A note on the extension of isotopy

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This note is the answer to the question : Can we follow the connected components of the complementary of a subset which is tranformed by an isotopy ? More precisely, we have :

Proposition 1 *Let U an open set of $\mathbb{R}^n$ and $K \subset U$ a compact subset of $\mathbb{R}^n$, $M > 0$ a real constant. Let $i : [0,1] \times K \mapsto U$ a homotopy of K such that, $i(0) = Id_{|K}$, $\partial_t i$ is Lipschitz on $[0,1] \times K$ of constant M and for each $t \in [0,1]$, $i(t)$ is a homeomorphism bi-Lipschitz of K of Lipschitz constant M, then there exists an extension of i, $\tilde{i} : [0,1] \times U \mapsto U$ such that :*

- $\tilde{i}(t,.)_{|K} = i(t,.)$
- *$\tilde{i}$ is continuous and for each $t \in [0,1]$,*
- *$\tilde{i}(t)$ is a homeomorphism.*

Proof : Consider the vector field $(1, \partial_t i)$ defined on $i_t([0,1] \times K)$, it can be extended on $[0,1] \times U$ by the Lipschitz extension theorem provided this vector field is Lipschitz :
Consider (s, x_s) and (t, y_t) such that $x_s \in i(s)(K)$ and $y_t \in i(t)(K)$ then, introducing $x_r = i(r)(i(s)^{-1}(x_s))$ for $r \in [0,1]$, we have :

$$|x_0 - y_0| \leq M|x_s - y_s| \leq M|x_s - y_t| + M^2|t - s|.$$

This inequality implies :

$$\begin{aligned} |\partial_t i(s, i(s)^{-1}(x_s)) - \partial_t i(t, i(t)^{-1}(y_t))| &\leq M|i(s)^{-1}(x_s) - i(t)^{-1}(y_t)| + M|t - s| \\ &\leq M^2|x_s - y_t| + (M + M^2)|t - s|. \end{aligned}$$

Hence, we get the result with $\tilde{i}$ defined by the flow of the time vector field $(1, \partial_t i) \in \mathbb{R} \times \mathbb{R}^n$. $\square$

This theorem is known as the extension isotopy theorem which can be found in [**?**] if i is a differentiable isotopy and the set K is more regular than above. The result is not true as soon as the isotopy is only continuous, a counter example is the horned sphere which is a deformation of the sphere such that the complementary of this deformation has a non trivial fundamental group. This implies that there does not exist an extension to the global space of

the isotopy because the two complementary components of the sphere has a trivial fundamental group.
As a direct consequence of this proposition, the injection $Id : U \setminus K \mapsto ([0,1] \times U) \setminus i([0,1] \times K)$ gives an isomorphism $Id_* : H_0(U \setminus K) \mapsto H_0([0,1] \times U \setminus i([0,1] \times K))$. We want to prove the same result for i a isotopy which is only continuous.

Lemme 1 *Let U an open set of $\mathbb{R}^n$ and $K \subset U$ a compact subset of $\mathbb{R}^n$, $M > 0$ a real constant. Let $i : [0,1] \times K \mapsto U$ a homotopy of K such that, $i(0) = Id_{|K}$ and for each $t \in [0,1]$, $i(t)$ is a homeomorphism of K onto $i(t)(K)$, then the injection*

$$Id : U \setminus K \mapsto ([0,1] \times U) \setminus i([0,1] \times K)$$

induces an injection :

$$Id_* : H_0(U \setminus K) \mapsto H_0([0,1] \times U \setminus i([0,1] \times K)). \tag{1}$$

Proof : Let $c : [0,1] \mapsto [0,1] \times U \setminus i([0,1] \times K)$ such that $c(0) \in 0 \times U \setminus K$ and $c(0) \in 0 \times U \setminus K$ in a different connected component (of $U \setminus K$) than $c(0)$. Then for every approximation (ϵ is given) j differentiable and verifying the isotopy extension proposition we have the existence of $z_\epsilon \in [0,1] \times K$ and $t_\epsilon \in [0,1]$ such that $j(z_\epsilon) = c(t_\epsilon)$. By compacity an continuity we find z_0 and t_0 such that $i(z_0) = c(t_0)$. Hence, $c(0)$ and $c(1)$ are in two distinct connected components of $[0,1] \times U \setminus i([0,1] \times K)$, and the injectivity is proved. □

From this lemma, we deduce that there exists a natural injection in a neighborhood of $t = 0$, namely

$$Id_* : H_0(U \setminus K) \mapsto H_0(U \setminus i(t)(K)),$$

deduced from the above application. The following result gives a sense to the obvious view of "following" the connected components of $U \setminus i(t)(K)$ with respect to t, and the resulting identification does not depend on the chosen isotopy.

Proposition 2 *With the hypothesis of the lemma 1 and the additional hypothesis that $dim(H_0(U \setminus K)) < +\infty$, there exists a canonical isomorphism :*

$$Id_* : H_0(U \setminus K) \mapsto H_0(U \setminus i(1)(K)),$$

which does not depend on the isotopy between $i(0)$ and $i(1)$.

Proof : First, choose $(x_1, \dots, x_n)$ representing the connected components of $U \setminus K$, then : there exists $\epsilon > 0$ such that for each $i \in [1,n]$, $B((0,x_i), 2\epsilon) \subset [0,1] \times U \setminus i([0,1] \times K)$, then we can define :

$$\begin{array}{rcl} Id_* : H_0(U \setminus K) & \mapsto & H_0(U \setminus i(\epsilon)(K)) \\ cl(B((0,x_i), 2\epsilon) \cap (U \setminus K)) & \mapsto & cl(B((0,x_i), 2\epsilon) \cap (U \setminus i(\epsilon)(K)))), \end{array}$$

with the notation $cl(Z)$ for the connected component containing the connected set Z. To prove that this application is well defined, we see that this is the natural restriction of the map in lemma 1. As a consequence, it is an injection. Considering $i(t)^{-1}$ and $i(\epsilon)(K)$, with the same proof we get that this injection is an isomorphism. The first conclusion of the proposition follows by compacity of $[0, 1]$ and the fact that it does not depend on the isotopy is easily deduced from the injectivity proved in the lemma 1. $\square$