A note on the extension of isotopy

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This note is the answer to the question: Can we follow the connected components of the complementary of a subset which is tranformed by an isotopy? More precisely, we have:

**Proposition 1** Let $U$ an open set of $\mathbb{R}^n$ and $K \subset U$ a compact subset of $\mathbb{R}^n$, $M > 0$ a real constant. Let $i : [0, 1] \times K \rightarrow U$ a homotopy of $K$ such that, $i(0) = Id|_K$, $\partial_t i$ is Lipschitz on $[0, 1] \times K$ of constant $M$ and for each $t \in [0, 1]$, $i(t)$ is a homeomorphism bi-Lipschitz of $K$ of Lipschitz constant $M$, then there exists an extension of $i$, $\tilde{i} : [0, 1] \times U \rightarrow U$ such that:

- $\tilde{i}(t, \cdot)|_K = i(t, \cdot)$
- $\tilde{i}$ is continuous and for each $t \in [0, 1]$, $\tilde{i}(t)$ is a homeomorphism.

**Proof:** Consider the vector field $(1, \partial_t i)$ defined on $i([0, 1] \times K)$, it can be extended on $[0, 1] \times U$ by the Lipschitz extension theorem provided this vector field is Lipschitz:

Consider $(s, x_s)$ and $(t, y_t)$ such that $x_s \in i(s)(K)$ and $y_t \in i(t)(K)$, then, introducing $x_r = i(r)(i(s)^{-1}(x_s))$ for $r \in [0, 1]$, we have:

$$|x_0 - y_0| \leq M|x_s - y_t| \leq M|x_s - y_t| + M^2|t - s|.$$  

This inequality implies:

$$|\partial_t i(s, i(s)^{-1}(x_s)) - \partial_t i(t, i(t)^{-1}(y_t))| \leq M|i(s)^{-1}(x_s) - i(t)^{-1}(y_t)| + M|t - s| \leq M^2|x_s - y_t| + (M + M^2)|t - s|.$$  

Hence, we get the result with $\tilde{i}$ defined by the flow of the time vector field $(1, \partial_t i) \in \mathbb{R} \times \mathbb{R}^n$. $\square$

This theorem is known as the extension isotopy theorem which can be found in [?] if $i$ is a differentiable isotopy and the set $K$ is more regular than above. The result is not true as soon as the isotopy is only continuous, a counter example is the horned sphere which is a deformation of the sphere such that the complementary of this deformation has a non trivial fundamental group. This implies that there does not exist an extension to the global space of
the isotopy because the two complementary components of the sphere has a trivial fundamental group.

As a direct consequence of this proposition, the injection $Id : U \setminus K \mapsto ([0,1] \times U) \setminus i([0,1] \times K)$ gives an isomorphism $Id_\ast : H_0(U \setminus K) \mapsto H_0([0,1] \times U \setminus i([0,1] \times K))$. We want to prove the same result for $i$ a isotopy which is only continuous.

**Lemma 1** Let $U$ an open set of $\mathbb{R}^n$ and $K \subset U$ a compact subset of $\mathbb{R}^n$, $M > 0$ a real constant. Let $i : [0,1] \times K \mapsto U$ a homotopy of $K$ such that, $i(0) = Id|_K$ and for each $t \in [0,1]$, $i(t)$ is a homeomorphism of $K$ onto $i(t)(K)$, then the injection

$$Id : U \setminus K \mapsto ([0,1] \times U) \setminus i([0,1] \times K)$$

induces an injection :

$$Id_\ast : H_0(U \setminus K) \mapsto H_0([0,1] \times U \setminus i([0,1] \times K)).$$  \hspace{1cm} (1)

**Proof :** Let $c : [0,1] \mapsto [0,1] \times U \setminus i([0,1] \times K)$ such that $c(0) \in 0 \times U \setminus K$ and $c(0) \in 0 \times U \setminus K$ in a different connected component (of $U \setminus K$) than $c(0)$. Then for every approximation ($\epsilon$ is given) $j$ differentiable and verifying the isotopy extension proposition we have the existence of $z_\epsilon \in [0,1] \times K$ and $t_\epsilon \in [0,1]$ such that $j(z_\epsilon) = c(t_\epsilon)$. By compacity an continuity we find $z_0$ and $t_0$ such that $i(z_0) = c(t_0)$. Hence, $c(0)$ and $c(1)$ are in two distinct connected components of $[0,1] \times U \setminus i([0,1] \times K)$, and the injectivity is proved. $\square$

From this lemma, we deduce that there exists a natural injection in a neighborhood of $t = 0$, namely

$$Id_\ast : H_0(U \setminus K) \mapsto H_0(U \setminus i(1)(K)),$$

deduced from the above application. The following result gives a sense to the obvious view of “following” the connected components of $U \setminus i(t)(K)$ with respect to $t$, and the resulting identification does not depend on the chosen isotopy.

**Proposition 2** With the hypothesis of the lemma 1 and the additional hypothesis that $\text{dim}(H_0(U \setminus K)) < +\infty$, there exists a canonical isomorphism :

$$Id_\ast : H_0(U \setminus K) \mapsto H_0(U \setminus i(1)(K)),$$

which does not depend on the isotopy between $i(0)$ and $i(1)$.

**Proof :** First, choose $(x_1, \ldots, x_n)$ representing the connected components of $U \setminus K$, then : there exists $\epsilon > 0$ such that for each $i \in [1,n]$, $B((0,x_i),2\epsilon) \subset [0,1] \times U \setminus i([0,1] \times K)$, then we can define :

$$Id_\ast : H_0(U \setminus K) \mapsto H_0(U \setminus i(\epsilon)(K))$$

$$cl(B((0,x_i),2\epsilon) \cap (U \setminus K)) \mapsto cl(B((0,x_i),2\epsilon) \cap (U \setminus i(\epsilon)(K)))),$$
with the notation \( d(Z) \) for the connected component containing the connected set \( Z \). To prove that this application is well defined, we see that this is the natural restriction of the map in lemma 1. As a consequence, it is an injection. Considering \( i(t)^{-1} \) and \( i(e)(K) \), with the same proof we get that this injection is an isomorphism. The first conclusion of the proposition follows by compactness of \([0,1]\) and the fact that it does not depend on the isotopy is easily deduced from the injectivity proved in the lemma 1. □