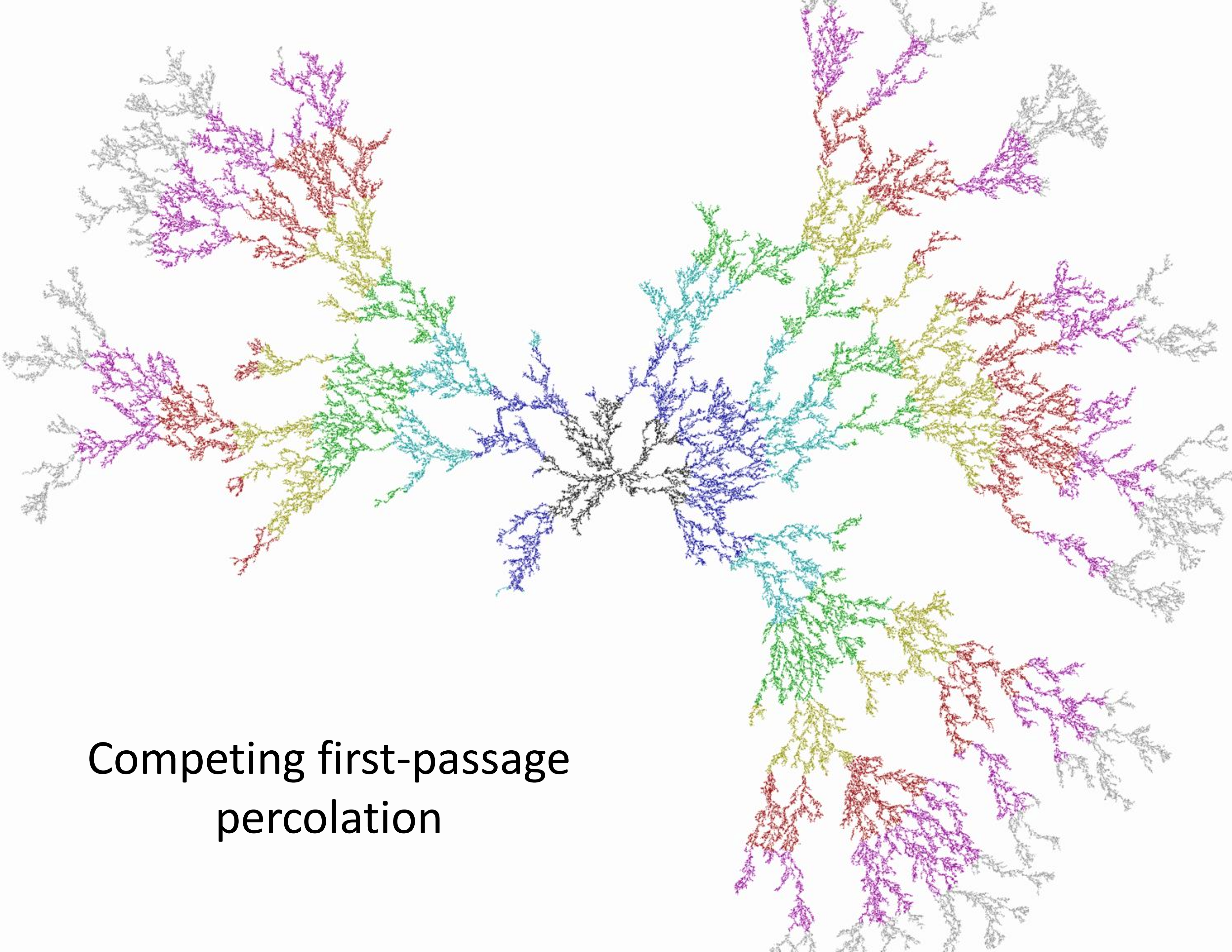
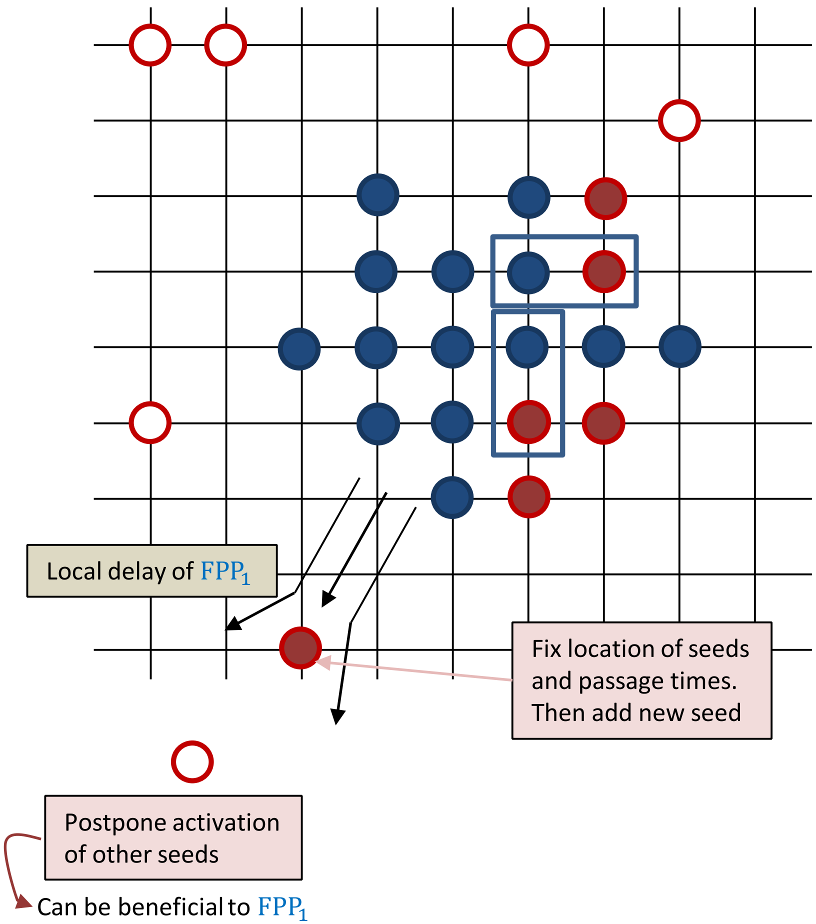


Competing first-passage percolation

The model

λ = *rate of* FPP$_\lambda$
p = *density of* FPP$_\lambda$ *seeds*



FPP$_1$ starts from the origin

- Perform FPP at rate 1

FPP$_\lambda$ starts from seeds of i.i.d. Bern(p), which

- Do not evolve from time 0
- get *activated* when FPP$_1$ or FPP$_\lambda$ try to occupy it
- After activation, evolve as FPP at rate λ

Main questions

- Which type survives (produces an infinite cluster)?
  (FPP$_\lambda$ is always an infinite set)
- Is there coexistence?

Focus on case $p < 1 - p_c^{\text{site}}$ (i.e., $1 - p > p_c^{\text{site}}$)

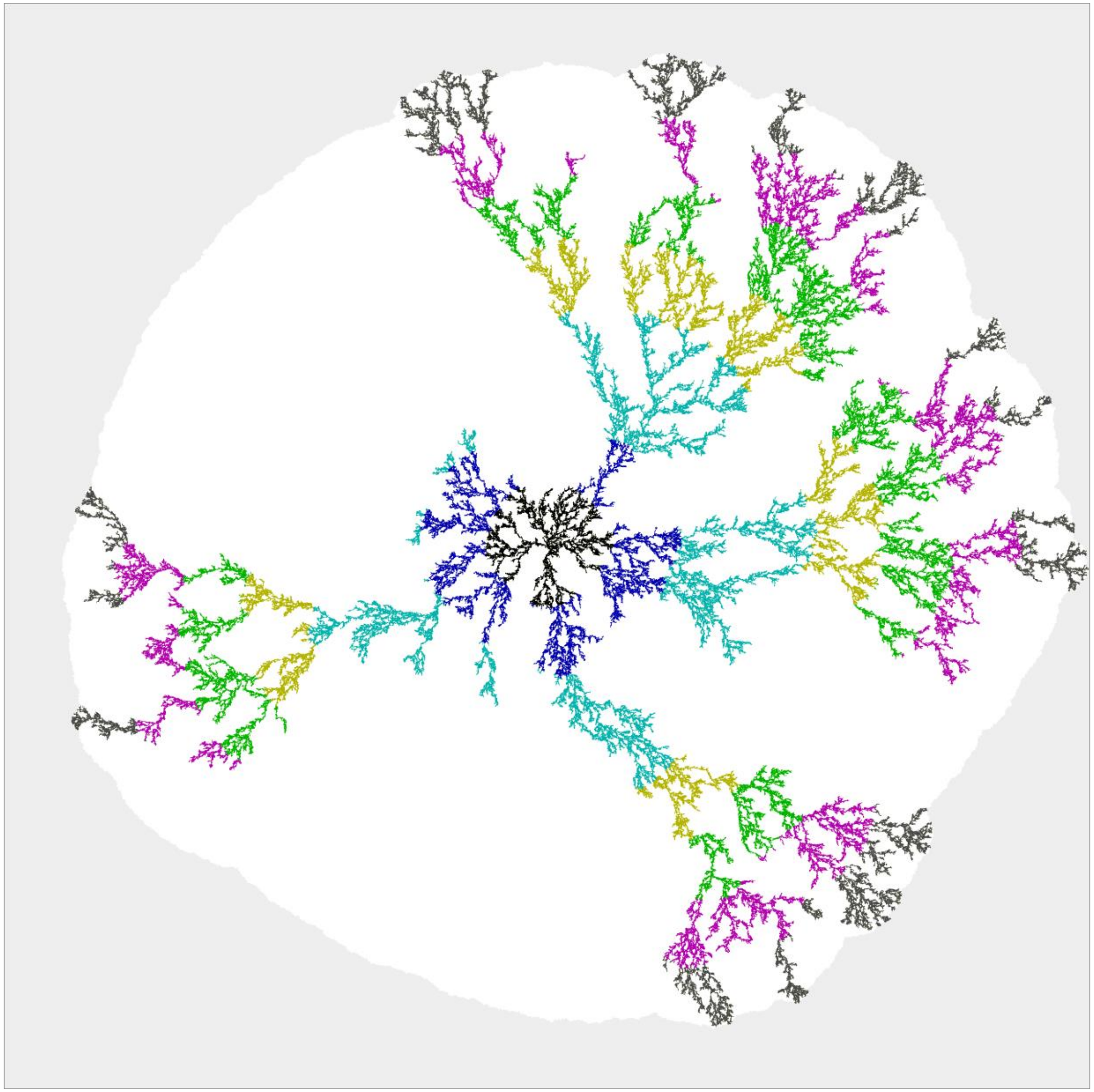
Theorem [Candellero, S. 21]
There exist graphs for which $\mathbb{P}$(FPP$_1$ survives) is not a monotone function of p and λ

- *multiple phase transitions*
- *Quasi-transitive graphs*

Monotonicity?
Is increasing p or λ beneficial to FPP$_\lambda$*?*

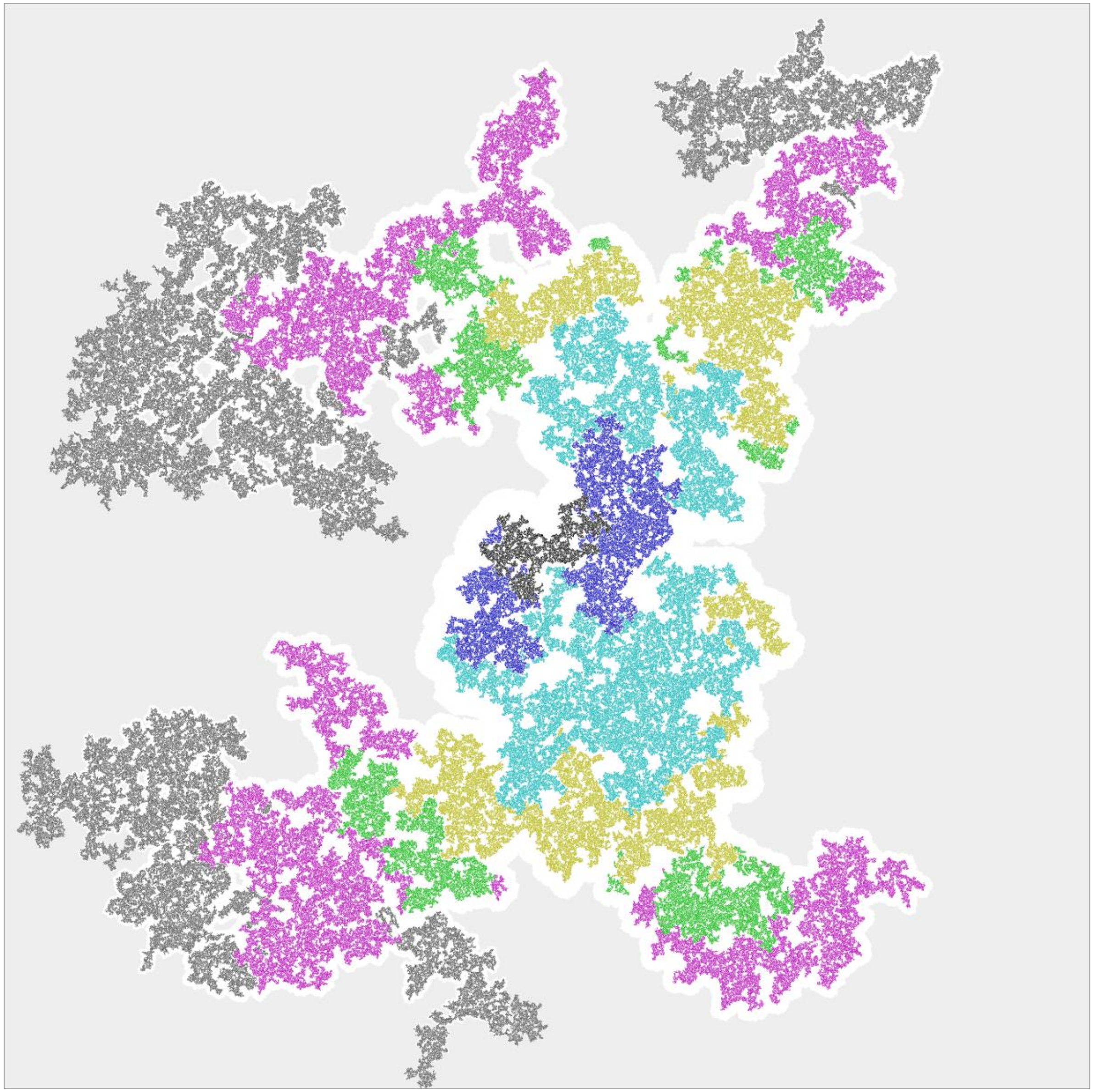
Simulation: $p = 0.03$ and $\lambda = 0.7$

λ = *rate of* FPP_λ
p = *density of* FPP_λ *seeds*



Simulation: $p = 0.4$ and $\lambda = 0.008$

λ = *rate of* FPP_λ
p = *density of* FPP_λ *seeds*



First Result

λ = *rate of* FPP_λ
p = *density of* FPP_λ *seeds*

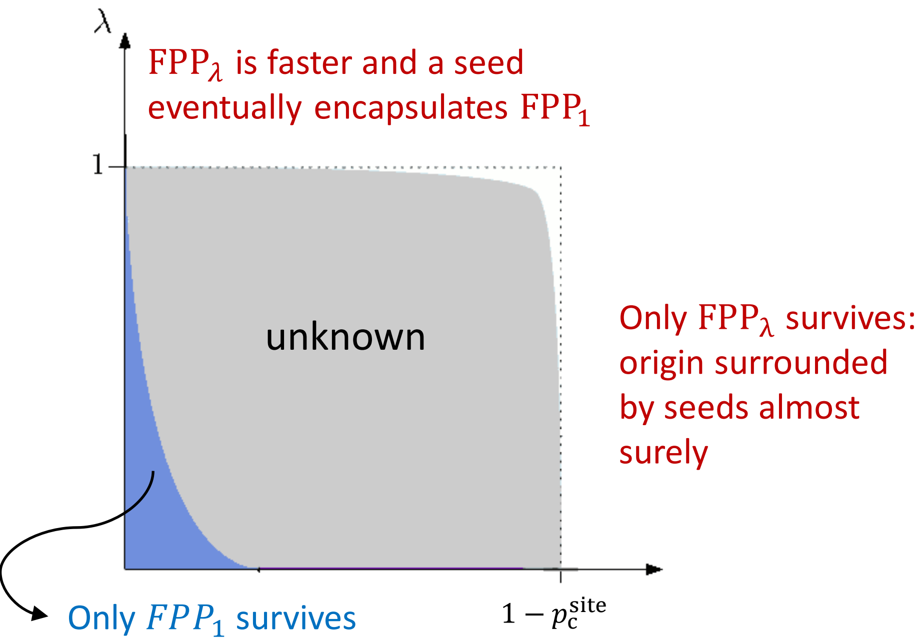
Theorem 1 [Sidoravicius, S. 2019]

For any $\lambda < 1$, there exists $p_0 \in (0,1)$ such that $\forall p < p_0$

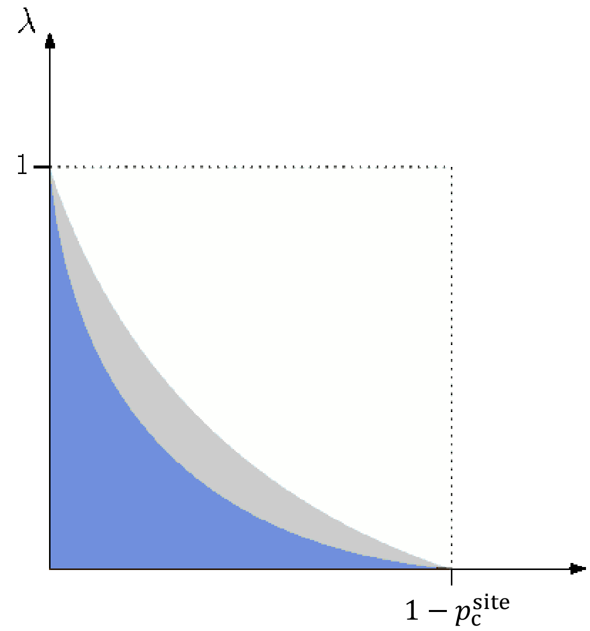
1. $\mathbb{P}(\mathrm{FPP}_1 \text{ survives}) > 0$
2. $\mathbb{P}(\mathrm{FPP}_1 \text{ survives and } \mathrm{FPP}_\lambda \text{ dies out}) > 0$
3. $\mathbb{P}(\forall t \geq 0,\ \overline{\mathrm{FPP}_1}(t) \supset \mathrm{Ball}(ct)) > 0,$

where $\overline{\mathrm{FPP}_1} = \mathrm{FPP}_1 \cup \text{finite components of } \mathrm{FPP}_1^c$

Known behavior:



Expected behavior:



New Result

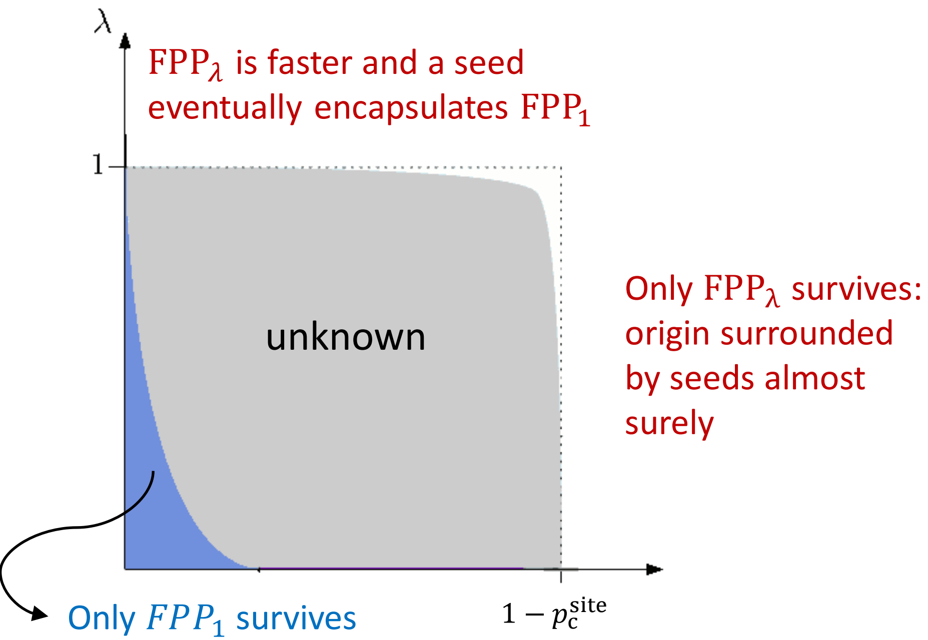
λ = *rate of* FPP_λ
p = *density of* FPP_λ *seeds*

Theorem 2 [Finn, S.]

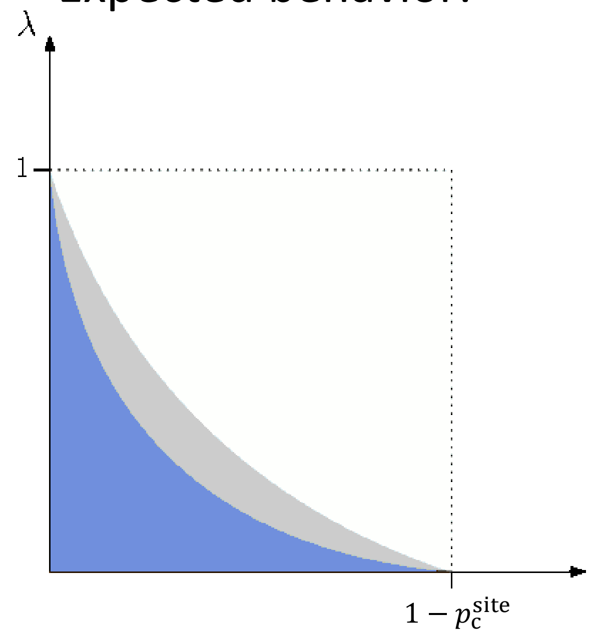
For any $p < 1 - p_c^{\text{site}}$, there exists $\lambda_0 > 0$ such that $\forall \lambda < \lambda_0$

$$\mathbb{P}(\mathrm{FPP}_1 \text{ survives}) > 0$$

Known behavior:



Expected behavior:



Theorem 2 [Finn, S.]

$\forall p < 1 - p_c^{\text{site}}, \exists \lambda_0 > 0$ s.t. $\forall \lambda < \lambda_0$

$$\mathbb{P}(\text{FPP}_1 \text{ survives}) > 0$$

Theorem 1 [Sidoravicius, S. 2019]

$\forall \lambda < 1, \exists p_0$ s.t. $\forall p < p_0$

$$\mathbb{P}(\text{FPP}_1 \text{ survives}) > 0$$

Remark:

$p_c^{\text{site}} < \frac{1}{2} < 1 - p_c^{\text{site}}$ when $d \geq 3$

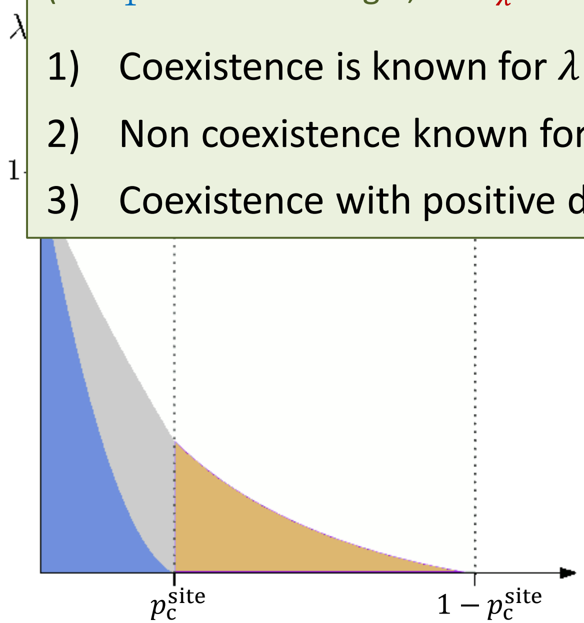
Theorem 2 gives coexistence when $d \geq 3$ and $p \in (p_c^{\text{site}}, 1 - p_c^{\text{site}})$

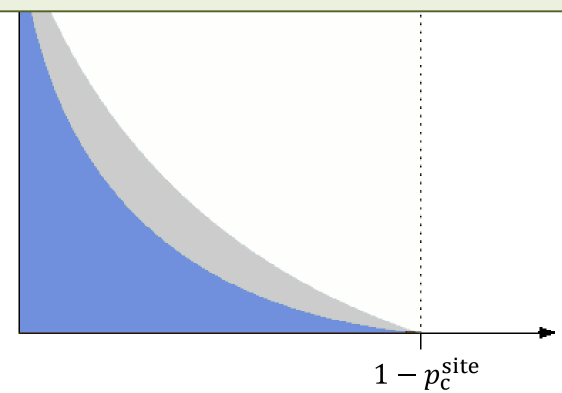
Strong coexistence: Both types produce a positive density

Compare with two type Richardson model:

(FPP_1 starts from origin, FPP_λ starts from neighbor of origin, no seed)

1) Coexistence is known for $\lambda = 1$ (Conjecture: coexistence occurs iff $\lambda = 1$)
2) Non coexistence known for all but countably many λ
3) Coexistence with positive density for both types is impossible

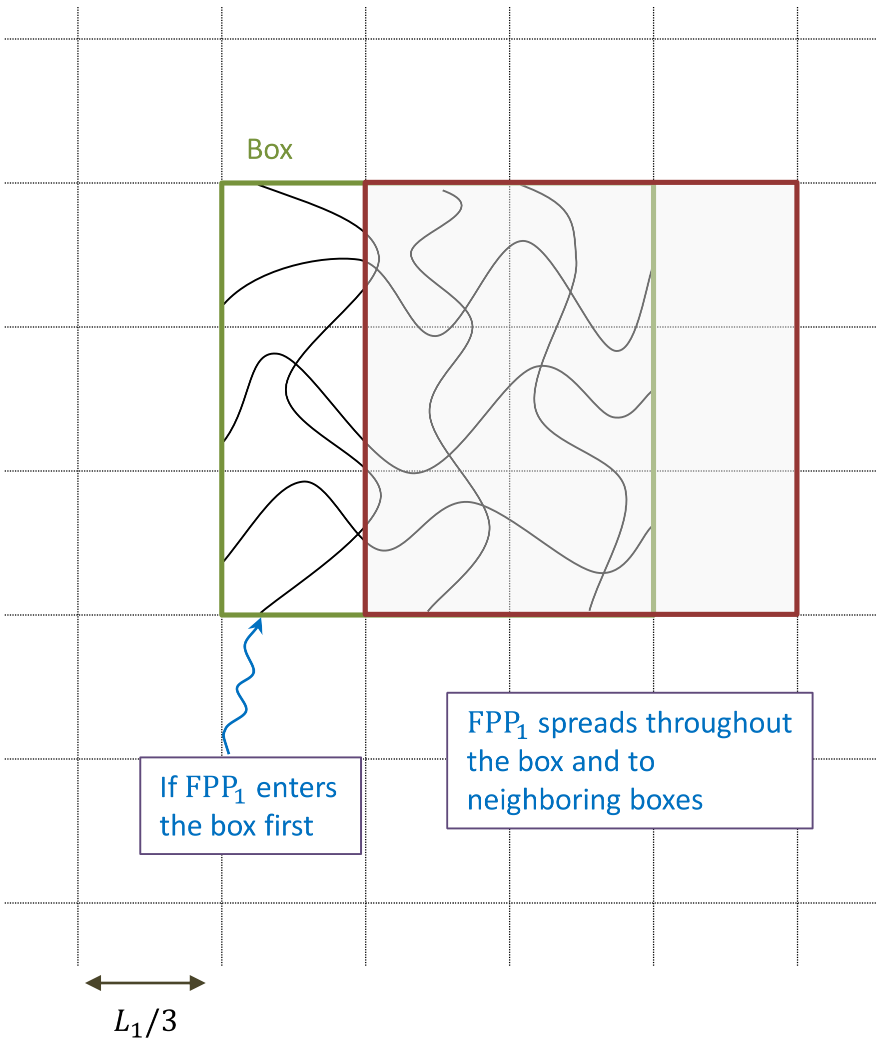




The proof:
multi-scale analysis with non-equilibrium feedback [Finn, S.]

Theorem 2 [Finn, S.]

$$\forall p < 1 - p_c^{\text{site}}, \exists \lambda_0 > 0 \text{ s.t. } \forall \lambda < \lambda_0$$
$$\mathbb{P}(\text{FPP}_1 \text{ survives}) > 0$$



Tessellate $\mathbb{Z}^d$ into cubes of side length $L_1/3$
Boxes are cubes of side length L_1

A box is called **good** if:

i. Non-seeds form a cluster of order L_1^d
ii. 2nd largest cluster of non-seeds is $O(\log^2 L_1)$
iii. Chemical distance (distance after removing seeds) has same order as ℓ_1-distance
iv. FPP_1-distance (distance weighed by passage times) between non-seeds has same order as ℓ_1-distance
v. Passage time of FPP_λ through each edge $\geq \sqrt{\frac{1}{\lambda}}$

For λ small, a good box implies that

- If FPP_1 enters the box first, FPP_1 spreads through the entire box while FPP_λ does not occupy any non-seed

Non-local event although definition of good is a local event

The proof:
multi-scale analysis with non-equilibrium feedback [Finn, S.]

Theorem 2 [Finn, S.]

$$\forall p < 1 - p_c^{\text{site}}, \exists \lambda_0 > 0 \text{ s.t. } \forall \lambda < \lambda_0$$

$$\mathbb{P}(\text{FPP}_1 \text{ survives}) > 0$$

Further classify good boxes into positive or negative feedback
Bad boxes are not further classified

Defn: a box of scale 1 is of positive feedback if FPP_1 occupies a site far from the boundary in time $\leq rL_1$ from the entrance time of the box

Note that positive feedback is not a local event

- × We will not estimate $\mathbb{P}(\text{box has positive feedback})$
- ✓ We will just assess the consequence of finding a box of negative feedback

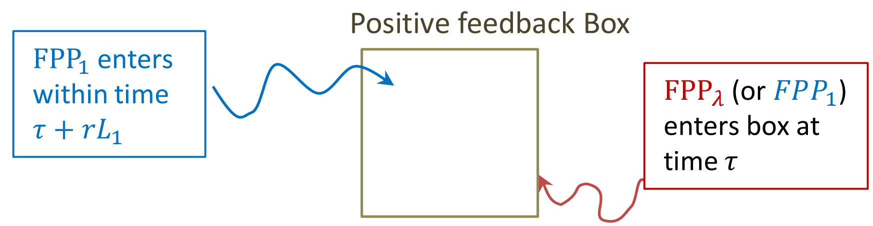
This is why we call it a *feedback:*

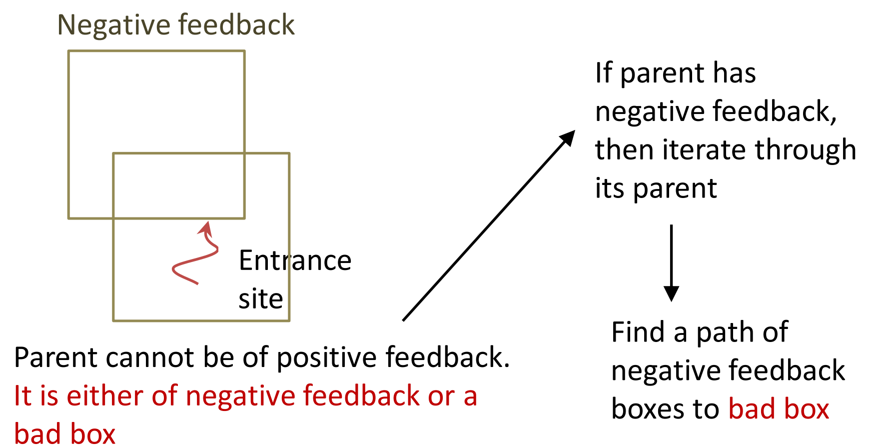
- we regard it as an information that is given to us.

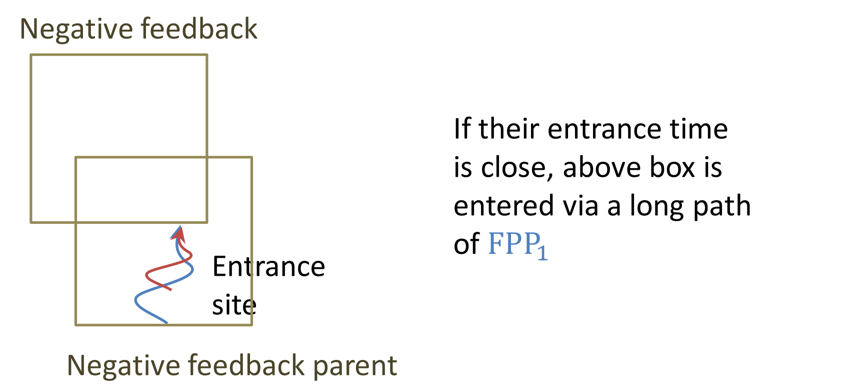
Property 1: positive feedback box spread FPP_1 quickly to neighboring boxes

Main property: a negative feedback box has a parent box that is bad or of negative feedback

Property 3: negative feedback box with negative feedback parent has delayed entrance time





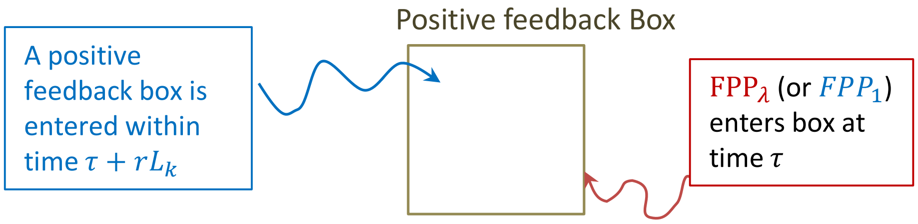


The proof:
multi-scale analysis with non-equilibrium feedback [Finn, S.]

Theorem 2 [Finn, S.]
$\forall p < 1 - p_c^{\text{site}}, \exists \lambda_0 > 0$ s.t. $\forall \lambda < \lambda_0$
$$\mathbb{P}(\text{FPP}_1 \text{ survives}) > 0$$

Higher scale

Defn: a box of scale k is of positive feedback if it contains a (k-1)-box of positive feedback far from its boundary that is entered in time $\leq rL_k$ from the entrance time of the box



If *k*-box is good (i.e., has few bad *(k-1)*-boxes), same three properties from scale 1 hold in scale k:

- A positive feedback *k*-box is mostly occupied by positive feedback (k-1)-boxes
- A negative feedback k-box has a parent of negative feedback or bad
- Negative feedback k-box with negative feedback parent has delayed entrance time

The proof:
multi-scale analysis with non-equilibrium feedback [Finn, S.]

Theorem 2 [Finn, S.]

$$\forall p < 1 - p_c^{\text{site}}, \exists \lambda_0 > 0 \text{ s.t. } \forall \lambda < \lambda_0$$
$$\mathbb{P}(\mathrm{FPP}_1 \text{ survives}) > 0$$

How to apply this analysis to other models?

Proof is divided into two parts:

I. A standard multi-scale analysis controlling good/bad boxes (which are local events only)
II. Introduction of non-local events which we call positive/negative feedback

Positive/negative feedback have the following goals:

- Positive feedback spreads fast to neighbors
- Negative feedback has a delayed spread to another negative feedback box.
- Negative feedback box can be associated to a nearby bad box (controlled via standard multi-scale).

➢ Positive feedback forms a strongly supercritical percolation

FPPHE used to analyse other models

1) MDLA [Sidoravicius, S.'19]
2) SI with different rates [Dauvergne Sly'22+]
3) SIR [Dauvergne Sly]