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ADVANCED ANALYSIS
Exercise sheet 13 – 7-8-9.02.2023

Ex.1 (Case $n = 1$) Let $V \in L^1(\mathbb{R}) + L^\infty(\mathbb{R})$ and $\psi \in H^1(\mathbb{R})$, define

$$\mathcal{E}(\psi) = \int_{\mathbb{R}} |\nabla \psi|^2 + \int_{\mathbb{R}} V|\psi|^2$$

and

$$E_0 = \inf \left\{ \mathcal{E}(\psi) \middle| \psi \in H^1(\mathbb{R}) \quad \|\psi\|_{L^2(\mathbb{R})} = 1 \right\}. \tag{1}$$

1. *Stability*
   (a) Recall the Sobolev inequality for $H^1(\mathbb{R})$ and justify that $V|\psi|^2 \in L^1(\mathbb{R})$ for $\psi \in H^1(\mathbb{R})$ and $V \in L^1(\mathbb{R}) + L^\infty(\mathbb{R})$.
   (b) Prove that $E_0 > -\infty$.
2. *Weak continuity of the potential.*
   (a) Assume that for all $\eta > 0$

   $$|\{x : \quad |V(x)| > \eta\}| < \infty$$

   Show that if $\psi_j \rightharpoonup \psi_0$ in $H^1(\mathbb{R})$ then

   $$\int V|\psi_j|^2 \underset{j\to\infty}{\longrightarrow} \int V|\psi|^2$$

   (b) *Existence of minimizers.* Assume that $E_0 < 0$.
      i. Show that there exists $\psi_0 \in H^1(\mathbb{R})$ such that $\|\psi_0\|_2 = 1$ and $\mathcal{E}(\psi_0) = E_0$.
      ii. Show that ψ_0 satisfies

      $$-\Delta\Psi_0 + V\Psi_0 = E_0\Psi_0 \quad \text{ in } D'(\mathbb{R}).$$

Ex.2 (Case $n = 2$)
Let $\varepsilon > 0$, $V \in L^{1+\varepsilon}(\mathbb{R}^2) + L^\infty(\mathbb{R}^2)$ and $\psi \in H^1(\mathbb{R}^2)$, define

$$\mathcal{E}(\psi) = \int_{\mathbb{R}} |\nabla \psi|^2 + \int_{\mathbb{R}^2} V|\psi|^2$$

and

$$E_0 = \inf \left\{ \mathcal{E}(\psi) \middle| \psi \in H^1(\mathbb{R}^2), \quad \|\psi\|_{L^2(\mathbb{R})^2} = 1 \right\}.$$

1. *Stability.*

   (a) Recall the Sobolev inequality for $H^1(\mathbb{R}^2)$ and justify that $V|\psi|^2 \in L^1(\mathbb{R}^2)$ for $\psi \in H^1(\mathbb{R}^2)$ and $V \in L^{1+\varepsilon}(\mathbb{R}^2) + L^\infty(\mathbb{R}^2)$ for all $\varepsilon > 0$.

   (b) Prove that $E_0 > -\infty$.

2. *Weak continuity of the potential.*

   (a) Assume that for all $\eta > 0$

   $$|\{x : \quad |V(x)| > \eta\}| < \infty$$

   Show that if $\psi_j \rightharpoonup \psi_0$ in $H^1(\mathbb{R}^2)$ then

   $$\int V|\psi_j|^2 \underset{j\to\infty}{\longrightarrow} \int V|\psi|^2$$

Ex.3 (Regularity of eigenfunctions) Let $n \geq 1$ and assume that $V \in C^\infty(\mathbb{R}^n)$. Let $E \in \mathbb{R}$ and $\psi \in L^2(\mathbb{R}^n)$ such that

$$-\Delta\psi + V\Psi = E\psi, \qquad \text{in } D'(\mathbb{R}^n).$$

1. Show that for any $R > 0$, $\Psi \in W^{2,2}(B_R)$.

2. Show that $\psi \in C^\infty(\mathbb{R}^n)$.

Ex.4 (The hydrogen atom)
For $\Psi \in H^1(\mathbb{R}^3)$, define

$$\mathcal{E}(\Psi) = \int_{\mathbb{R}^3} |\nabla\psi|^2 - \int_{\mathbb{R}^3} \frac{1}{|x|}|\psi(x)|^2$$

and define E_0 as in (1).

1. Let $\Psi_{gs}(x) = e^{-\frac{1}{2}|x|}$. Compute $\mathcal{E}(\Psi_{gs})$.

2. Show that there exists $\psi_0 \in H^1(\mathbb{R}^3)$ such that $\mathcal{E}(\psi_0) = E_0$ and $\|\psi_0\|_{L^2} = 1$.

3. Show that ψ_0 is C^∞ on $\mathbb{R}^3 \setminus \{0\}$ and C^0 on $\mathbb{R}^3$.

4. Denoting $\eta = \psi_0\Psi_{gs}^{-1}$, show that

   $$\mathcal{E}(\psi_0) = \int_{\mathbb{R}^3} |\Psi_{gs}|^2|\nabla\eta|^2 - \frac{1}{4}.$$

5. Deduce that there exists $c \in \mathbb{C}$ such that $\psi_0 = c\Psi_{gs}$.

Ex.4 (The non-linear Schrödinger equation)
Let $a \in \mathbb{R}$ and define

$$\mathcal{E}_{\mathrm{nls}}(\psi) = \int_{\mathbb{R}^3} |\nabla\psi(x)|^2 + \int_{\mathbb{R}^3} |x|^2 |\psi(x)|^2 + \frac{a}{2}\int_{\mathbb{R}^3} |\psi(x)|^4$$

and

$$E_0 = \inf\left\{ \mathcal{E}_{\mathrm{nls}}(\psi) \middle| \psi \in H^1(\mathbb{R}^n), \quad \|\psi\|_{L^2(\mathbb{R})} = 1 \right\}. \tag{2}$$

1. *Repulsive case.* Assume $a > 0$.

   (a) Show that there exists ψ_0 that solves the minimization problem (2).

   (b) Show that

   $$-\Delta\psi_0 + |x|^2\psi_0 + |\psi_0|^2\psi_0 = \mu\psi_0, \qquad \text{in } D'(\mathbb{R}^n)$$

   for some $\mu \in \mathbb{R}$ to be determined.

2. *Attractive case.* Assume $a < 0$.

   (a) Show that $E_0 = -\infty$. *Hint:* one could consider $\psi_\lambda = \lambda^{3/2}\psi(\lambda\cdot)$ for $\lambda > 0$ and some $\psi \in H^1(\mathbb{R}^3)$.