Mathematisches Institut LMU München
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ADVANCED ANALYSIS
Exercise sheet 11 – 26.01.2023

Ex.1.1 (Hardy inequality)
Let $n \geq 3$. We want to prove that there is some constant C_n such that for $f \in H^1(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} |f(x)|^2 \frac{1}{|x|^2} \mathrm{d}x \leq C_n \int_{\mathbb{R}^n} |\nabla f(x)|^2 \mathrm{d}x. \tag{1}$$

1. Recall the Hardy-Littlewood-Sobolev inequality.
2. Let $f, g \in C_c^\infty(\mathbb{R}^n)$, for $\lambda > 0$ denote $g_\lambda(x) = \lambda^{-n} g(\lambda^{-1}x)$. Applying the HLS inequality to f, g_λ and $|x|^{-2}$ and letting $\lambda \to 0$ show that
$$\int_{\mathbb{R}^n} |f(x)| \frac{1}{|x|^2} \mathrm{d}x \leq C_{HLS} \|f\|_{L^{4n/(n-2)}}.$$
3. Using the Sobolev inequality, show that (1) holds for $f \in C_c^\infty$.
4. Using the density of C_c^∞ in $H^1(\mathbb{R}^n)$, show that (1) holds for $f \in H^1(\mathbb{R}^n)$.

Ex.1.2 (Particle in a well)
Let us define

$$\mathcal{S} = \{f \in H^1(\mathbb{R}^3), \text{ such that } \int_{\mathbb{R}^3} |f|^2 |x|^2 < \infty \text{ and } \|f\|_{L^2} = 1\},$$

$$\mathcal{E}(f) = \int_{\mathbb{R}^3} |\nabla f(x)|^2 \mathrm{d}x + \int_{\mathbb{R}^3} |f(x)|^2 |x|^2 \mathrm{d}x$$

and

$$E_0 = \inf \{\mathcal{E}(f), \quad f \in \mathcal{S}\}.$$

1. Let $\{f_j\} \subset \mathcal{S}$ such that $\mathcal{E}(f_j) \to E_0$ when $j \to \infty$. Show that
$$\sup_j \|f_j\|_{H^1(\mathbb{R}^3)} < \infty.$$
2. Deduce from it that, up to a subsequence, $\{f_j\}$ converges weakly and almost everywhere to some $\varphi \in H^1$.
3. From the Fatou lemma, deduce that
$$\int_{\mathbb{R}^3} |\varphi|^2 |x|^2 \leq 1 \text{ and } \int_{\mathbb{R}^3} |\varphi|^2 |x|^2 < \infty.$$

4. For $A > 0$, show that

$$\int_{B(0,A)^c} |f_j(x)|^2 \leq \frac{1}{A^2}\mathcal{E}(f_j).$$

5. Deduce from this and from Theorem 8.6 that for some constant $C > 0$ and for all $A > 0$

$$\|\varphi\|_{L^2} \geq 1 - \frac{C}{A^2}.$$

6. Conclude that $\|\varphi\|_{L^2} = 1$ and that φ is a minimizer of $\mathcal{E}$ on $\mathcal{S}$, that is $\varphi \in \mathcal{S}$ and for all $f \in \mathcal{S}$, $\mathcal{E}(f) \geq \mathcal{E}(\varphi)$.