On the diffeomorphisms group generated by gaussian vector fields

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Abstract

This note gives a partial answer to a question asked during the workshop[1] *Mathematics on Shape Spaces*.

We will denote by $\mathcal{H}_\sigma$ the reproducing kernel Hilbert space of vector fields on $\mathbb{R}^d$ generated by a Gaussian kernel $k_\sigma(x,y) = e^{-\|x-y\|^2/\sigma^2}\,\mathrm{Id}_{\mathbb{R}^d}$ for a positive real parameter σ. Let us first recall an analytical characterization of the space $\mathcal{H}_\sigma$, denoting $\hat{f}$ the Fourier transform of $f \in L^2(\mathbb{R}^d,\mathbb{R}^d)$:

$$\mathcal{H}_\sigma = \left\{ f \in L^2(\mathbb{R}^d,\mathbb{R}^d) \,\middle|\, \|f\|^2_{\mathcal{H}_\sigma} = \frac{\sigma^d}{2^d\pi^{d/2}} \int_{\mathbb{R}^d} |\hat{f}(\omega)|^2 \exp\left(\frac{\sigma^2|\omega|^2}{4}\right) \mathrm{d}\omega < \infty \right\} . \quad (1)$$

The group $\mathcal{G}_{\mathcal{H}_\sigma}$ consists of all flows that can be generated by $\mathcal{H}_\sigma$ vector fields,

$$\mathcal{G}_{\mathcal{H}_\sigma} = \left\{ \varphi(1) \,:\, \varphi(t) \text{ is the solution of (2) with } u \in L^1([0,1],\mathcal{H}_\sigma) \right\} .$$

Given a time-dependent vector field $u \in L^1([0,1],\mathcal{H}_\sigma)$, there exists a unique curve $\varphi \in C([0,1],\mathrm{Diff}^1(\mathbb{R}^d))$ solving

$$\partial_t \varphi(t) = u(t) \circ \varphi(t) \,, \qquad \varphi(0) = \mathrm{Id} \,, \quad (2)$$

for $t \in [0,1]$ almost everywhere. Let us recall what is proved in [You10]:

- Since the space $\mathcal{H}_\sigma$ can be continuously embedded in the space of $H^n(\mathbb{R}^d)$ vector fields (Sobolev space of order $n \geq 1$), the flow is contained in $\mathrm{Diff}^\infty(\mathbb{R}^d)$.
- Since the kernel k_σ is positive definite on $\mathbb{R}^d$, the group $\mathcal{G}_{\mathcal{H}_\sigma}$ acts $n-$transitively on $\mathbb{R}^d$ if $d \geq 2$, i.e. for any two ordered sets of n distinct points $(x_1,\ldots,x_n)$ and $(y_1,\ldots,y_n)$ in $\mathbb{R}^d$ there exists an element $\varphi \in \mathcal{G}_{\mathcal{H}_\sigma}$ such that for each $i \in [\![1,n]\!]$, $\varphi(x_i) = y_i$.

These groups are widely used in application such as diffeomorphic image matching [BMTY05, RVW+11, SLNP11], although their understanding is less developed than the group of Sobolev diffeomorphisms [KLMP11, MP10, BV14]. A possible generalization of the previous property is the following question:

Question 1. *Does the group $\mathcal{G}_{\mathcal{H}_\sigma}$ acts transitively on the space of compactly supported smooth densities* $\mathrm{Dens}^\infty(\mathbb{R}^d) := \{\rho \in C^\infty(\mathbb{R}^d,\mathbb{R}) \mid \int_{\mathbb{R}^d} \rho(x)\,\mathrm{d}x = 1\}$*?*

Recall that this property is well-known in the case of smooth diffeomorphisms by the so-called *Moser trick*. As we will prove in this note, the answer to question 1 is negative. Indeed, we have:

Proposition 1. *The group $\mathcal{G}_{\mathcal{H}_\sigma}$ is contained in the group of real analytic diffeomorphisms of $\mathbb{R}^d$, $\mathcal{G}_{\mathcal{H}_\sigma} \subset \mathrm{Diff}^\omega(\mathbb{R}^d)$. More precisely, any element of $\mathcal{G}_{\mathcal{H}_\sigma}$ admits a holomorphic extension on a cylindrical open set of $\mathbb{R}^d$ in $\mathbb{C}^d$, namely $C(r) = \{z \in \mathbb{C}^d \,|\, \forall i \in [\![1,d]\!] \,|\, |\mathrm{Im}(z_i)| \leq r\}$ for $r > 0$ sufficiently small.*

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Before proving the proposition, we briefly describe a counter-example for question 1. Let us define the set of singular points of a function by being the complementary of the set of points where the function is analytic. By definition, this set is closed; we will denote it by $S(\rho)$ for a given density ρ. The topology of $S(\rho)$ is preserved under the action of the group $\mathcal{G}_{\mathcal{H}_\sigma}$. Indeed, its action on densities only involves multiplication and composition by real analytic functions which preserve analyticity. Last, there exists densities whose singular set do not have the same topology (for instance connectedness).

Proof of the proposition. The Gaussian kernel has a complex extension (see [SHS06])

$$k_\sigma^{\mathbb{C}}(z,z') = \exp\left(-\frac{1}{\sigma^2}\sum_{i=1}^{d}(z_i - \overline{z_i'})^2\right) \tag{3}$$

where $z = (z_1, \dots, z_d) \in \mathbb{C}^d$ and we denote the associated reproducing kernel Hilbert space by $\mathcal{H}_\sigma^{\mathbb{C}}(\mathbb{C}^d)$ or simply $\mathcal{H}_\sigma^{\mathbb{C}}$. This space can be explicitely described by

$$\mathcal{H}_\sigma^{\mathbb{C}} = \{f : \mathbb{C}^d \to \mathbb{C}^d \,|\, f \text{ is holomorphic and } \|f\|_\sigma < \infty\}$$

where

$$\|f\|_\sigma^2 = \int_{\mathbb{C}^d} \|f(z)\|^2 e^{\frac{1}{\sigma^2}\sum_{i=1}^{d}(z_i - \overline{z_i})^2}\, dz\,.$$

Note that $\|f\|_\sigma$ is the norm of f in the space $\mathcal{H}_\sigma^{\mathbb{C}}$ up to a multiplicative constant.

The restriction of $k_\sigma^{\mathbb{C}}$ to the real line is k_σ which means (see paragraph 5 of part 1 in [Aro50]) that the space $\mathcal{H}_\sigma$ can be described as a subspace of $\mathcal{H}_\sigma^{\mathbb{C}}(\mathbb{C}^d)$. More precisely, for every $f \in \mathcal{H}_\sigma$ there exists a unique $\tilde{f} \in \mathcal{H}_\sigma^{\mathbb{C}}$ minimizing the norm $\|\tilde{f}\|_{\mathcal{H}_\sigma^{\mathbb{C}}}$ among the functions $\tilde{f} \in \mathcal{H}_\sigma^{\mathbb{C}}$ such that $\tilde{f}_{|\mathbb{R}^d} = f$. In particular, this shows that every element of $\mathcal{H}_\sigma$ is an analytic function.

We now consider a time dependent vector field $v_t \in L^2([0,1], \mathcal{H}_\sigma)$ and we denote by $\tilde{v}_t$ its lift in $L^2([0,1], \mathcal{H}_\sigma^{\mathbb{C}})$. The flow associated with $\tilde{v}_t$ may not exist since the Lipschitz constant of v_t may be infinite on $\mathbb{C}^d$, however this Lipschitz constant is uniformly bounded on any cylindrical neighborhood of $\mathbb{R}^d$: As proven in [SHS06] in *proof of lemma 3.1*

$$\|f(z)\|^2 \leq \frac{c}{(2\sigma^2)^d}\|f\|_\sigma^2 \tag{4}$$

with $c(r) = \max\{e^{-\frac{1}{\sigma^2}\sum_{i=1}^{d}(z_i - \overline{z_i'})^2} \,|\, \max_{i=1,\dots,d} \mathrm{Im}(z_i)| \leq r\}$, which implies the Lipschitz property on $C(r)$ using the Cauchy formula.

In order to show the result, we consider a smooth function η on $\mathbb{C}^d$ such that $\eta(z) = 1$ for $z \in C(1)$ and $\eta(z) = 0$ for $z \notin C(2)$. The time dependent vector field $u_t(z) = \eta(z)v_t(z)$ is holomorphic on $C(1)$ and globally Lipschitz with a constant that depends linearly on the Lipschitz constant of v_t on $C(2)$. Then, the flow ϕ_t of u_t is well defined and applying Gronwall's lemma we have:

$$\|\phi_t(z) - \phi_t(z')\| \leq \|z - z'\| \exp\left(\int_0^t \|u_s\|_{1,\infty}\, ds\right), \tag{5}$$

where $\|u\|_{1,\infty}$ denotes the sup norm of u and its first derivative. Since there exists a constant M such that $\|u_t\|_{1,\infty} \leq M\|\tilde{v}_t\|_\sigma$, we have

$$\|\phi_t(z) - \phi_t(z')\| \leq \|z - z'\| \exp\left(\int_0^t M\|\tilde{v}_t\|_\sigma\, ds\right), \tag{6}$$

In particular, there exists $\varepsilon > 0$ such that for all $t \in [0,1]$, $\phi_t(z) \in C(1)$ if $z \in C(\varepsilon)$. Since $u_t(z) = \tilde{v}_t(z)$ for $z \in C(1)$, ϕ_1 is holomorphic on $C(\varepsilon)$ being the flow of a vector field v_t which is holomorphic $\phi_t(C(\varepsilon)) \subset C(1)$. $\square$

In the proof we used the complex extension of the Gaussian kernel but it is probably possible to prove the analyticity by direct estimations.

As a conclusion, the initial question 1 might be reformulated as follows

Question 2. *Does the group $\mathcal{G}_{\mathcal{H}_\sigma}$ acts transitively on the space of analytical densities* $\mathrm{Dens}^\omega(\mathbb{R}^d) := \{\rho \in C^\omega(\mathbb{R}^d, \mathbb{R}) \mid \int_{\mathbb{R}^d} \rho(x)\,\mathrm{d}x = 1\}$*?*

However, the answer to this question is no in dimension 1 since this would imply that $\mathcal{G}_{\mathcal{H}_\sigma} = \mathrm{Diff}^\omega(\mathbb{R})$. Indeed, let F denote the cumulative distribution function of a Gaussian density which is an analytical diffeomorphism between $\mathbb{R}$ and $]0,1[$ and $\varphi \in \mathrm{Diff}^\omega(\mathbb{R})$ to which we associate the analytical cumulative distribution function $F \circ \varphi$. Using transitivity of $G_{\mathcal{H}_\sigma}$ on analytical densities, there exists $\psi \in G_{\mathcal{H}_\sigma}$ such that $F \circ \varphi \circ \psi = F$. This implies $\varphi \circ \psi = \mathrm{Id}$ and the result.

Therefore, this would even be interesting to answer the following simple question:

Question 3. *Even if $\mathcal{H}_\sigma \subsetneq \mathcal{H}_{\sigma'}$ when $\sigma' < \sigma$, is it true that $\mathcal{G}_{\mathcal{H}_\sigma} \subsetneq \mathcal{G}_{\mathcal{H}_{\sigma'}}$?*

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