Mathematisches Institut LMU München
Arnaud Triay SoSe 23

Introduction to the Calculus of Variations
Exercise sheet 1

Ex.1.1 (Properties of convex functions) Let $f : C^2(\mathbb{R}^n) \to \mathbb{R}$, show that the following assertions are equivalent.

1. For all $x, y \in \mathbb{R}^n$ and $0 \le \lambda \le 1$,
$$f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda) f(y).$$
2. For all $x, y \in \mathbb{R}^n$,
$$f(x) \ge f(y) + \langle \nabla f(y), x - y\rangle.$$
3. For all $x, y \in \mathbb{R}^n$,
$$\langle \nabla f(x) - \nabla f(y), x - y \rangle \ge 0$$
4. For all $x, v \in \mathbb{R}^n$,
$$\langle \nabla^2 f(x) v, v\rangle \ge 0.$$

Ex.1.2 (Implicit function theorem)
Let $f : \mathbb{R}^2 \to \mathbb{R}$, C^1 and such that $f(0,0) = 0$ and $\partial_y f(0,0) \neq 0$. We want to prove that there exist $\varepsilon > 0$ and a C^1 function $\varphi :]-\varepsilon, \varepsilon[\to]-\varepsilon, \varepsilon[$ such that for all $x, y \in]-\varepsilon, \varepsilon[$
$$f(x,y) = 0 \iff y = \varphi(x).$$
Without loss of generality, we assume $\partial_y f(0,0) > 0$.

1. Show that there exists $\varepsilon > 0$, such that for all $x \in [-\varepsilon, \varepsilon]$, $[-\varepsilon, \varepsilon] \ni y \mapsto f(x,y)$ is strictly increasing.
2. Deduce that for all $x \in [-\varepsilon, \varepsilon]$, there exists a unique $\varphi(x) \in [-\varepsilon, \varepsilon]$ such that $f(x,y) = 0$ if and only if $y = \varphi(x)$.
3. Using a Taylor expansion of f around $(0,0)$, show that φ is differentiable at 0.
4. Show that it is in fact differentiable on $]-\varepsilon, \varepsilon[$ and C^1 on this set.

Ex.1.3 (Weierstrass example)
Let $f(x, \xi) = x\xi^2$ for $x, \xi \in \mathbb{R}$ and consider, for $\varepsilon \in [0,1)$,
$$(P_\varepsilon) \qquad m_\varepsilon = \inf_{u \in X} \left\{ I(u) := \int_\varepsilon^1 f(x, u'(x)) \mathrm{d}x \right\},$$
$$X_\varepsilon = \left\{ u \in C^1([0,1]), \quad u(\varepsilon) = 1, u(1) = 0 \right\}.$$

1. Show that for $\varepsilon \in (0,1)$ there is a unique minimizer of (P) in $C^2 \cap X_\varepsilon$.
2. Show that for $\varepsilon = 0$ there is no minimizer of (P) in $X_0 \cap C^2$.
3. Find a sequence $\{u_n\} \subset C^1_p$ (piecewise C^1) such that $u_n(0) = 1, u_n(1) = 0$ and $I(u_n) \to 0$ as $n \to \infty$.
4. Show that $m_0 = 0$ and that there is therefore no minimizer of (P) in X.

Ex.1.4 (Lagrange multipliers: finite dimensional case)
Let $n \geq 1$, $\Omega \subset \mathbb{R}^n$ open and $f, g : \Omega \to \mathbb{R}^n$ C^1 functions. Assume that

- f has a local minimum at $x_0 \in \Omega$ subject to the condition $g(x) = 0$, that is

$$\exists \varepsilon > 0, \quad |x - x_0| \leq \varepsilon \text{ and } g(x) = 0 \implies f(x) \geq f(x_0).$$

- $\nabla g(x_0) \neq 0$.

Show that there exists $\lambda \in \mathbb{R}$ such that

$$\nabla f(x_0) = \lambda \nabla g(x_0).$$

Hint: Adapt the proof of the theorem from the lecture with the isoperimetrical constraint.

Ex.1.5 (Lagrange multipliers: application) Let $A \in \mathcal{M}_n(\mathbb{R})$ be a non-negative matrix $A \geq 0$ (that is $\langle x, Ax \rangle \geq 0$ for all $x \in \mathbb{R}^n$). Define

$$m := \inf_{x \in X} \{I(u) = \langle x, Ax \rangle\},$$
$$X = \{x \in \mathbb{R}^n, \text{ such that } \|x\| = 1\}.$$

Using the implicit function theorem, prove that the minimization problem has a solution x_0 and that x_0 is en eigenvector of A with eigenvalue m.

Ex.1.6 (Geodesics of the Euclidean space are straight lines)
Show that the geodesics (path of minimum distance between two points) of the Euclidean space are straight lines (at least among C^1 paths).
Hint: For a C^2 path $\gamma : [0,1] \to \mathbb{R}^n$, for some $n \geq 1$, defines the length of γ, $L(\gamma)$, and compute the Euler–Lagrange equation.

Ex.1.7 (Geodesics of the cylinder are helices)
Consider $\Sigma = \{(x, y, z) \in \mathbb{R}^3, |z| = 1\}$.

1. Show that the geodesics on Σ are helices, that is they can be parametrized by $\gamma(t) = (\cos(\omega t), \sin(\omega t), \alpha t + \beta)$ for some $\omega, \alpha, \beta \in \mathbb{R}$.

   Hint: There are always many ways to parametrize a path, a smart way is to pick one parametrized by arclength, that is $|\gamma'(s)| = 1$ for all s.
2. What is the shortest path on Σ from $(1,0,0)$ to $(1,0,1)$?

Ex.1.8 (Lagrangian formalism)

Let $n \geq 1$, $V : \mathbb{R}^n \to \mathbb{R}$ be C^2 and define $f(t, u, \xi) = \frac{1}{2}m\xi^2 - V(u)$, for $u, \xi \in \mathbb{R}^n$. For some $X_0, X_1 \in \mathbb{R}^n$, consider the minimization problem

$$(P) \qquad m = \inf_{u \in X} \left\{ I(u) := \int_0^1 f(t, u(t), u'(t))\mathrm{d}t \right\},$$
$$X = \left\{ u \in C^1([0,1]), \quad u(0) = X_0, u(1) = X_1 \right\}.$$

Assume that $u_0 : [0,1] \to \mathbb{R}^n$ is C^2 and solves the above minimization problem.

1. Show that the energy $H(u, \xi) = \frac{1}{2}m\xi^2 + V(u)$ is preserved along the trajectory u_0.

2. Show that u_0 satisfies Newton's principle, that is

$$mu'' = F(u)$$

   where $F : \mathbb{R}^n \to \mathbb{R}^n$ is a function to be determined in terms of V.

In this setting f is called the Lagrangian of the system and I the action. The formalism of Lagrange is, to some extent, equivalent to the ones of Newton and Hamilton.

Ex.1.9 (Fermat's principle)

A light beam goes from $(0, 1) \in \mathbb{R}^2$ to $(1, -1) \in \mathbb{R}^2$. In the upper half plane $\{y > 0\}$, the speed of light is c/n_1 and c/n_2 in the lower half plane $\{y < 0\}$, for some indices $n_1, n_2 \geq 1$. The trajectory of light follows the path of shortest time. Show that when it crosses the plane $\{y = 0\}$, we have $n_1 \sin\theta_1 = n_2 \sin\theta_2$ where we have denoted by θ_1 and θ_2 the angles of incidence.