Mathematisches Institut LMU München WiSe 22/23
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ADVANCED ANALYSIS
Exercise sheet 6 – 8.12.2022

Let $\Omega \subset \mathbb{R}^n$ be an open set.

Ex.1.1 (Some examples of distributions)

1. Let $\Omega = (-1, 1)$, and let $f(x) = |x|$. Compute the derivative of f in the distributional sense. What can you tell on f ?

2. Same question for $g(x) = \mathrm{sign}(x)$.

Ex.1.2 (Functions are uniquely determined by distributions)
Recall that for $f \in L^1_{\mathrm{loc}}(\Omega)$, we define the distribution $T \in D'(\Omega)$ by

$$T_f(\phi) = \int f\phi$$

for any $\phi \in D(\Omega)$. Prove that if $f, g \in L^1_{\mathrm{loc}}(\Omega)$ and $T_f = T_g$ then $f = g$.

Ex.1.3(Distributions with zero derivatives are constants)
Let $T \in D'(\Omega)$ and assume Ω to be connected. Show that if $\partial_i T = 0$ in $D'(\Omega)$ for all $1 \leq i \leq n$, then $T = C$ for some constant C.

Ex.1.4(Multiplication by C^∞ and convolution with C_c^∞)
Let $T \in D'(\Omega)$, show that

1. for any $\psi \in C^\infty(\Omega)$, $T_\psi(\phi) := T(\psi\phi)$, for all $\phi \in D(\Omega)$, defines a distribution.

2. for any $j \in C_c^\infty(\Omega)$, $j * T := T(j_R * \phi)$, for all $\phi \in D(\Omega)$ and where $j_R(x) = j(-x)$, defines a distribution.

Ex.1.6(The null space of a distribution is of codimension 1)
For any $T \in D'(\Omega)$ define

$$\mathcal{N}_T = \{\phi \in D(\Omega),\, T(\phi) = 0\}.$$

Show that there exists $\phi_0 \in D(\Omega)$, such that all $\phi \in D(\Omega)$ can be decomposed

$$\phi = \lambda\phi_0 + \psi_T$$

for some $\lambda \in \mathbb{C}$ and $\psi_T \in \mathcal{N}_T$.

Ex.1.6(Lagrange multipliers for distributions)
Let $T, S_1, \ldots, S_N \in D'(\Omega)$ such that

$$\bigcap_{i=1}^{N} \mathcal{N}_{S_i} \subset \mathcal{N}_T.$$

We want to show that there are $c_1, \dots, c_N \in \mathbb{C}$ such that

$$T = \sum_{i=1}^{N} c_i S_i. \tag{1}$$

1. Justify why it is enough to assume that $(S_i)_i$ is linearly independent.
2. For $\phi \in D(\Omega)$ denote $\underline{S}(\phi) = (S_1(\phi), \dots, S_N(\phi))$

   $$V = \left\{ \underline{S}(\phi), \phi \in D(\Omega) \right\}.$$

   Show that V is a vector space of dimension N.
3. Show that there are some $u_1, \dots, u_N \in D(\Omega)$, such that $\underline{S}(u_1), \dots, \underline{S}(u_N)$ are linearly independent and therefore span V.
4. Deduce from this that the matrix whose coefficients are $M_{i,j} = S_i(u_j)$ is invertible.
5. For any $\phi \in D(\Omega)$, denote

   $$\lambda_i(\phi) = \sum_{j=1}^{N} (M^{-1})_{i,j} S_j(\phi)$$

   and check that

   $$\phi - \sum_{i=1}^{N} \lambda_i(\phi) u_i \in \bigcap_{i=1}^{N} \mathcal{N}_{S_i}.$$
6. Prove that (1) holds for some $c_1, \dots, c_N \in \mathbb{C}$.

Ex.1.7($C^\infty(\Omega)$ is dense in $W^{1,p}_{\text{loc}}(\Omega)$).

Let $f \in W^{1,p}_{\text{loc}}(\Omega)$. Let $\mathcal{O} \subset K \subset \Omega$ with $\mathcal{O}$ open and K compact. Show that there is a sequence $\{f_k\} \subset C^\infty(\mathcal{O})$ such that

$$f_k \underset{k \to \infty}{\longrightarrow} f \quad \text{strongly in } W^{1,p}(\mathcal{O}).$$