Mathematisches Institut LMU München
Arnaud Triay SoSe 23

Introduction to the Calculus of Variations
Exercise sheet 1

Ex. 1 (Geodesics on surfaces of revolution: Clairaut's invariant)

For $\theta \in \mathbb{R}$, we denote by $u_\theta = (\cos\theta, \sin\theta, 0), v_\theta = \partial_\theta u_\theta = (-\sin\theta, \cos\theta, 0)$ and $k = (0,0,1)$. Let $f, g \in C^2(\mathbb{R})$ and define

$$\begin{cases} \sigma : \ \mathbb{R}^2 & \longrightarrow \ \mathbb{R}^3 \\ \quad (\theta, \lambda) & \longmapsto \ f(\lambda)u_\theta + g(\lambda)k. \end{cases}$$

and $\Sigma = \sigma(\mathbb{R}^2)$. We assume that $f \neq 0$ and $f'^2 + g'^2 \neq 0$. Let $\gamma \in C^2([0,1], \mathbb{R}^3)$ such that $\gamma(t) \in \Sigma$ for all $t \in [0,1]$, we assume that γ is parametrized by arclength (i.e. $\|\gamma'\| = 1$). The goal of this exercise is to show by two "different" ways that if γ is a geodesic on Σ then

$$\begin{aligned} &r(t)\cos\phi(t) = \text{constant, where} \\ &r(t) = \sqrt{\gamma_1(t)^2 + \gamma_2(t)^2}, \qquad \cos\phi(t) = \langle \gamma'(t), v_\theta \rangle. \end{aligned}$$

Here we have denoted $\gamma = (\gamma_1(t), \gamma_2(t), \gamma_3(t))$ and $\|(x,y,z)\| = \sqrt{x^2+y^2+z^2}$ for any $x, y, z \in \mathbb{R}$.

1. Show that there exist $\lambda, \theta \in C^2([0,1])$ such that

$$\gamma(t) = f(\lambda(t))u_{\theta(t)} + g(\lambda(t))k, \quad \forall t \in [0,1].$$

   We can abuse notation and write $\gamma = f(\lambda)u_\theta + g(\lambda)k$. In particular note that $r = f(\lambda)$.

2. Recall what it means for γ to be a geodesic of Σ (minimization problem + Lagrangian + constraint).

3. Let $X = (x,y,z) \in \Sigma$, show that $(\partial_\lambda\sigma(X), \partial_\theta\sigma(X))$ is an orthogonal basis of $T_X\Sigma$, the tangent plane to Σ at X.

4. Show that the Euler-Lagrange equations of γ associated to the minimization problem in question 1. are equivalent to the system

$$\begin{cases} \langle \partial_\lambda\sigma, \gamma'' \rangle = 0, \\ \langle \partial_\theta\sigma, \gamma'' \rangle = 0. \end{cases}$$

5. Deduce from the above that $r(t)\cos\phi(t) = \text{constant}$.

6. Use Noether's theorem to obtain the same result from the formulation of question 1. <u>*Hint:*</u> *What are the symmetries of the Lagrangian in 1. ? Is it invariant by some transformation ?*

General hint: Check the course, it is often a reformulation of it.

Ex.1.2 (Properties of the Legendre transform)
Let $f : \mathbb{R}^n \to \mathbb{R} \cap \{+\infty\}$ and define for $x^* \in \mathbb{R}^n$

$$f^*(x^*) = \sup_{x\in\mathbb{R}^n} \{x^* \cdot x - f(x)\}.$$

Show that

1. For all $x, x^* \in \mathbb{R}^n$, $f(x) + f^*(x^*) \geq x^* \cdot x$.
2. If $g : \mathbb{R}^n \to \mathbb{R} \cap \{+\infty\}$ and $g \geq f$ then $f^* \geq g^*$.
3. The Legendre transform f^* is a convex function and $f^{**} \leq f$.
4. If f is convex and C^1 (therefore we assume it to be finite, i.e. to not take the value $+\infty$), then $f(x) + f^*(\nabla f(x)) = x \cdot \nabla f(x)$ for all $x \in \mathbb{R}^n$. *Hint: one can use the inequality* $f(y) - f(x) \geq \nabla f(x) \cdot (y - x)$.
5. Deduce from the above that $f^{**} = f$.
6. If f is strictly convex and $f(x)/|x| \to \infty$ as $|x| \to \infty$, then f^* is C^1.
7. If f and f^* are C^1 and if f is convex then we have the equivalence

$$f(x) + f^*(x^*) = x^* \cdot x \iff x^* = \nabla f(x) \iff x = \nabla f^*(x^*).$$

Ex.1.3 (Regularity of the Hamiltonian)
Let $f : [a, b] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, C^2 and such that

-
$$D^2_\xi f(t, u, \xi) = \left(\frac{\partial^2 f}{\partial\xi_i\partial\xi_j}(t, u, \xi)\right)_{i,j} > 0$$

  for all $t, u, \xi \in [a, b] \times \mathbb{R}^n \times \mathbb{R}^n$.

- there exist ω, g continuous, $\omega \geq 0$, such that $\omega(\theta)/\theta \to \infty$ as $\theta \to \infty$ and such that

$$(t, u, \xi) \geq \omega(|\xi|) + g(x, u)$$

  for all $t, u, \xi \in [a, b] \times \mathbb{R}^n \times \mathbb{R}^n$.

For all $t, u \in [a, b] \times \mathbb{R}^n$ define

$$H(t, u, p) = \sup_{\xi\in\mathbb{R}^n} \{\xi \cdot v - f(t, u, \xi)\}.$$

Show that

1. Show that for all $t, u \in [a, b] \times \mathbb{R}^n$ there exists a unique $\xi(t, u, p)$ such that

$$H(t, u, p) = \xi(t, u, p) \cdot v - f(t, u, \xi(t, u, p))$$

   and that $\xi \in C^1([a, b] \times \mathbb{R}^n \times \mathbb{R}^n)$. *Hint: Implicit function theorem.*

2. Deduce from this that H is in fact C^2.

Ex.1.4 (Condition for Euler-Lagrange solutions to be minimizers)

Assume the same hypotheses as in Ex. 1.3 and moreover that there is a solution $S \in C^2(\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^3)$ to the Hamilton-Jacobi equation

$$\partial_t S(t,u) + H(t,u,\nabla_u S(t,u)) = 0, \quad \forall (t,u) \in \mathbb{R} \times \mathbb{R}^n,$$

and some u_0 satisfying

$$u_0'(t) = -\nabla_v H(t, u_0(t), \nabla_q S(t, u_0(t))), \quad \forall t \in [a,b].$$

1. Show that u_0 satisfies the Euler-Lagrange equation associated to f, that is

$$\frac{\mathrm{d}}{\mathrm{d}t} \nabla_\xi f(t, u_0, u_0') = \nabla_u f(t, u_0, u_0').$$

2. Show that for all $u \in C^2([a,b], \mathbb{R}^n)$, it holds that

$$\frac{\mathrm{d}}{\mathrm{d}t} S(t, u(t)) \leq f(t, u(t), u'(t)).$$

3. Conclude that for all $u \in C^2([a,b], \mathbb{R}^n)$ with $u(a) = u_0(a)$ and $u(b) - u_0(b)$, it holds that

$$\int_a^b f(t, u(t), u'(t)) \mathrm{d}t \geq \int_a^b f(t, u_0(t), u_0'(t)) \mathrm{d}t.$$

Ex.1.4 (Damped harmonic oscillator)

Consider the Hamiltonian

$$H(t,q,p) = \frac{1}{2m} p^2 e^{-\Gamma t} + \frac{m}{2} \omega_0^2 q^2 e^{\Gamma t}$$

for some $m, \omega_0, \Gamma > 0$.

1. Consider the generating function $S(t,q,Q) = e^{\Gamma t/2} qQ$. Compute the associated the new coordinates (Q,P) and show that the new Hamiltonian in this system of coordinates is

$$\widetilde{H}(t,Q,P) = \frac{1}{2m} Q^2 + m\omega_0^2 P^2 + \frac{\Gamma}{2} QP.$$

2. What remarkable property does $\widetilde{H}$ satisfy ? Look for a solution to the Hamilton-Jacobi equation of the form

$$\widetilde{S}(Q,\alpha) = \psi(Q,\alpha) - \alpha t$$

and solve the Hamiltonian dynamics of $\widetilde{H}$. One may distinguish different cases depending on the relative values of the parameters m, ω_0 and Γ.

3. Deduce from the above the form of solutions to the Hamiltonian equations associated to H.