Mathematisches Institut LMU München
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# ADVANCED ANALYSIS
## Exercise sheet 3 – 16.11.2022

**Ex.1.1** (The dual of $L^1$)

We assume known that $L^p(\Omega)^* = L^{p'}(\Omega)$ for $1 < p < \infty$, in the sense that for any $L \in L^p$ there is some $g \in L^{p'}$ such that $L(f) = \int gf$ for all $f \in L^p$. We want to prove the same thing for $p = 1$ when $\mu$ is $\sigma$-finite.

Let $L \in L^1(\Omega)^*$.

1. Assume for the moment $\mu(\Omega) < \infty$. Prove that for all $p \geq 1$, there is $g_p \in L^{p'}$ such that $L(f) = \int g_p f$ for all $f \in L^p(\Omega)$.

2. Prove that $g_p$ is independent of $p$, we will denote $g := g$.

3. Prove that there is some $C > 0$ such that for all $p \geq 1$,
$$\|g\|_p \leq C\mu(\Omega)^{1/q}.$$

4. Prove that $g \in L^\infty$ and that $\|g\|_{L^\infty} \leq C$.

5. Prove that for $f \in L^1$, we have
$$L(f) = \int gf \tag{1}$$

6. We know release the assumption $\mu(\Omega) < \infty$ and use the $\sigma$-finiteness of $\mu$. Prove that there exists some $g \in L^\infty$ such that (1) holds.

**Ex.1.2**

Consider $L^1(\mathbb{R}, (1 + x^2)\mathrm{d}x)$ $(\mathrm{d}mu(x) = (1 + x^2)\mathrm{d}x)$ and
$$K = \left\{g \in L^1(\mathbb{R}, (1 + x^2)\mathrm{d}x) \int g = 1\right\}.$$

1. Prove that $K$ is closed and convex.

2. Prove that the distance from $K$ to $f = 0$ is not attained.

**Ex.1.3** (Convolutions) Let us take $\Omega = \mathbb{R}^n$ and $d\mu(x) = \mathrm{d}x$.

1. Let $f \in L^p$ and $g \in L^q$ with $\frac{1}{p} + \frac{1}{q} = 1$. Prove that for all $\varepsilon > 0$, there is some $R_\varepsilon > 0$ such that
$$\sup_{|x|>R_\varepsilon} |f * g(x)| < \varepsilon.$$

2. Let $f \in L^p$, $g \in L^q$ and $h \in L^r$ with $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$, show that

$$\left| \int fgh \right| \leq \|f\|_p \|g\|_q \|h\|_r.$$

3. Let $f \in L^p$, $g \in L^q$ and $h \in L^r$ with $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 2$, show that

$$\left| \iint f(x)g(x-y)h(y)\mathrm{d}x\mathrm{d}y \right| \leq \|f\|_p \|g\|_q \|h\|_r.$$

*Hint: W.l.o.g. we can assume $f, g, h \geq 0$. Then, one could use (2) with the functions $\alpha(x,y) = f(x)^{p'/r} g(x-y)^{q/r'}$, $\beta(x,y) = g(x-y)^{q/p'} h(y)^{r/p'}$ and $\gamma(x,y) = f(x)^{p/q'} h(y)^{r/q'}$.*