Mathematisches Institut LMU München
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Introduction to the Calculus of Variations
Homework 2, due date 25.05

Ex. 1 (Geodesics on surfaces of revolution: Clairaut's invariant)

For $\theta \in \mathbb{R}$, we denote by $u_\theta = (\cos\theta, \sin\theta, 0), v_\theta = \partial_\theta u_\theta = (-\sin\theta, \cos\theta, 0)$ and $k = (0,0,1)$. Let $f, g \in C^2(\mathbb{R})$ and define

$$\begin{cases} \sigma : \mathbb{R}^2 & \longrightarrow \mathbb{R}^3 \\ \quad (\theta, \lambda) & \longmapsto f(\lambda)u_\theta + g(\lambda)k. \end{cases}$$

and $\Sigma = \sigma(\mathbb{R}^2)$. We assume that $f \neq 0$ and $f'^2 + g'^2 \neq 0$. Let $\gamma \in C^2([0,1], \mathbb{R}^3)$ such that $\gamma(t) \in \Sigma$ for all $t \in [0,1]$, we assume that γ is parametrized by arclength (i.e. $\|\gamma'\| = 1$). The goal of this exercise is to show by two "different" ways that if γ is a geodesic on Σ then

$$\begin{aligned} &r(t)\cos\phi(t) = \text{constant, where} \\ &r(t) = \sqrt{\gamma_1(t)^2 + \gamma_2(t)^2}, \qquad \cos\phi(t) = \langle \gamma'(t), v_\theta \rangle. \end{aligned}$$

Here we have denoted $\gamma = (\gamma_1(t), \gamma_2(t), \gamma_3(t))$ and $\|(x,y,z)\| = \sqrt{x^2+y^2+z^2}$ for any $x, y, z \in \mathbb{R}$.

1. Show that there exist $\lambda, \theta \in C^2([0,1])$ such that
$$\gamma(t) = f(\lambda(t))u_{\theta(t)} + g(\lambda(t))k, \quad \forall t \in [0,1].$$
We can abuse notation and write $\gamma = f(\lambda)u_\theta + g(\lambda)k$. In particular note that $r = f(\lambda)$.

2. Recall what it means for γ to be a geodesic of Σ (minimization problem + Lagrangian + constraint).

3. Let $X = (x,y,z) \in \Sigma$, show that $(\partial_\lambda\sigma(X), \partial_\theta\sigma(X))$ is an orthonormal basis of $T_X\Sigma$, the tangent plane to Σ at X.

4. Show that the Euler-Lagrange equations of γ associated to the minimization problem in question 1. are equivalent to the system
$$\begin{cases} \langle \partial_\lambda\sigma, \gamma'' \rangle = 0, \\ \langle \partial_\theta\sigma, \gamma'' \rangle = 0. \end{cases}$$

5. Deduce from the above that $r(t)\cos\phi(t) = \text{constant}$.

6. Use Noether's theorem to obtain the same result from the formulation of question 1. *Hint: What are the symmetries of the Lagrangian in 1. ? Is it invariant by some transformation ?*

General hint: Check the course, it is often a reformulation of it.