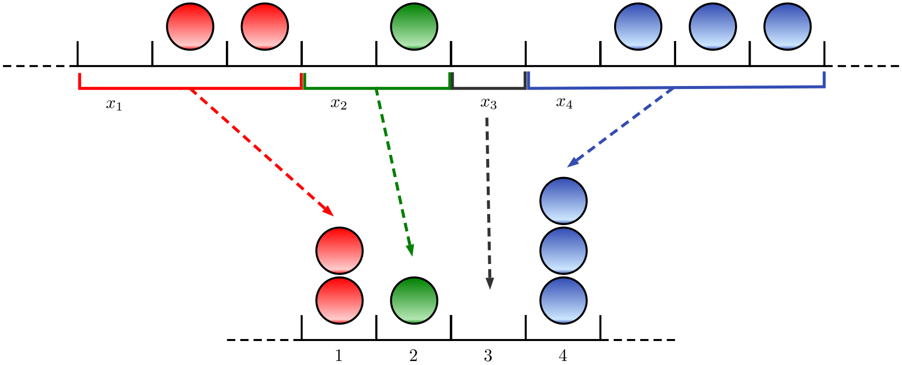
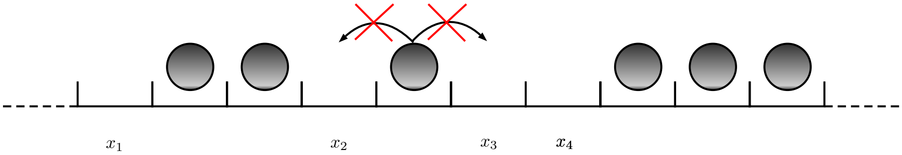
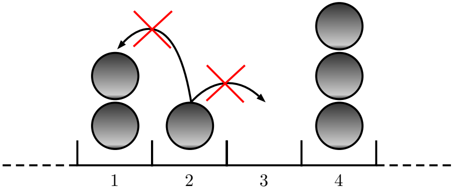
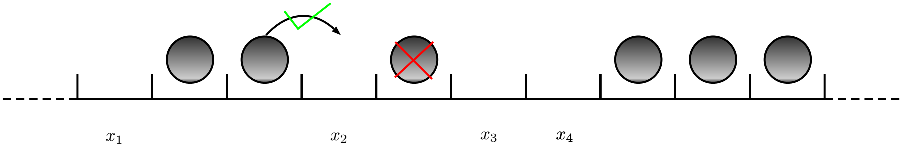


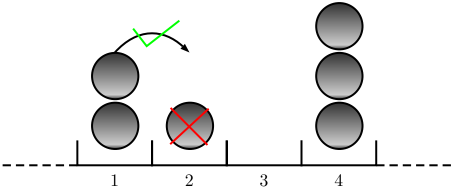
Mapping with FZRP : dynamics





Mapping with FZRP : dynamics





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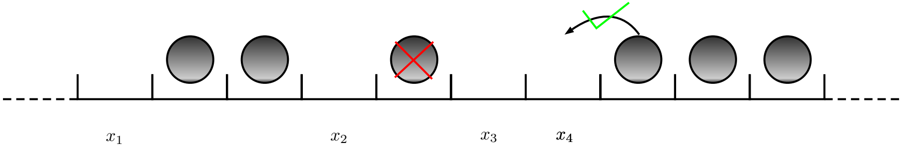

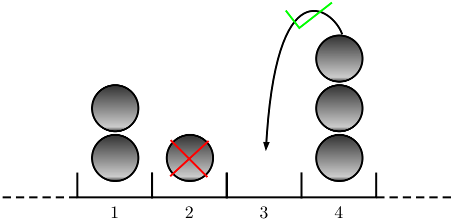

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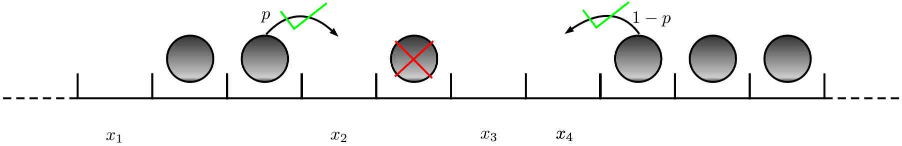
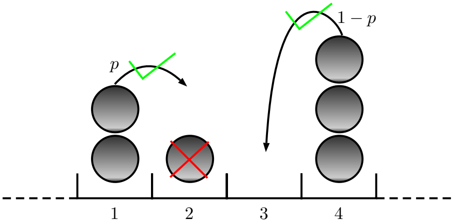




⇒ If the exclusion process is driven by the facilitated generator, the corresponding **facilitated zero-range process (FZRP)** *seen from the tagged empty site* is driven by the generator

$$\mathcal{L}^{zr} g(\omega) = \sum_{y \in \mathbb{Z}} \mathbf{1}_{\{\omega_y \geq 2\}} \Big\{ p g(\omega^{y,y+1}) + (1-p) g(\omega^{y,y-1}) - g(\omega) \Big\}.$$

Properties of the FZRP

- Since the function $k \mapsto \mathbf{1}_{\{k\geq 2\}}$ is non-decreasing, this "facilitated" zero-range process is **attractive**: the evolution of two such processes ω and ζ can be coupled in such a way that

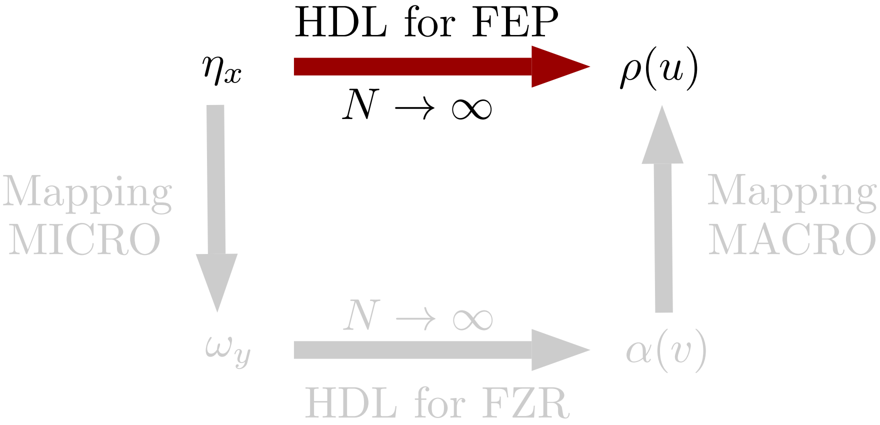
$$\omega(0) \leq \zeta(0) \quad \Rightarrow \quad \omega(t) \leq \zeta(t) \quad \forall t.$$

- The equilibrium/stationary distributions for the FZRP are product geometric measures **with no empty sites**, and density $\alpha > 1$, i.e. with marginals

$$\nu_\alpha(\omega_0 = k) = \mathbf{1}_{\{k\geq 1\}} \frac{1}{\alpha}\left(1 - \frac{1}{\alpha}\right)^{k-1}$$

- Even with attractiveness, coupling arguments are tricky, because the **process is not ergodic**: filling an empty site with a particle is irreversible for the FZRP, and equilibrium states only exist in the supercritical phase $\alpha > 1$.

Strategy of proof, HDL for the FEP



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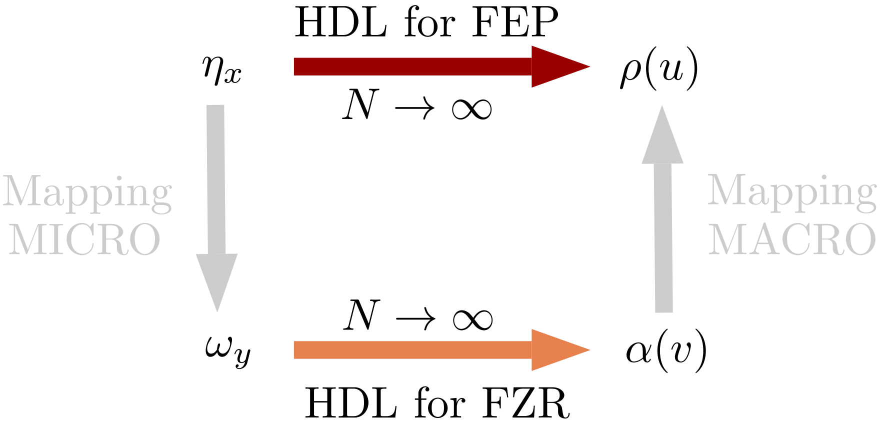

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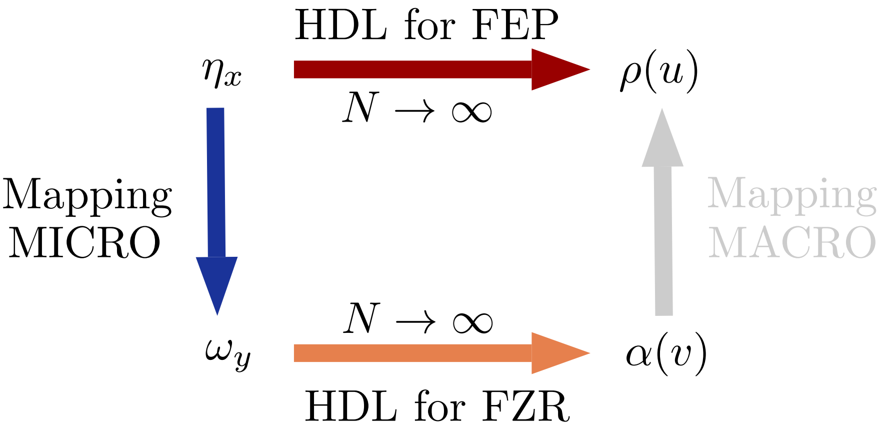

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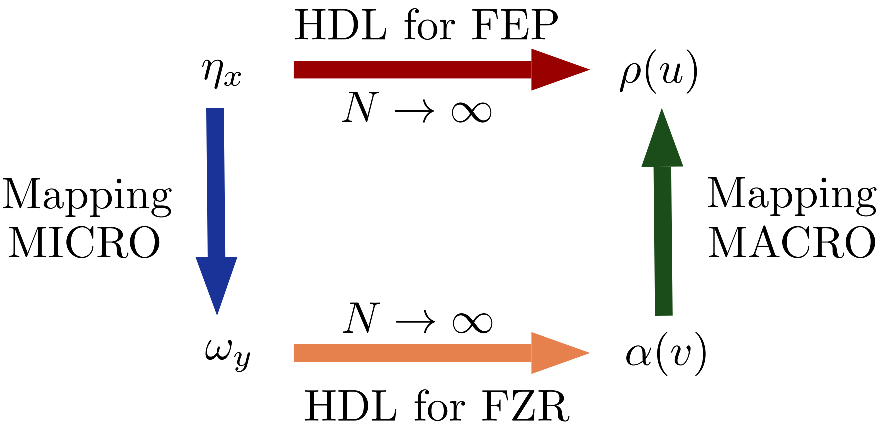

Hydrodynamics for the FZRP

Theorem (E', Simon, Zhao 2022)

Given an initial profile α_0, consider the ***asymmetric*** *($p \in (1/2, 1]$) FZRP $\omega(t)$. Assuming that for any smooth compactly supported H, under the initial distribution,*

$$\frac{1}{N}\sum_{y\in\mathbb{Z}} H(y/N)\omega_y \xrightarrow[N\to\infty]{\mathbb{P}} \int_{\mathbb{R}} H(v)\alpha_0(v)dv$$

then for any $t > 0$

$$\frac{1}{N}\sum_{y\in\mathbb{Z}} H(y/N)\omega_y(tN) \xrightarrow[N\to\infty]{\mathbb{P}} \int_{\mathbb{R}} H(v)\alpha(t,v)dv$$

where α is the ***unique entropy solution*** *to the hyperbolic Stefan problem*

$$\partial_t\alpha + (2p-1)\partial_v\left\{\frac{(\alpha-1)}{\alpha}\mathbf{1}_{\{\alpha\geq 1\}}\right\} \qquad \alpha(0,u) = \alpha_0(u).$$

$\mapsto$ *Hydrodynamic limit for attractive particle systems on $\mathbb{Z}^d$*, F. Rezakhanlou.

Macroscopic mapping

- Denote $X_0 = X_0(t)$ the position of the tagged empty site in the FEP, and $\nu_t[\rho] = \lim_{N\to\infty} X_0(t)/N$ its macroscopic position at time t.
- The macroscopic position of the tagged empty site is formally written as

$$\nu_t[\alpha] = \nu_0 + \int_0^\infty \alpha_0(v) - \alpha(t,v)dv.$$

- Space variable y for ω corresponding to x in η ? **Number of empty sites** between X_0 and x. At the **macroscopic scale** $u = x/N$, $v = y/N$, we can write

$$y = y(x) = \sum_{x'=X_0}^{x}(1-\eta_{x'}) \quad \Rightarrow \quad v = v(u) = \int_{\nu_t}^{u}(1-\rho(u'))du'$$

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$$\Rightarrow \quad \boxed{\rho(u) = \frac{\alpha}{1+\alpha}(v)}$$

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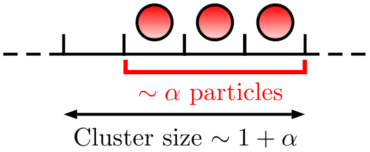
- 

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MACROSCOPIC MAPPING

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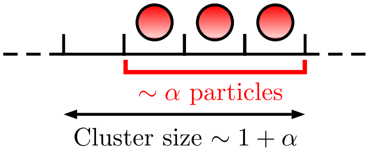
- 

$\Rightarrow$ $$\boxed{\rho(u) = \frac{\alpha}{1+\alpha}(v)}$$

Mapping Hydrodynamics

Now, to prove the HDL for the FEP given that of the FZRP, one can use that

$$\frac{1}{N}\sum_{x\in\mathbb{Z}}\eta_x(tN^2)H(x/N)\simeq\frac{1}{N}\sum_{y\in\mathbb{Z}}\omega_y(tN^2)[H\circ u_t](y/N)+O(1/N),$$

where $u=u_t(v)$ is the inverse mapping of $v=v_t(u)$ seen earlier. Assuming everything is smooth, thanks to the HDL for the FZRP

$$\frac{1}{N}\sum_{y\in\mathbb{Z}}\omega_y(tN^2)[H\circ u_t](y/N)\simeq\int_{\mathbb{R}}\alpha(t,v)[H\circ u_t](v)dv,$$

and by a change of variable $v\mapsto u_t(v)$, the right hand side becomes

$$\int_{\mathbb{R}}\rho(t,u)H(u)du,$$

where ρ is given by $\rho(u)=\frac{\alpha}{1+\alpha}(v)$.

$\mapsto$ *Problem*: everything needs to be smoothed out, because of the hyperbolic equation, and because of the Stefan problem.

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$$\frac{1}{N}\sum_{x\in\mathbb{Z}}\eta_x(tN^2)H(x/N) \simeq \frac{1}{N}\sum_{y\in\mathbb{Z}}\omega_y(tN^2)[H\circ u_t](y/N) + O(1/N),$$

where $u = u_t(v)$ is the inverse mapping of $v = v_t(u)$ seen earlier. Assuming everything is smooth, thanks to the HDL for the FZRP

$$\frac{1}{N}\sum_{y\in\mathbb{Z}}\omega_y(tN^2)[H\circ u_t](y/N) \simeq \int_{\mathbb{R}}\alpha(t,v)[H\circ u_t](v)dv,$$

and by a change of variable $v \mapsto u_t(v)$, the right hand side becomes

$$\int_{\mathbb{R}}\rho(t,u)H(u)du,$$

where ρ is given by $\rho(u) = \frac{\alpha}{1+\alpha}(v)$.

$\mapsto$ *Problem*: everything needs to be smoothed out, because of the hyperbolic equation, and because of the Stefan problem.

Current work

- Phase transition(s) for the FEP/CLG in higher dimensions,
  with A. Roget, A. Shapira and M. Simon.

- Effect of boundary interactions on the FEP,
  with M. Simon.

Thanks for your attention !

Related works - FEP

- E., Simon, Zhao, arxiv - 2202.04469 (2022).
- Blondel, E. and Simon, *Probability and Mathematical Physics* (2021).
- Blondel, E., Sasada and Simon, *An. de l'IHP - Prob. et Stat.* (2020).

Attractiveness & Stefan problems

- Seppäläinen. Translation invariant exclusion processes (2008).
- Kipnis, Landim, *Scaling Limits of Interacting Particle Systems* (1999).
- Funaki, *An. de l'IHP - Prob. et Stat.* (1999).
- Rezakhanlou, Com. in math. phys. (1991).