Mathematisches Institut LMU München
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ADVANCED ANALYSIS
Exercise sheet 2 – 10.11.2022

We denote by (Ω, Σ, μ) a measure space.

Ex.1.1 (Strongly convergent convex combinations)

Let $1 \leq p < \infty$ and let $(f_n) \in L^p \cap L^\infty$ such that $f_n \rightharpoonup F \in L^p$. We want to show that we can find a sequence (F_n) of convex combinations of the (f_n) that converges strongly to F, that is $F_n \to F$ in L^p with

$$F_n = \sum_{k=1}^{n} c_k^{(n)} f_k$$

where $0 \leq c_k^{(n)}$ and $\sum_{k=1}^n c_k^{(n)} = 1$.

1. Denote $\widetilde{K}$ the set of finite convex combinations of $\{f_n\}$. Justify that $\widetilde{K}$ is convex.
2. Denote K the closure of $\widetilde{K}$, justify that K is convex and closed.
3. Show that our claim is equivalent to proving that $F \in K$.
4. Using the Lemma 2.8 in the Lieb-Loss (projection on convex sets), prove that there exists $\Pi_K(F) \in K$ such that
$$\int |F(x) - \Pi_K(F)(x)|^{p-2}(\overline{F(x)} - \overline{\Pi_K(F)(x)})(g(x) - \Pi_K(F)) \leq 0$$
for all $g \in K$.
5. Show that this implies that $F \in K$.

Ex.1.2

Let $1 \leq p \leq \infty$, for $L \in L^p(\Omega)^*$, define

$$\|L\| := \sup\left\{|L(f)|\, f \in L^p(\Omega) \,\text{such that}\, \|f\|_p \leq 1\right\}.$$

Show that $\|\cdot\|$ is a norm.

Ex.1.3

Let $L : L^p(\Omega) \to \mathbb{C}$ be linear. Show that L is continuous if and only if it is bounded, i.e. $\|L\| < \infty$, where $\|L\|$ was defined above.

Ex.1.4 (Gas of fermions at positive temperature)

We say that $m \in \mathcal{S}$ if $m \in L^1(\mathbb{R}^3 \times \mathbb{R}^3)$ and satisfies

- $0 \leq m(x,p) \leq 1$ for a.e. $x, p \in \mathbb{R}^3$ (Pauli exclusion principle)
- (density of N electrons in phase space)

$$\int_{\mathbb{R}^3\times\mathbb{R}^3} m(x,p)\mathrm{d}x\mathrm{d}p = 1 \tag{1}$$

For $Z \geq 1$ (atomic number), and $V \in L^\infty_{\mathrm{loc}}(\mathbb{R}^3)$ such that $e^{-V} \in L^1(\mathbb{R}^3)$, define the Vlasov energy of the gas described by the distribution m by

$$\mathcal{E}^T_{\mathrm{Vlasov}}(m) = \iint_{\mathbb{R}^3\times\mathbb{R}^3} \left(p^2 + V(x)\right) m(x,p)\mathrm{d}x\mathrm{d}p + T\iint m(x,p)\log(m(x,p))\mathrm{d}x\mathrm{d}p$$

1. Show that

$$\mathcal{E}_0(T) := \inf_{m\in\mathcal{S}} \mathcal{E}^T_{\mathrm{Vlasov}}(m) = -T\log\left[\iint_{\mathbb{R}^3\times\mathbb{R}^3} e^{-\frac{1}{T}(p^2+V(x))}\mathrm{d}x\mathrm{d}p\right]$$

and that this minimum is attained uniquely for

$$m_0(x,p) = \frac{1}{\mathcal{E}_0} e^{-\frac{1}{T}(p^2-V(x))}.$$

Hint: You could use Jensen's inequality with the measure m_0 and the convex function $s(x) = x\log x$.

2. The function $s_f(x) = x\log x + (1-x)\log(1-x)$ is also convex on $[0,1]$. Show it and find

$$\inf_{m\in\mathcal{S}} \mathcal{E}^{T,f}_{\mathrm{Vlasov}}(m)$$

where

$$\mathcal{E}^{T,f}_{\mathrm{Vlasov}}(m) = \iint_{\mathbb{R}^3\times\mathbb{R}^3} \left(p^2 + V(x)\right) m(x,p)\mathrm{d}x\mathrm{d}p + T\iint s_f(m(x,p))\mathrm{d}x\mathrm{d}p.$$

NB: The function s_f is called the fermionic entropy