Critical Bootstrap Percolation and Kinetically Constrained Models: Universality Results

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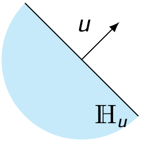
Reminders

We work on $\mathbb{Z}^2$.

Stable directions: If $u \in S^1$, let $\mathbb{H}_u = \{x \in \mathbb{Z}^2 : \langle x, u \rangle < 0\}$.
$u \in S^1$ is a stable direction if, starting with a configuration infected in $\mathbb{H}_u$ and healthy in $\mathbb{Z}^2 \setminus \mathbb{H}_u$, no other site can be infected by bootstrap percolation.
Otherwise, u is an unstable direction.



Definition

A constraint is critical if both:

- there exists no open semicircle of unstable directions,
- there exists an open semicircle with only a finite number of stable directions.

Reminders

Initial configuration with law μ: sites are independently infected with probability q.

We define τ^{BP} the first time at which the origin is infected in bootstrap percolation.
We denote T^{BP} the median of τ^{BP}.

Theorem (Bollobás, Smith, Uzzell, 2015)

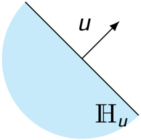
For any critical constraint, $T^{BP} = \exp(1/q^{\Theta(1)})$ *when* q *tends to 0.*

$\Rightarrow$ More results on critical models?

The Difficulty of a Direction

For $u \in S^1$, the difficulty $\alpha(u)$ of u is

- 0 if u is unstable,
- ∞ if u belongs to an interval of stable directions,
- the smallest cardinal of K such that starting with a configuration infected in $\mathbb{H}_u \cup K$, the bootstrap percolation dynamics infects infinitely many sites, if u is an isolated stable direction.



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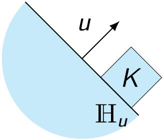

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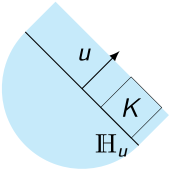


The Difficulty of a Constraint

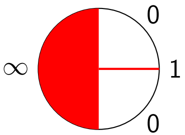
For C an open semicircle, we set $\alpha(C) = \max_{u \in C} \alpha(u)$.

The difficulty α of a constraint is

$$\alpha = \min_{C \text{ open semicircle}} \alpha(C).$$

Example: the Duarte model

red=stable directions
white=unstable directions



$\alpha(◗) = 1$
$\alpha(C) = \infty$ if $C \neq$ ◗

$\Rightarrow$ We get $\alpha = 1$.

There is an open semicircle in which all directions have difficulty $\leq \alpha$.

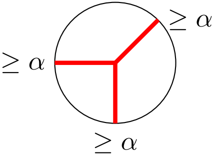
A Refinement for Critical Bootstrap Percolation

Theorem (Bollobás, Duminil-Copin, Morris, Smith, 2014)

For any critical constraint, $T^{BP} = \exp((\frac{1}{q})^{\alpha+o(1)})$ *when* q *tends to 0.*

$$\alpha = \min_{C \text{ open semicircle}} \alpha(C)$$

$\Rightarrow$ In each open semicircle there is a direction of difficulty $\geq \alpha$.
$\Rightarrow$ $\exists$ set of directions of difficulty $\geq \alpha$ whose convex envelope contains the origin.



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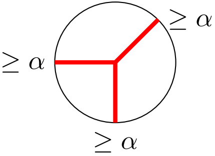

Heuristics for Critical Bootstrap Percolation

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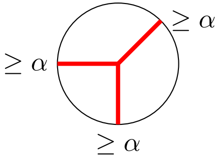

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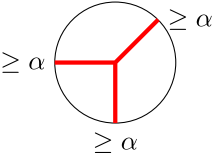

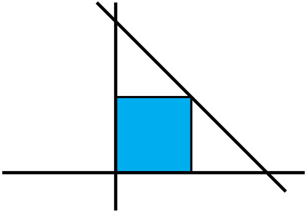

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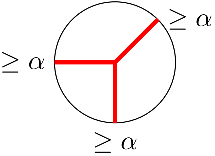

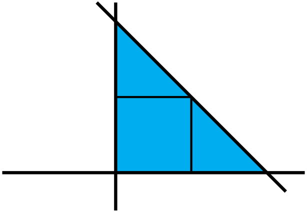

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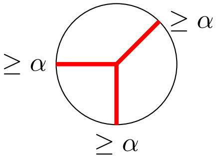

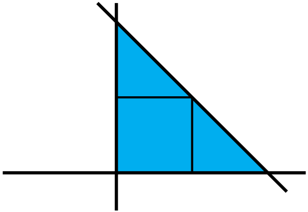

To get significant new infection, one needs to find a group of α infected sites near the triangle.
$\Rightarrow$  of size $\Theta(1/q^{\alpha})$.

Heuristics for Critical Bootstrap Percolation

$\Theta(1/q^{\alpha})$

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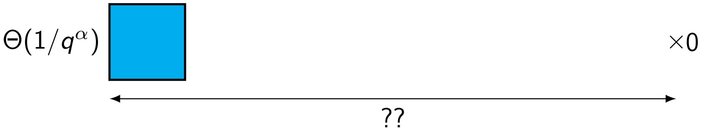
Heuristics for Critical Bootstrap Percolation

$\Theta(1/q^\alpha)$  $\times 0$

Heuristics for Critical Bootstrap Percolation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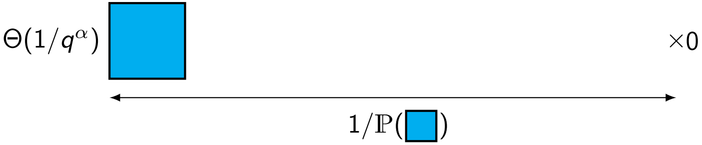

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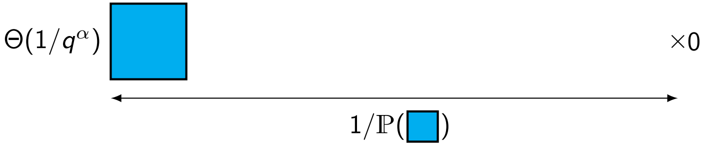
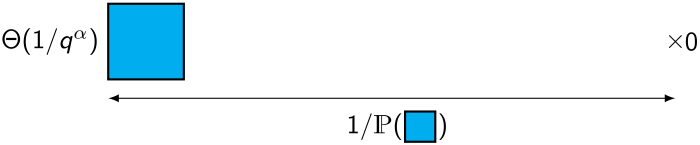


$$T^{BP} = 1/\mathbb{P}\left(\square\right).$$



$$T^{BP} = 1/\mathbb{P}\left(\square\right).$$

$$\Rightarrow \mathbb{P}\left(\square\right) = q^{\Theta(1/q^\alpha)} = \exp(-1/q^{\alpha+o(1)}).$$
$$\Rightarrow T^{BP} = 1/\mathbb{P}\left(\square\right) = \exp(1/q^{\alpha+o(1)}).$$

The Infection Time for Critical KCMs

We define τ^{KCM} the first time at which the origin is infected in the KCM.

We have $\mathbb{E}(\tau^{KCM}) = \Omega(T^{BP})$ when $q \to 0$.

$\Rightarrow \mathbb{E}_\mu(\tau^{KCM}) \geq \exp((\frac{1}{q})^{\alpha+o(1)})$ when $q \to 0$.

$\Rightarrow$ Do we have $\mathbb{E}_\mu(\tau^{KCM}) = \exp((\frac{1}{q})^{\alpha+o(1)})$ when $q \to 0$?

Universality for Critical KCMs

$T^{BP} = \exp((\frac{1}{q})^{\alpha+o(1)})$ when $q \to 0$.

Theorem (Hartarsky, M., Toninelli, 2020 + Martinelli, Morris, Toninelli, 2019 + Hartarsky, Martinelli, Toninelli, 2021)

If the constraint has:

- a finite number of stable directions, $\mathbb{E}_\mu(\tau^{KCM}) = \exp((\frac{1}{q})^{\alpha+o(1)})$ when $q \to 0$,
- an infinite number of stable directions, $\mathbb{E}_\mu(\tau^{KCM}) = \exp((\frac{1}{q})^{2\alpha+o(1)})$ when $q \to 0$.

Heuristics for Critical KCMs

$\Theta(1/q^\alpha)$ 

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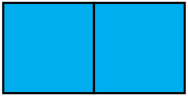

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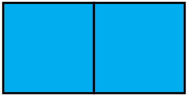
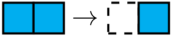
$\Theta(1/q^{\alpha})$ 

Heuristics for Critical KCMs with a Finite Number of Stable Directions

Models with a finite number of stable directions:
 possible.

FA-1f mechanism.

$\times 0$

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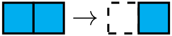 possible.

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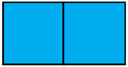

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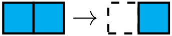 possible.

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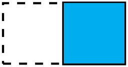

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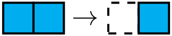 possible.

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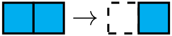 possible.

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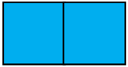

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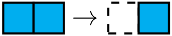 possible.

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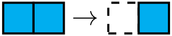 possible.

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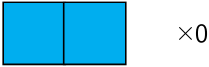

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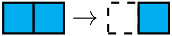 possible.

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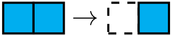 possible.

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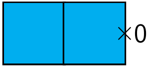

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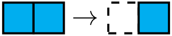 possible.

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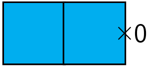

$\Rightarrow$ Energy barrier $\mu\left(\square\right)^2$.
$\Rightarrow \mathbb{E}_\mu(\tau^{KCM}) = 1/\mu\left(\square\right)^2 = 1/\exp(-1/q^{\alpha+o(1)}) = \exp(1/q^{\alpha+o(1)})$.

Models with an infinite number of stable directions:
$\blacksquare\blacksquare \to \square\blacksquare$ impossible.

Only moves: $\blacksquare\square \to \blacksquare\blacksquare$ and $\blacksquare\blacksquare \to \blacksquare\square$.

East mechanism.

The dynamics has to go through a configuration with at least n $\blacksquare$,
where $n = \Theta(\ln(\text{distance between the origin and the closest initial } \blacksquare))$.
$\Rightarrow n = \Theta(\ln(1/\mu(\blacksquare)))$.

$\Rightarrow$ Energy barrier $\mu\left(\blacksquare\right)^n$.
$\Rightarrow \mathbb{E}_\mu(\tau^{KCM}) = 1/\mu\left(\blacksquare\right)^n = 1/\mu\left(\blacksquare\right)^{\ln(1/\mu(\blacksquare))} = \exp(1/q^{2\alpha+o(1)})$.

One can do better than $T^{BP} = \exp((\frac{1}{q})^{\alpha+o(1)})$!

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Definition

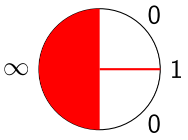
A critical constraint is called

- *balanced* if there exists a closed semicircle in which directions have difficulty $\leq \alpha$,
- *unbalanced* otherwise.

Examples: red=stable directions, white=unstable directions

Duarte model

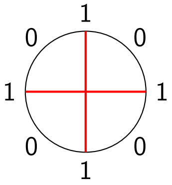
$\alpha = 1$



Unbalanced.

FA-2f model

$\alpha = 1$



Balanced.

Theorem (Bollobás, Duminil-Copin, Morris, Smith, 2014)

If the constraint is:

- balanced, then $T^{BP} = \exp(\Theta((\frac{1}{q})^{\alpha}))$ when $q \to 0$,
- unbalanced, then $T^{BP} = \exp(\Theta((\frac{1}{q})^{\alpha} \ln(\frac{1}{q})^2))$ when $q \to 0$.

Universality for Critical KCMs with an Infinite Number of Stable Directions

$\mathbb{E}_\mu(\tau^{KCM}) = \exp((\frac{1}{q})^{2\alpha+o(1)})$ when $q \to 0$.

Universality for Critical KCMs with an Infinite Number of Stable Directions

$\mathbb{E}_\mu(\tau^{KCM}) = \exp((\frac{1}{q})^{2\alpha+o(1)})$ when $q \to 0$.

Theorem (Hartarsky, M., 2022 + Martinelli, Morris, Toninelli, 2019 + Hartarsky, 2021)

Critical constraints with an infinite number of stable directions satisfy:

- if they are balanced, $\mathbb{E}_\mu(\tau^{KCM}) = \exp(\Theta((\frac{1}{q})^{2\alpha}))$ when $q \to 0$;
- if they are unbalanced, $\mathbb{E}_\mu(\tau^{KCM}) = \exp(\Theta((\frac{1}{q})^{2\alpha} \ln(\frac{1}{q})^4))$ when $q \to 0$.

Heuristics for Critical Bootstrap Percolation

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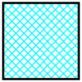
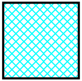


Heuristics for Critical Bootstrap Percolation



Balanced families:

$$\mu\left(\square\right) = \exp(-\Theta(1/q^{\alpha})).$$

$$\Downarrow$$

$$T^{BP} = 1/\mu\left(\square\right) = \exp(\Theta(1/q^{\alpha})).$$

Unbalanced families:

$$\mu\left(\square\right) = \exp(-\Theta((\tfrac{1}{q})^{\alpha}\ln(\tfrac{1}{q})^{2})).$$

$$\Downarrow$$

$$T^{BP} = 1/\mu\left(\square\right) = \exp(\Theta((\tfrac{1}{q})^{\alpha}\ln(\tfrac{1}{q})^{2})).$$

Heuristics for Critical KCMs with an Infinite Number of Stable Directions

East dynamics with  instead of .

$\mathbb{E}_\mu(\tau^{KCM}) = 1/\mu\,($  $)^{\ln(1/\mu(}$  $^{))}$ becomes $1/\mu\,($  $)^{\ln(1/\mu(}$  $^{))}$.

Heuristics for Critical KCMs with an Infinite Number of Stable Directions

East dynamics with $\boxtimes$ instead of $\blacksquare$.

$\mathbb{E}_\mu(\tau^{KCM}) = 1/\mu\left(\blacksquare\right)^{\ln(1/\mu(\blacksquare))}$ becomes $1/\mu\left(\boxtimes\right)^{\ln(1/\mu(\boxtimes))}$.

- Balanced families: $\mu\left(\boxtimes\right) = \exp(-\Theta(1/q^\alpha))$.
  $\Rightarrow \mathbb{E}_\mu(\tau^{KCM}) = \exp(\Theta(1/q^{2\alpha}))$.
- Unbalanced families: $\mu\left(\boxtimes\right) = \exp(-\Theta((\frac{1}{q})^\alpha \ln(\frac{1}{q})^2))$.
  $\Rightarrow \mathbb{E}_\mu(\tau^{KCM}) = \exp(\Theta((\frac{1}{q})^{2\alpha} \ln(\frac{1}{q})^4))$.

Universality for Critical KCMs with a Finite Number of Stable Directions, Unbalanced

Definition

A critical constraint is called:

- *rooted* if there exist two non-opposite directions of difficulty $> \alpha$,
- *unrooted* otherwise.

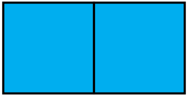
Theorem (Hartarsky, M., 2022 + Hartarsky, Martinelli, Toninelli, 2019 + Hartarsky, 2021)

Critical constraints with a finite number of stable directions and unbalanced satisfy:

- if they are unrooted, $\mathbb{E}_\mu(\tau^{KCM}) = \exp(\Theta((\frac{1}{q})^\alpha \ln(\frac{1}{q})^2))$ when $q \to 0$,
- if they are rooted, $\mathbb{E}_\mu(\tau^{KCM}) = \exp(\Theta((\frac{1}{q})^\alpha \ln(\frac{1}{q})^3))$ when $q \to 0$.

Heuristics for Critical KCMs with a Finite Number of Stable Directions, Unbalanced

Heuristics for Critical KCMs with a Finite Number of Stable Directions, Unbalanced



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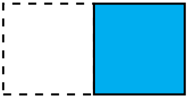

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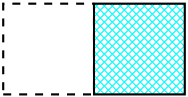

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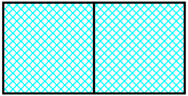

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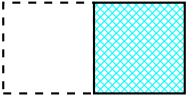

Heuristics for Critical KCMs with a Finite Number of Stable Directions, Unbalanced, Unrooted

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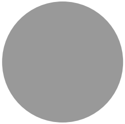

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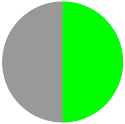

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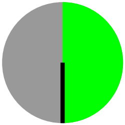

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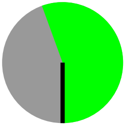

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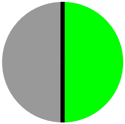

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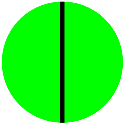

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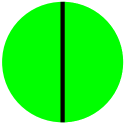

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$\Rightarrow$ Infection can propagate on the left as well as on the right.

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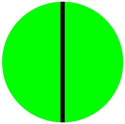

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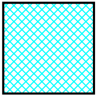

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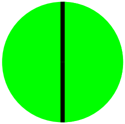

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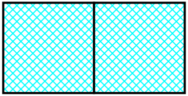

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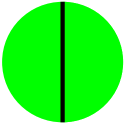

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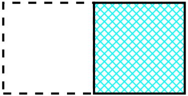

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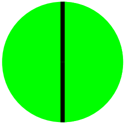

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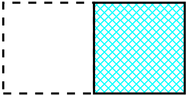

$\Rightarrow$ FA-1f mechanism with 
$\Rightarrow \mathbb{E}_\mu(\tau^{KCM}) = 1/\mu$ ()^2.

μ () $= \exp(-\Theta((\frac{1}{q})^\alpha \ln(\frac{1}{q})^2)) \Rightarrow \mathbb{E}_\mu(\tau^{KCM}) = \exp(\Theta((\frac{1}{q})^\alpha \ln(\frac{1}{q})^2))$.

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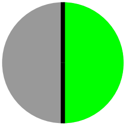

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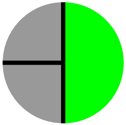

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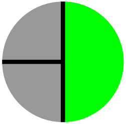

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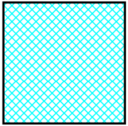
Mechanism to propagate the infection on the left:



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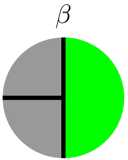

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Mechanism to propagate the infection on the left:

β infected sites

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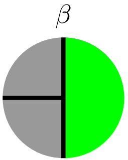

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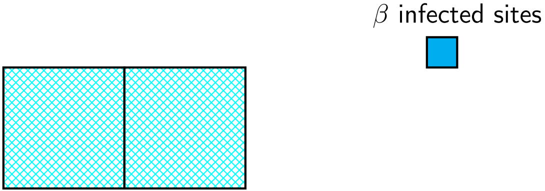

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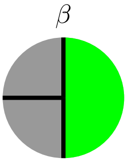

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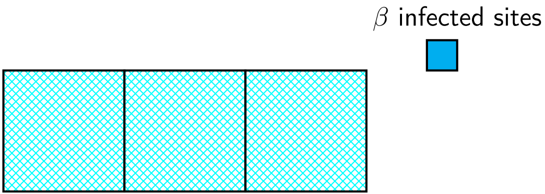

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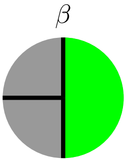

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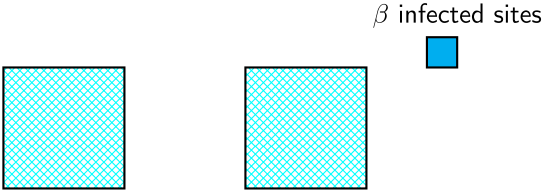

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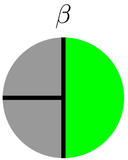

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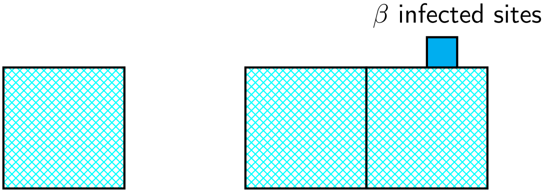

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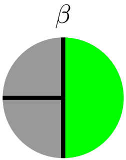

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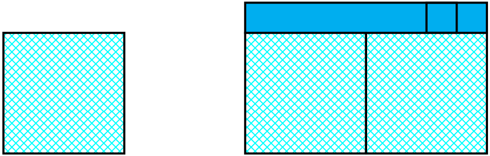

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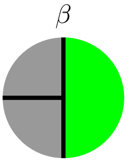

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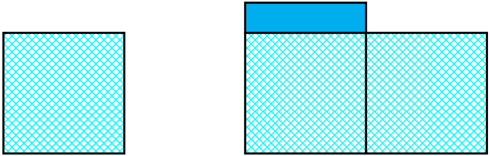

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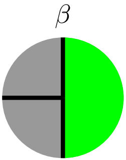

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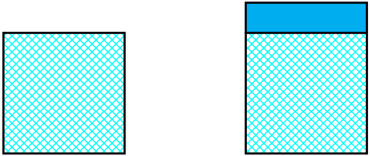

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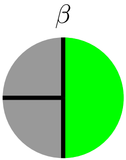

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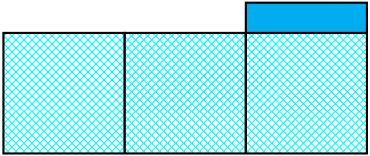

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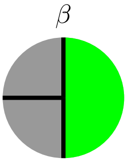

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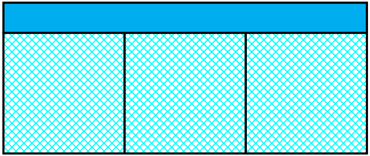

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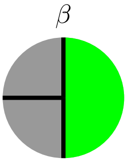

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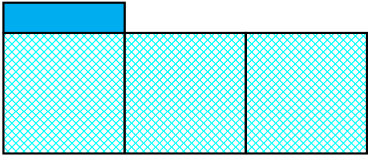

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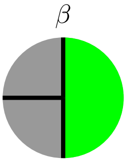

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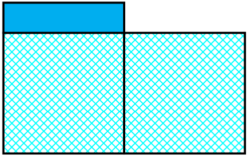

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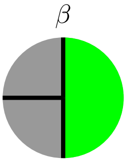

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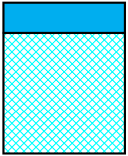
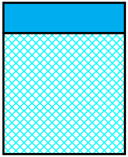


Heuristics for Critical KCMs with a Finite Number of Stable Directions, Unbalanced, Rooted

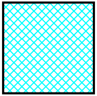
Mechanism to propagate the infection on the left:



Propagation to the right $\Rightarrow$ propagation to the top.
Propagation to the top $\Rightarrow$ propagation to the left.

Heuristics for Critical KCMs with a Finite Number of Stable Directions, Unbalanced, Rooted

Mechanism to propagate the infection on the left:



β infected sites

Propagation to the right $\Rightarrow$ propagation to the top.
Propagation to the top $\Rightarrow$ propagation to the left.

$\Rightarrow$ East dynamics to reach the β infected sites.
Distance to cross $= \Theta(1/q^{\beta})$.
$\Rightarrow$ The dynamics has to go through a configuration with at least n $\boxtimes$,
where $n = \Theta(\ln(1/q^{\beta})) = \Theta(\ln(1/q))$.

$\Rightarrow$ Energy barrier $\mu\left(\boxtimes\right)^{\Theta(\ln(1/q))}$.
$\mu\left(\boxtimes\right) = \exp(-\Theta((\frac{1}{q})^{\alpha}\ln(\frac{1}{q})^{2})) \Rightarrow \mathbb{E}_{\mu}(\tau^{KCM}) = \exp(\Theta((\frac{1}{q})^{\alpha}\ln(\frac{1}{q})^{3}))$.

Universality for Critical KCMs with a Finite Number of Stable Directions, Balanced

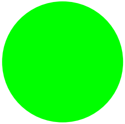
Theorem (Hartarsky, M., 2022 + Hartarsky, 2021)

Critical constraints with a finite number of stable directions and balanced can have:

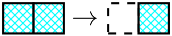
- no direction of difficulty $> \alpha$, then they are called *isotropic* and $\mathbb{E}_\mu(\tau^{KCM}) = \exp(\Theta((\frac{1}{q})^\alpha))$ when $q \to 0$;
- exactly one direction of difficulty $> \alpha$, then they are called *semi-directed* and $\mathbb{E}_\mu(\tau^{KCM}) = \exp(\Theta((\frac{1}{q})^\alpha \ln(\ln(\frac{1}{q}))))$ when $q \to 0$;
- at least two directions of difficulty $> \alpha$, then they are rooted and $\mathbb{E}_\mu(\tau^{KCM}) = \exp(\Theta((\frac{1}{q})^\alpha \ln(\frac{1}{q})))$ when $q \to 0$;

Heuristics for Critical KCMs with a Finite Number of Stable Directions, Balanced, Isotropic

Isotropic = no direction of difficulty $> \alpha$.



gray=unknown
green=difficulty $\leq \alpha$
black=difficulty $> \alpha$

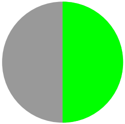
$\Rightarrow$ Semicircles on the right AND the left: directions with difficulty $\leq \alpha$.
$\Rightarrow$ Infection can propagate on the left as well as on the right.
$\Rightarrow$  straightforward.

$\Rightarrow$ FA-1f mechanism with .
$\Rightarrow \mathbb{E}_\mu(\tau^{KCM}) = 1/\mu$ ( $)^2$.

μ () $= \exp(-\Theta(1/q^\alpha)) \Rightarrow \mathbb{E}_\mu(\tau^{KCM}) = \exp(\Theta(1/q^\alpha))$.

Universality for Critical KCMs with a Finite Number of Stable Directions, Balanced, at Least 2 Directions with Difficulty $> \alpha$

Balanced $=$ $\exists$ a closed semicircle in which directions have difficulty $\leq \alpha$.
At least 2 directions of difficulty $> \alpha$.



gray=unknown
green=difficulty $\leq \alpha$
black=difficulty $> \alpha$

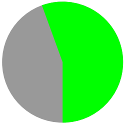
2 non-opposite directions with difficulty $> \alpha$
$\Rightarrow$ The model is rooted.

As in unbalanced rooted models, East dynamics on a scale $1/q^{\Theta(1)}$.

$\Rightarrow$ Energy barrier μ () $^{\Theta(\ln(1/q))}$.
μ () $= \exp(-\Theta(1/q^{\alpha})) \Rightarrow \mathbb{E}_{\mu}(\tau^{KCM}) = \exp(\Theta((\frac{1}{q})^{\alpha} \ln(\frac{1}{q})))$.

Universality for Critical KCMs with a Finite Number of Stable Directions, Balanced, at Least 2 Directions with Difficulty $> \alpha$

Balanced $= \exists$ a closed semicircle in which directions have difficulty $\leq \alpha$.
At least 2 directions of difficulty $> \alpha$.



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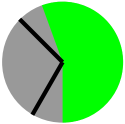
Universality for Critical KCMs with a Finite Number of Stable Directions, Balanced, at Least 2 Directions with Difficulty $> \alpha$

Balanced = $\exists$ a closed semicircle in which directions have difficulty $\leq \alpha$.
At least 2 directions of difficulty $> \alpha$.



gray=unknown
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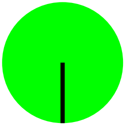
2 non-opposite directions with difficulty $> \alpha$
$\Rightarrow$ The model is rooted.

As in unbalanced rooted models, East dynamics on a scale $1/q^{\Theta(1)}$.

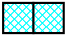
$\Rightarrow$ Energy barrier $\mu\left(\square\right)^{\Theta(\ln(1/q))}$.
$\mu\left(\square\right) = \exp(-\Theta(1/q^{\alpha})) \Rightarrow \mathbb{E}_{\mu}(\tau^{KCM}) = \exp(\Theta((\frac{1}{q})^{\alpha} \ln(\frac{1}{q})))$.

Heuristics for Critical KCMs with a Finite Number of Stable Directions, Balanced, Semi-directed

Semi-directed = exactly 1 direction with difficulty $> \alpha$.



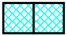
gray=unknown
green=difficulty $\leq \alpha$
black=difficulty $> \alpha$

 $\rightarrow$  requires to go through  which has probability $\exp(-\Theta((\frac{1}{q})^{\alpha} \ln(\ln(\frac{1}{q}))))$.

$\Rightarrow \mathbb{E}_{\mu}(\tau^{KCM}) = \exp(\Theta((\frac{1}{q})^{\alpha} \ln(\ln(\frac{1}{q}))))$.

Conclusion

Energy barriers at different scales:

- Semi-directed constraints: passage through .
  $\Rightarrow$ Inside a .
- Rooted constraints:  $\rightarrow$  requires the creation of auxiliary .
  $\Rightarrow$ Around a .
- Constraints with an infinite number of stable directions: global East mechanism for .
  $\Rightarrow$ Global scale.

$\Rightarrow$ Much more complex behavior than in bootstrap percolation.

Open questions

Universality in higher dimension?

Done for bootstrap percolation, open for KCMs...

Thanks for your attention.