Mathematisches Institut LMU München
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# ADVANCED ANALYSIS
## Exercise sheet 1 – 03.11.2022

We denote by $(\Omega, \Sigma, \mu)$ a measure space.

**Ex.1.1** (Good notation)
Let $1 \leq p < \infty$ and let $f \in L^p \cap L^\infty$. Show that

- $f \in L^q$ for any $p \leq q \leq \infty$
- $\lim_{q\to\infty} \|f\|_q = \|f\|_\infty$.

**Ex.1.2** (Convex functions have subgradients (in $1d$))
Let $I \subset \mathbb{R}$ be an open interval and $f : I \to \mathbb{R}$ and convex functions.

1. Show that the function $\tau$ defined for all $x \neq y$ by
$$\tau(x, y) = \frac{f(x) - f(y)}{x - y}$$
is non-decreasing in both its variables.

2. Show that $f$ has a subderivative at each point of $I$, i.e. for any $x \in I$, there is some $V_x \in \mathbb{R}$ such that for all $y \in I$
$$f(y) \geq f(x) + V_x(y - x).$$

3. Show the above property in the case where $f$ is $C^2$ (we take for granted that, since $f$ is convex, it satisfies $f'' \geq 0$).

**Ex.1.3** (Gas of fermions (electrons))
Let $N \geq 1$, we say that $m \in \mathcal{S}$ if $m \in L^1(\mathbb{R}^3 \times \mathbb{R}^3)$ and satisfies

- $0 \leq m(x, p) \leq 1$ for a.e. $x, p \in \mathbb{R}^3$ (Pauli exclusion principle)
- (density of $N$ electrons in phase space)
$$\frac{1}{(2\pi)^3} \int_{\mathbb{R}^3\times\mathbb{R}^3} m(x, p)\mathrm{d}x\mathrm{d}p = N \tag{1}$$

For $Z \geq 1$ (atomic number), define the Vlasov energy of the gas described by the distribution $m$ by
$$\mathcal{E}_{\text{Vlasov}}(m) = \iint_{\mathbb{R}^3\times\mathbb{R}^3} \left(p^2 - \frac{Z}{|x|}\right) m(x, p)\mathrm{d}x\mathrm{d}p + \int_{\mathbb{R}^3} \frac{\rho_m(x)\rho_m(y)}{|x - y|}\mathrm{d}x\mathrm{d}y,$$
where $\rho_m$ is the position density associated to $m$ and is defined by
$$\rho_m(x) = \frac{1}{(2\pi)^3} \int m(x, p)\mathrm{d}p.$$

Using the bathtub principle (Theorem 1.14 of the Lieb-Loss), show that

$$\inf_{m\in\mathcal{S}}\mathcal{E}_{\text{Vlasov}}(m)=\inf_{\substack{0\le\rho\in L^1(\mathbb{R}^3)\\ \int_{\mathbb{R}^3}\rho=N}}\mathcal{E}_{\text{TF}}(\rho)$$

where the Thomas-Fermi energy is defined by

$$\mathcal{E}_{\text{TF}}(\rho)=C_{\text{TF}}\int_{\mathbb{R}^3}\rho^{5/3}(x)-Z\int_{\mathbb{R}^3}\frac{\rho(x)}{|x|}+\iint_{\mathbb{R}^3\times\mathbb{R}^3}\frac{\rho(x)\rho(y)}{|x-y|}\mathrm{d}x\mathrm{d}y.$$

*Hint:* One could use that

$$\inf_{m\in\mathcal{S}}\mathcal{E}_{\text{Vlasov}}(m)=\inf_{\substack{0\le\rho\in L^1(\mathbb{R}^3)\\ \int_{\mathbb{R}^3}\rho=N}}\;\inf_{\substack{m\in\mathcal{S}\\ \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}m(\cdot,p)\mathrm{d}p=\rho}}\mathcal{E}_{\text{Vlasov}}(m),$$

where by $\frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}m(\cdot,p)\mathrm{d}p=\rho$, it is meant

$$\frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}m(x,p)\mathrm{d}p=\rho(x)$$

for almost every $x\in\mathbb{R}^3$.

*NB: The Thomas-Fermi energy is a model for the energy of electrons in atoms and molecules. It is a theory that depends only on the spatial density of the electrons (this is very surprising !). Here, we see that it is obtained as a special case of the Vlasov theory that models electrons by a density measure on phase space $\mathbb{R}^3\times\mathbb{R}^3$ (space $\times$ momentum). It is gives some qualitative properties of such systems but not always (atoms don't bind in TF, so there is no molecule, sad...). Physicists now use other models but mathematicians love playing with it and its extensions (other terms can be added to try to make it more accurate).*

**Ex.1.4** (Some other counter examples)
Consider the following sequences in $L^p(\mathbb{R})$, with $1<p<\infty$:

1. $f_k(x):=\begin{cases}\sin kx & \text{for } 0\le x\le 1,\\ 0 & \text{otherwise.}\end{cases}$
2. $g_k(x):=k^{\frac{1}{p}}g(kx)$, where $g$ is any fixed function in $L^p(\mathbb{R})$.
3. $h_k(x):=g(x+k)$ for some fixed function $g$ in $L^p(\mathbb{R})$.

Prove that $f_k$, $g_k$, $h_k$ converge weakly to 0 but do not converge strongly to 0 (or to anything else).

**Ex.1.5** (Projection on convex sets, Lemma 2.8 of the Lieb-Loss)
Let $1<p<\infty$ and let $K\subset L^p(\Omega)$ be convex and norm closed set (i.e. if $(f_n)\in K$ is a sequence such that $\|f_n-f\|_p\to 0$ then $f\in K$). Let $f\in L^p(\Omega)\setminus K$ and define the distance to $K$ as

$$D=\mathrm{dis}(f,K)=\inf_{g\in K}\|f-g\|_p. \tag{2}$$

We wan to show that there is a function $h\in K$ such that the distance is attained, i.e.

$$D=\|h-f\|_p.$$

We will assume the Hanner inequality (2.5 in LL).

**Lemma 1** *(Hanner's inequality, more useful than it looks) For any $f, g \in L^p$ and $1 \le p \le 2$, then*

$$\|f+g\|_p^p + \|f-g\|_p^p \ge (\|f\|_p + \|g\|_p)^p + \big|\|f\|_p - \|g\|_p\big|^p \tag{3}$$

$$\big(\|f+g\|_p + \|f-g\|_p\big)^p + \big|\|f+g\|_p - \|f-g\|_p\big|^p \le 2^p(\|f\|_p^p + \|g\|_p^p). \tag{4}$$

*If $2 \le p \le \infty$, the inequalities are reversed.*

1. Assume $1 \le p \le 2$.
   - (a) Explain why without loss of generality, we can assume $f = 0$.
   - (b) Let $(g_n)$ be a minimizing sequence for (2), show that $\|g_n + g_m\|_p \to D$ when $n, m \to \infty$.
   - (c) Using (4), prove that $(g_n)$ is a Cauchy-sequence, you can use a contradiction argument.
   - (d) Conclude.
2. Prove the case $2 \le p \le \infty$.