Stochastic Replicator Dynamics

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Introduction

The concept of natural selection that emerged in 1858 with the theory of evolution in the book of Darwin *On the Origin of the Evolution* (1858) is a cornerstone of evolutionary theory. However, from the biologic perspective, evolution is not limited to selection: reproduction, mutations, random drift and other elements providing diversity are also part of evolutionary processes. Also, it is sometimes disputed among biologists whether or not the process of selection is a neutral process closer to random drift or if it is in accordance to natural selection which states that features of a species with reproductive advantages get selected over time.

Along with the growing emergence of evolutionary theory, not only biologists addressed the issue but mathematicians did as well. In the 1920's and 1930's, Ronald Fisher, J.B.S. Haldane and Sewall Wright built sophisticated mathematical models of evolution. Following these endeavour to model evolutionary processes, evolutionary game theory emerged as an application of the mathematical theory of games to biological contexts. It was first developed by Fisher (1930, *The Genetic Theory of Natural Selection*) in his attempt to explain the approximate equality of the sex ratio for mammals, even if Fisher's did not state it in terms of game theory. R.C. Lewontin (1961) made the first explicit application of game theory to evolutionary biology. He was concerned with the evolution of genetic mechanisms, which he viewed as a game played between species and nature.

In 1973, the concept of an evolutionarily stable strategy (ESS) emerged with the publication "The Logic of Animal Conflict" (1973) by John Maynard Smith and Georges R. Price. ESS became a central notion in evolutionary game theory as it provided a subtler means to help in the issue of equilibrium selection. With Maynard Smith, evolutionary game theory stopped to be seen as a game between species and nature (as supposed Lewontin) but as a game between members of a same species or populations of different species. His book *Evolution and the Theory of Games* (1982) became a reference to evolutionary game theory as it provided a recap of the scientific advances on evolutionary game theory.

One of the most important mathematical model of selection comes from evolutionary game theory and is called "replicator dynamics". The notion emerged in 1978 with the introduction of differential equation in evolutionary game theory by Taylor and Jonker. Foster and Young (1990) were the first to introduce perturbations in the model and to present a stochastic version of replicator dynamics. This first stochastic model was followed by the model of Fudenberg and Harris (1992) which has then been generalized and extended by Cabrales (2000) and other researchers such as Imhof and Hofbauer (2009). A most recent reference that records current knowledge on evolutionary game theory is the *Handbook of Game Theory with Economic Applications* (2015, volume 4)[1].

Replicator dynamics adressed many issues. For instance, the game generated by replicator dynamics can be studied in the perspective of equilibrium selection, that is to say, to determine which equilibrium should be selected among a collection of equilibria. This was one of the perspectives chosen by Fudenberg and Harris (1992). Another issue is the elimination of (strictly) dominated strategies, in other words, to determine whether or not (strictly) dominated strategies are eliminated from the game. In our report, we will concentrate on this last approach for both deterministic and stochastic versions of replicator dynamics.

After reminding the basic concepts of evolutionary game theory and setting the framework of our report, we will define the deterministic model of replicator dynamics. Then, following the historic of the introduction of stochasticity in the replicator dynamics, we will develop a stochastic version of replicator dynamics based on the works of Fudenberg and Harris (1992) and Cabrales (2000). Finally, we will consider the elimination of strictly dominated strategies regarding the deterministic and then stochastic replicator dynamics issued by Cabrales (2000) and conclude by briefly mentioning the approach of Hofbauer and Imhof (2009) concerning this issue regarding stochastic replicator dynamics.

[1] See Chapter 6- "Stochastic Evolutionary Game Dynamics" by Chris Wallace and H. Peyton Young, Chapter 11 - "Evolutionary Game Theory in Biology" by Peter Hammerstein and Olof Leimar, Chapter 13 - "Population Games and Deterministic Evolutionary Dynamics" by William H. Sandholm.

Part I. Prerequisites on game theory and framework for replicator dynamics

1 Definition and concept: a game in game theory

A *game* is defined by listing the strategies available to the players and the payoffs associated to matches between players and their specific strategies. A game needs interactions between agents, that we call players. Generally it takes the form of a confrontation between two players. These players choose a strategy among the collection they have in order to get the best benefit, if saw as rational players. In evolutionary game theory, the players are individuals, members of a species (or a population) which aim to survive[2]. Accordingly, the benefit they get from a game is associated with reproductive success.
A game is *symmetric* when the roles of both players are exchangeable.

Pure strategies are single strategies from the strategy set of the game while *mixed strategies* are a probability distribution over the set of pure strategies. For example in the case of a two strategy game with set of pure strategies $I = \{A, B\}$, on the one hand A and B are pure strategies, and on the other hand a mixed strategy would be for example that the player plays A or B with probability 0.5.

If the game has n pure strategies, we can represent any mixed strategy as a n-dimensional vector giving by its kth coordinate the probability assigned to the kth strategy by the player. In the example above, the mixed strategy is represented by $(0.5, 0.5)$. The mixed strategies belongs to the unit $(n-1)$-dimensional simplex.

Remark In population biology, *strategies* can be seen as genotypes or phenotypes (i.e. the expression of a genotype), which determine for instance the appearance of a species such as morphology or color, or the type of behaviours such as aggressive or non-aggressive.

2 Notations and initial framework

Hereinafter, we consider a symmetric two-player game with a population (or a species) divided in n subpopulations of players adopting a specific strategy. Each subpopulation contains a continuum of players. The size of the ith subpopulation is r_i and the total number of players is $R = \sum_i r_i$. Plus, denote $r := (r_1, ..., r_n)$ the vector of population sizes. R and r_i depend on time in the dynamic and so will be denoted $R(t)$ and $r_i(t)$ at time t.

Every member of a same subpopulation i plays the same pure strategy s_i taken from the finite set of pure strategies of the game $\{s_1, ..., s_n\}$. By a slight abuse of notation we will denote s_i by i, and so denote the set of pure strategies by $S := \{1, ..., n\}$. We will call a player from the ith subpopulation, playing the pure strategy i, a *player or individual of type* i, and the ith subpopulation the *subpopulation of type* i.

As we want to measure the evolutionary success of a strategy, let us consider the frequency of the strategies. Denote $x_i(t) := \frac{r_i(t)}{R(t)}$ the proportion of players of type i, that is to say, the frequency of strategy i. The vector $x(t) := (x_1(t), ..., x_n(t))$ is the vector of frequencies of the strategies and is called the *population state* at time t. We denote by S_n the $n-1$ dimensional simplex over I: $S_n = \left\{x \in \mathbb{R}^n_+; \sum_{i \in I} x_i = 1\right\}$. The function $x(\cdot)$ is a differentiable function of t and lives in the simplex S_n.

Remark As x lives in the simplex S_n, it may be seen as a mixed strategy played by a fictitious player embodying the population repartition.

3 Definitions and concepts: payoff and fitness

To understand what represents *payoff* in a game, let us see it as a gain (or a loss when it's negative) for a player who plays a strategy A against another player playing strategy B (the second player could play the same strategy as its opponent). In the original economic formulation payoffs were understood as utilities, but for the biological formulation, Maynard Smith reinterpreted them in terms of fitness. The concept of *fitness* was introduced in the 1920's by J.B.S Haldane and Sewall Wright in order to help giving to natural selection a mathematical precise form. They defined fitness as the expected number of offspring of an individual that reaches adulthood.

[2] Actually there is no aim in biology since survival happens to select species and/or strategies, and, except maybe for human species, there is no such thing as final goal in life. However, it is convenient to speak of survival as an aim.

Denote by $u_i(s)$ the associated payoff to individual of type i against strategy $s \in S_n$. For any pure strategy i, the *payoff function* of individual of type i is defined by the continuous function $u_i : S_n \to \mathbb{R}$.

Hereafter, the payoffs will be associated to fitness as we consider a biologic interpretation of replicator dynamics and we will use the notation given in the definition.

Part II. Replicator dynamics

According to Nowak (2006), the main ingredients of evolutionary dynamics are reproduction, mutation, selection, random drift and spatial movement. Evolutionary processes can however be restricted to two elements: a *mutation mechanism* that provides varieties, and a *selection mechanism* that favors some variety over others. Introduced in 1978 by Taylor and Jonker, *replicator dynamics* is one of the most important mathematical models used for selection.

4 The model

Replicator dynamics first models selection, and so, in the context of evolutionary dynamics, *natural selection.* In order to build a realistic model, the replicator equation is based on the Darwinian principle of natural selection, in other words, if a strategy gets disadvantaged in terms of reproductive success (compared to the others), it should be eliminated over time.

We want to explicit the growth rate of the size of the population as well as the frequency of players of type i, in other words, we want to explicit the derivative over time of x_i and r_i.

Suppose that each offspring inherits its single parent's strategy, (i.e. strategies breed true)[3]. If reproduction takes place continuously over time[4], then the *birthrate* at any time t and of any individuals of type i, is given by $B[t, r(t)] + u_i[x(t)]$, where $B \geq 0$ is the *background fitness* (the number of successor independent of the outcomes of the game). After reproduction, a fraction of the individuals of any types dies, except the newborns. Let suppose that the *death rate* is the same for all individuals and given by $D[t, r(t)] \geq 0$. Finally, for any populations of type i, the population dynamics results in:

$$\dot{r}_i(t) = r_i(t)\Big(B[t, r(t)] + u_i[x(t)] - D[t, r(t)]\Big) \tag{1}$$

The corresponding dynamic for the frequency x_i is then:

$$\dot{x}_i(t) = x_i(t) \cdot \left(u_i(x(t)) - \sum_j x_j(t)u_j(t)\right) \tag{2}$$

and is called the **replicator equation**[5].

Interpretations

The replicator equation describes the evolution of the frequencies of subpopulation types taking into account their mutual influence on their fitness. Since $\sum_j x_j u_j(x)$ can be seen as the *average payoff* of the population, the term $u_i(x) - \sum_j x_j u_j(x)$ gives the payoff of individuals type i relative to the average payoff. That way, the frequency of a strategy increases when it has above average payoff. It models the fact that the more successful a strategy is, the faster it grows within the population.

5 Assumptions

The replicator dynamic requires assumptions to be accurate: the population has to be *infinite* (or large enough), *well-mixed* and with *no mutations.* Indeed, concerning the last point, the deterministic replicator equation only describes selection and does not take into account mutations. "Well-mixed" population means that every individuals interact with every other one or at least has the same probability to interact with any other individuals

[3] Remark: the replicator dynamics has the drawback to disregards the complexity of sexual reproduction. The model is thus restricted to asexual reproduction.

[4] The discrete-time version of replicator dynamics has the benefit to take into account the repartition of reproduction over time which is not continuous for most of the species. However, we will not study this model in our report.

[5] The computation of x_i is easy to do by derivation of the function $x_i(t) = r_i(t)/R(t)$.

of the population. Finally, when considering a large or infinite population, we can make the assumption that any individuals interacts with a player who uses the average strategy within the population.

Another important assumption is that the game has to *start from the interior of the simplex.* Indeed, since the replicator dynamics is a selection dynamics, if the game doesn't start in the interior of the simplex, that is to say, if a strategy is not used at the beginning of the game, then it will not be used in the game ever after even if this strategy is the best considering the others.

Part III. Introduction of stochasticity in replicator dynamics

Replicator dynamics is a model for selection. However, in a biological context, even if deterministic models can provide a first approach and understanding to population dynamics in nature, it is rare or impossible to find an ecosystem where determinism is the rule. Indeed, many perturbations of different scales (such as the weather or natural disaster, competition or predation) influence the impact of strategies on survival. Thus, it seems relevant to take into account stochastic effects in the replicator equation and then observe whether or not it changes the outcome of the dynamics.

6 Preliminaries: elements of stochasticity

6.1 A tool to model aggregate shocks: Wiener process

In this report, we study the continuous time version of replicator dynamics. As a consequence, the first postulate that has to be made to model perturbations is that it must occur in continuous time. They are also supposed to be uncorrelated across time.

Individual perturbations and aggregate shocks

There exists different types of perturbations. In this report, we will concentrate on exterior perturbations.

Regarding a same subpopulation i, there are perturbations that affect independently each individual of type i and those that affect in the same way every individual of the population or of same type. The latter are called *aggregate shocks*.

As we do not consider a game between individuals but between individuals of a same type, the payoff of the individual of type i is actually a mean over the payoff of every member of a same subpopulation, that is to say, an expected payoff. Suppose that the independent perturbations $\big(\eta_i(k)\big)_{k\in\{1,...,r_i\}}$ are i.i.d. and follow a law of mean zero and denote by $i_1, ..., i_{r_i}$ the individuals of the subpopulation i, by $\eta_i(k)$ the perturbation that affects the player i_k, and by $\eta(i)$ the aggregate shocks that affects all players of type i. The payoff of the individual i_k is $u_i(x(t)) + \eta_i(k) + \eta(i)$. As a consequence, the expected payoff for a type i is given by

$$u_i(x(t)) + \frac{1}{r_i}\sum_{k=1}^{r_i}\eta_i(k) + \eta(i).$$

By the law of large numbers and since we consider a large enough population, the term $\frac{1}{r_i}\sum\limits_{k=1}^{r_i}\eta_i(k)$ can be neglected as it tends to its mean, that is, zero. Subsequently, hereinafter we will neglect the "independent" shocks and will only consider the aggregate shocks in the expected payoff (and so the stochastic version of replicator dynamics).

To sum-up, aggregate shocks are supposed to affect the payoffs associated to players of a same strategy in the same way. For simplicity, in our model, we suppose that aggregate shocks are independent from one subpopulation type to another[6].

[6] In Cabrales' model, the aggregate shocks are correlated and the same over the population.

Remark In the case of a small sized population, the perturbations on each individuals are of paramount importance in the outcome of a game due to the impossibility to counterbalance their impacts contrary to a large sized population, in which we can assume that the shocks are offset.

To conclude, *Wiener process* seems appropriate as a stochastic process to model aggregate shocks since it meets all the assumptions.

Definition: stochastic process and Wiener process

A *stochastic process* is a collection of random variables $\{X_t\}_{t \in T}$ defined on a probability space $(\Omega, \mathcal{F}, P)$ and assuming values in $\mathbb{R}^n$. The parameter space T is usually the halfline $[0, \infty)$. For each $t \in T$ fixed, we have a random variable $\omega \in \Omega \to X_t(\omega)$. On the other hand, fixing $\omega \in \Omega$, we can consider the function $t \in T \to X_t(\omega)$ which is called a *path* of X_t.

Wiener process, also called *Brownian motion*, is a continuous time stochastic process (i.e. it has a continuous path and t is seen as a time parameter), with independent increments across time and with mean zero. Plus, the random variables of a Wiener process are Gaussian process (i.e. they follow a (multi)normal distribution).

Notation Hereafter, we denote a n-dimensional Wiener process by $W = (W_1, ..., W_n)$.

6.2 The Itô's formula

Let $W(t,\omega) = \big(W_1(t,\omega), ..., W_n(t,\omega)\big)$ denote a n-dimensional Wiener process. If we have:

$$dX(t) = udt + vdW(t)$$

where

$$X(t) = \begin{pmatrix} X_1(t) \\ \vdots \\ X_n(t) \end{pmatrix}, \; u = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}, \; v = \begin{pmatrix} v_{11} & \cdots & v_{1n} \\ \vdots & & \vdots \\ v_{n1} & \cdots & v_{nn} \end{pmatrix}, \; dW(t) = \begin{pmatrix} dW_1(t) \\ \vdots \\ dW_n(t) \end{pmatrix}$$

and in the particular case where $u_i(t)$ and $v_{ij}(t)$ are integrable functions,
then, $X(t)$ is an *n-dimensional Itô process*.

Theorem: The general Itô formula

Let $dX(t) = udt + vdW(t)$ be an n-dimensional Itô process as above. Let $f(t,x) = \big(f_1(t,x), ..., f_p(t,x)\big)$ be a C^2 map from $[0,\infty) \times \mathbb{R}^n$ into $\mathbb{R}^p$. Then the process,

$$Y(t,\omega) = f\big(t, X(t)\big)$$

is again an Itô process, whose component number i, Y_i, is given by

$$dY_i = \frac{\partial f_i}{\partial t}(t,X)\mathrm{d}t + \sum_j \frac{\partial f_i}{\partial x_j}(t,X)\mathrm{d}X_j + \frac{1}{2}\sum_{j,k} \frac{\partial^2 f_i}{\partial x_j \partial x_k}(t,X)\mathrm{d}X_j\mathrm{d}X_k \qquad (IF)$$

where $dW_i dW_j = \delta_{ij}dt$, and $dW_i dt = dtdW_i = 0$.

Remark The theorem will not be proven in this report. For details, see Oksendal (2000).

7 Brief historic approach to model stochasticity in replicator dynamics

To understand the research method that led to stochastic version of replicator dynamics, let's look at the two first and principal approaches to stochasticity in replicator dynamics.

7.1 The first introduction of stochastic process in replicator dynamics Foster and Young (1990)

Foster and Young, in their article "Stochastic Evolutionary Game Dynamics" (1990) were the first to consider a stochastic-differential-equation model of evolutionary dynamics. Willing to add stochasticity in the replicator dynamics, they chose to develop a model in which the stochastic perturbation is directly added to the equation. However, one of the problems they confronted is the possibility to have a solution with negative population shares which is an impossible case regarding the biologic interpretation. In order to bypass this problem, they added to their model a supplementary term helping to make it well-defined.

7.2 The model: first conclusive approach Fudenberg and Harris (1992)

In 1992, Fudenberg and Harris presented an alternative to Foster and Young stochastic model in their article "Evolutionary Dynamics with Aggregate Shocks". Instead of adding the stochastic perturbations directly to the replicator equation, they added the perturbations − a n-dimensional Wiener process W − in the population growth rate, more precisely to the expected payoff. This way, the population sizes remain positive for any realization of W_i. Furthermore, biologically speaking, it gets a better interpretation as they are seen as *aggregate shocks*. Fudenberg and Harris used the example of the randomness of the weather to illustrate the impacts of these perturbations.

In the replicator equation, the stochasticity appears after the derivation of $x_i(t) = \frac{r_i(t)}{R(t)}$. To derivate, they suppose that r is an Itô process and then apply the Itô's formula to $x_i(t) = f_i\big(r(t)\big)$. As they studied the equilibrium selection problem, they considered two by two games[7] and do not generalized the model to n players and strategies by convenience.

8 The model: general stochastic version of replicator dynamics

In his article "Stochastic Replicator Dynamics" (2000), Cabrales generalizes the model of Fudenberg and Harris (1992). Instead of considering a symmetric game with two players and two strategies, he gives a model for a game with N populations (the game is not symmetric in this case). The players of the ith population has n_i pure strategies. We will consider the case of a single population ($N = 1$) with n pure strategies ($n_i = n$). In this case, the game is symmetric.

In his model, Cabrales uses a d-dimensional Wiener process W and consider a weighten sum of the components of W, with weights that depend on the strategies, to model aggregate shocks. The same way Fudenberg and Harris (1992) added the perturbations to the payoff, in the model of Cabrales (2000), the total payoff of an individual of type i is given by

$$u_i[x(t)]dt + \sum_{l=1}^{d} \sigma_{il} dW_l(t).$$

where σ_{il} is a d-dimensional vector of positive constants.[8]

We consider the case where $d = N$, and $\sigma_{il} = \sigma_i$ if $i = l$ and 0 otherwise. Thus, in our model, the **stochastic version of population growth** is given by:

$$dr_i(t) = r_i(t)\Big(\big(B[t, r(t)] - D[t, r(t)]\big)dt + u_i[x(t)]dt + \sigma_i dW_i(t)\Big) \tag{3}$$

where B is the background fitness and D is the death rate, u_i the payoff function, σ_i a n-dimensional vector of positive constants, and W a n-dimensional Wiener process.

By applying the general Itô's formula to the Itô process $x_i(t) = r_i(t)/R(t) = f_i(r(t))$ we get the **stochastic version of replicator equation**:

$$dx_i = x_i\left(u_i(x)dt + \sigma_i dW_i - \sum_{j\in I} x_j\big(u_j(x)dt + \sigma_j dW_j\big) - x_i\sigma_i^2 dt + \sum_{j\in I} x_j^2\sigma_j^2 dt\right) \tag{4}$$

Computation of (4): see appendix.

[7] Games with only two players and two strategies.

[8] The constant σ_{il} gives a weight to the impact of the perturbation according to the strategy chosen, in this case the impact of the lth perturbation on the ith pure strategies.
Remark: in Cabrales' model, there are d sources of shocks.

Part IV. The question of elimination of strictly dominated strategies under a replicator dynamic

Replicator dynamics provides a means to predict the outcome of a game, in other words, to determine the strategy that would be selected at the end of the game. In a rational standpoint, strictly dominated strategies (defined below) should be eliminated by the selection dynamics whichever the parameters of the game. However, it is not clear whether or not strictly dominated strategies are eliminated over time by the replicator equation since the form of the replicator equation "let" grow strictly dominated strategies within the population in the case they earn a payoff above the current population average. [9]

Questioning the elimination of strictly dominated strategies regarding replicator dynamics amounts to discuss whether or not rational type arguments can be made to predict outcomes of population dynamics in the case of selection. Besides, since the stochastic version of replicator dynamics provides a different model for selection than the deterministic version, it also raises the question of elimination or survival of strictly dominated strategies. Subsequently we want to know if the introduction of such shocks alters the properties of replicator dynamics.

9 Basic concepts and definitions

9.1 Strict domination of pure strategies

We say that the pure strategy i is *strictly dominated* if there exists a strategy $y \in S_n$ such that

$$u_i(x) < \sum_j y_j u_j(x) \quad \text{for all } x \in S_n.$$

In the present case, we say that the strategy y *strictly dominates* i.

Particular case : pure strategies vs pure strategies

The pure strategy i is *strictly dominated* by the pure strategy j if

$$u_i(x) < u_j(x) \quad \text{for all } x \in S_n.$$

Strict iterative domination

We say that i is *strictly iteratively dominated* if it is strictly dominated or becomes strictly dominated after removing the strictly dominated strategies from the game or more generally, if there exists a certain positive integer k such that after k iteratively repeated removal of strictly dominated strategies, the strategy we consider becomes strictly dominated.

9.2 Elimination of strategies over time

A pure strategy i is *eliminated* under a solution $x(\cdot)$ of (2) if $x_i(t) \longrightarrow 0$ as $t \to \infty$, that is to say, if the frequency of the population using the strategy i converges to zero. It *survives* otherwise.

A mixed strategy q is *eliminated* if

$$\min_{i \in I : q_i > 0} x_i(t) \longrightarrow 0 \quad \text{as } t \to \infty$$

$$\text{or} \qquad \prod_{i \in I} x_i^{q_i} \longrightarrow 0 \qquad \text{as } t \to \infty$$

10 Elimination under deterministic replicator dynamics

10.1 Pure strategy strictly dominated by a pure strategy

Theorem 1 *If a pure strategy i is strictly dominated by another pure strategy, then i is eliminated for any $x(0) \in \mathrm{int} S_n$.*

[9] Remark: In the case of discrete-time version of replicator dynamics, it has been demonstrated that strictly dominated strategies can actually survive to the selection dynamics of replicators. See Dekel and Scotchmer (1992).

PROOF: Suppose that the pure strategy i is strictly dominated by the pure strategy j. Let's define the function $v_i : \text{int}(S_n) \to \mathbb{R}$ by

$$v_i(x) = \log\left(\frac{x_i}{x_j}\right) = \log(x_i) - \log(x_j)$$

v_i is differentiable and its time derivative is :

$$\dot{v}_i(x) = \frac{\dot{x}_i}{x_i} - \frac{\dot{x}_j}{x_j} = u_i(x) - u_j(x) \quad \text{by definition of the replicator equation}$$

Since i is strictly dominated by j we have $u_i < u_j$. By continuity of $u_i - u_j$ and compactness of S_n there exists $\epsilon > 0$ such that $\dot{v}_i(x) < -\epsilon$. Thus we have,

$$v_i(x(t)) = \log\left(\frac{x_i(t)}{x_j(t)}\right) \longrightarrow -\infty \quad \text{as } t \to \infty$$

This implies $\dfrac{x_i(t)}{x_j(t)} \longrightarrow 0 \quad \text{as } t \to \infty$

And so $x_i(t) \longrightarrow 0 \quad \text{as } t \to \infty$

10.2 Pure strategy strictly dominated by a mixed strategy

Theorem 2 *If the pure strategy i is strictly dominated, then i is eliminated for any $x(0) \in \text{int}S_n$.*

PROOF: Assume that i is strictly dominated by $y \in S_n$. Let's define the function $v_i : \text{int}(S_n) \to \mathbb{R}$ by

$$v_i(x) = \log\left(\frac{x_i}{\prod_j x_j^{y_j}}\right) = \log(x_i) - \sum_j y_j \log(x_j)$$

v_i is differentiable and its time derivative is :

$$\dot{v}_i(x) = \frac{\dot{x}_i}{x_i} - \sum_j y_j \frac{\dot{x}_j}{x_j} = u_i(x) - \sum_j y_j u_j(x) \quad \text{by definition of the replicator equation}$$

Since i is strictly dominated by y we have $u_i < \sum_j y_j u_j$. By continuity of $u_i - \sum_j y_j u_j$ and compactness of S_n, there exists $\epsilon > 0$ such that

$$u_i(x) - \sum_j y_j u_j(x) < -\epsilon \qquad \text{for all } x \in S_n$$

Thus we have,

$$v_i(x(t)) = \log\left(\frac{x_i(t)}{\prod_j x_j^{y_j}(t)}\right) \longrightarrow -\infty \quad \text{as } t \to \infty$$

This implies $\dfrac{x_i(t)}{\prod_j x_j^{y_j}(t)} \longrightarrow 0 \quad \text{as } t \to \infty$

And so $x_i(t) \longrightarrow 0 \quad \text{as } t \to \infty$

11 Elimination under stochastic replicator dynamics - Cabrales' model (2000)

We mainly based the contents of this section on the article of Cabrales (2000). Cabrales formulated two propositions concerning the question of elimination of strictly dominated strategies. Regarding the propositions of Cabrales (2000) and using the same notations and assumptions as in parts II and III, we consider the case of strict domination and not strict iteratively domination (i.e. the case of a unique iteration)[10]. Under these assumptions, the propositions are stated as follows.

[10] With Cabrales' notations, we suppose $N = 1$ (i.e. there is a unique population), $d = n_i = n$ and $\sigma^i_{\alpha l} = 0$ if $\alpha \neq l$ (i.e. the number of Wiener process is equal to the number of pure strategies an the ith Wiener process affects only player i). We also suppose that there is no mutation, that is to say that $\lambda^i_{\alpha\beta} = 0$.

Proposition 3 Let strategy $p \in S_N$ be strictly dominated. There exists $\bar{\sigma}_p > 0$ such that if $\max_i\{\sigma_i\} < \bar{\sigma}_p$ then,

$$\lim_{t\to\infty} \prod_{i\in I} x_i(t)^{p_i} = 0 \quad \text{a.s.} \tag{5}$$

Proposition 4 Let pure strategy $i \in I$ be strictly dominated by a strategy $p \in S_N$. There exists $\bar{\sigma}_p > 0$ such that if $\max_i\{\sigma_i\} < \bar{\sigma}_p$ then,

$$\limsup_{t\to\infty} \mathbb{E}\big[x_i(t)\big] = 0 \tag{6}$$

Interpretations

Proposition 3 states that if there is little perturbations, the strict dominated mixed strategies are eliminated. Proposition 4 states that the frequency of a strictly dominated pure strategy tends in mean to zero, that is to say is eliminated over time when the population is large enough. The results given by Cabrales (2000) are thus intuitive.

12 Example of another perspective: Hofbauer and Imhof (2009)

The aim of the article "Time averages, recurrence and transience in the stochastic replicator dynamics" was to "provide further insight into the long-run behaviour" of $x(t)$ as a stochastic process. In order to do that, Hofbauer and Imhof introduced the modified payoff

$$\bar{u}_i\big(x(t)\big) = u_i\big(x(t)\big) - \frac{1}{2}\sigma_i^2 \tag{7}$$

where u_i is the payoff function associated to the individual of type i.[11]

The modified game has no biologic interpretation but has interesting mathematical properties that permit to study elimination of strictly dominated strategies. Indeed, from **Corollary 4.2**, they state that if a strategy is eliminated in the modified game, that is, in the game using the modified payoff, then this strategy is eliminated a.s.. As a consequence, strategies that are dominating in the unmodified game but strictly dominated in the modified game are getting eliminated. This surprising results shows that it is possible to "reverse" the domination in a game if perturbations are too important against a strategy.

[11] In the original paper, Hofbauer and Imhof use matrix notations (see equation (1.5) for the modified payoff matrix).

Conclusion

Even if the deterministic replicator equation does not provide a perfect model for natural selection, it has been a great advance in evolutionary game theory and as it is proved by its now extended to social sciences as well as other modelings than natural selection. For instance, instead of considering the inherited aspects of strategies from parent to offspring, it can be seen as a model for imitation or learning mechanisms. The introduction of stochasticity helped to improve the model by considering exterior parameters in the game and thus provided a mode which isl closer to reality. By means of stochastic processes added to expected payoffs, the impact of phenomenons associated to the randomness of the environment of a species such as the weather or the impact of the presence of other species on the population could be modeled by replicator dynamics.

The interest of replicator dynamics is not only to provide a better understanding of the mechanisms of selections of strategies through its modeling but also lies within the outcomes of this model. Considering which strategies are eliminated through the selection of replicator dynamics is one approach to predict the outcome of a game. We have seen that when it comes to the deterministic equation, replicator dynamics eliminates the strictly dominated strategies. A question raised was then to figure out if this statement is the same for the stochastic version of replicator dynamics. If we take as a basis the work of Cabrales, it appears that if the perturbations are little enough, strictly dominated strategies are also eliminated. Others, such as Hofbauer and Imhof (2009), addressed the question of elimination of strictly dominated strategies under stochastic replicator dynamics from a different perspective: by introducing a modified game, they come to the conclusion that strictly dominated strategies in the modified game are eliminated even if they are dominating the unmodified game. This way, it was shown that important perturbations can lead to the elimination of dominating strategies.

To conclude, the model can be enlarged to N populations as Cabrales did in 2000. In addition to the fact that perturbations can take different forms, not only we can consider exterior perturbations but shocks from the inside of the population. One way to add "interior" perturbations is to include mutations in the model: since the selected aspect of replicator dynamics does not permit the introduction of new strategies in the game, mutation terms can partly cover this problem. Finally, the models that we described in this report are a very little part of the literature on replicator dynamics and many approaches can be followed to make use of the properties of replicator dynamics.

Part V. Appendix

13 Computation of (4): Stochastic replicator equation

We want to compute, the stochastic version of the deterministic replicator equation (2).

We have $x_i(t) = \dfrac{r_i(t)}{R(t)} = f_i(r(t))$ with r_i given by (3). By the general Itô's formula (IF) we get:

$$\mathrm{d}x_i = \frac{\partial f_i(r)}{\partial t}\mathrm{d}t + \sum_j \frac{\partial f_i(r)}{\partial r_j}\mathrm{d}r_j + \frac{1}{2}\sum_{j,k}\frac{\partial^2 f_i(r)}{\partial r_j \partial r_k}\mathrm{d}r_j\mathrm{d}r_k \tag{4.1}$$

$$= \sum_j \frac{\partial f_i(r)}{\partial r_j}\mathrm{d}r_j + \frac{1}{2}\sum_{j,k}\frac{\partial^2 f_i(r)}{\partial r_j \partial r_k}\mathrm{d}r_k\mathrm{d}r_j \tag{4.2}$$

where $\mathrm{d}r_j\mathrm{d}r_k = r_j^2\sigma_j^2\mathrm{d}t$ if $j = k$, and 0 otherwise.

We have (4.2) because f_i is autonomous, and so the differential of f_i in time is equal to zero.
We develop the calculus. Let's remind that we have: $R(t) = \sum_{i=1}^{d} r_i(t)$.

$$\frac{\partial f_i(r)}{\partial r_j} = \frac{\partial}{\partial r_j}\left(\frac{r_i}{R}\right) = \frac{\partial r_i}{\partial r_j}\frac{1}{R} + r_i\frac{\partial}{\partial r_j}\left(\frac{1}{R}\right) = \frac{\partial r_i}{\partial r_j}\frac{1}{R} + r_i\frac{\partial}{\partial r_j}\left(\frac{1}{\sum_{i=1}^{d} r_i}\right) = \frac{\partial r_i}{\partial r_j}\frac{1}{R} - r_i\frac{1}{R^2}$$

$$\begin{aligned}
\frac{\partial^2 f_i(r)}{\partial r_k \partial r_j} &= \frac{\partial}{\partial r_k}\left[\frac{\partial r_i}{\partial r_j}\frac{1}{R} + r_i\frac{\partial}{\partial r_j}\left(\frac{1}{R}\right)\right] \\
&= \frac{\partial^2 r_i}{\partial r_k \partial r_j}\frac{1}{R} + \frac{\partial r_i}{\partial r_j}\frac{\partial}{\partial r_k}\left(\frac{1}{R}\right) + \frac{\partial r_i}{\partial r_k}\frac{\partial}{\partial r_j}\left(\frac{1}{R}\right) + r_i\frac{\partial}{\partial r_j \partial r_k}\left(\frac{1}{R}\right) \\
&= \frac{\partial^2 r_i}{\partial r_k \partial r_j}\frac{1}{R} - \frac{\partial r_i}{\partial r_j}\frac{1}{R^2} - \frac{\partial r_i}{\partial r_k}\frac{1}{R^2} - r_i\frac{\partial}{\partial r_k}\left(\frac{1}{R^2}\right) \\
&= \frac{\partial^2 r_i}{\partial r_k \partial r_j}\frac{1}{R} - \frac{\partial r_i}{\partial r_j}\frac{1}{R^2} - \frac{\partial r_i}{\partial r_k}\frac{1}{R^2} + 2r_i\frac{1}{R^3}
\end{aligned}$$

Then, we compute the global equation:

$$\begin{aligned}
dx_i &= \sum_j \frac{\partial f_i(r)}{\partial r_j}dr_j + \frac{1}{2}\sum_{j,k}\frac{\partial^2 f_i(r)}{\partial r_j \partial r_k}dr_k dr_j \\
&= \sum_j\left[\frac{\partial r_i}{\partial r_j}\frac{1}{R} - r_i\frac{1}{R^2}\right]dr_j + \frac{1}{2}\sum_{j,k}\left[\frac{\partial^2 r_i}{\partial r_k \partial r_j}\frac{1}{R} - \frac{\partial r_i}{\partial r_j}\frac{1}{R^2} - \frac{\partial r_i}{\partial r_k}\frac{1}{R^2} + 2r_i\frac{1}{R^3}\right]dr_k dr_j \\
&= \frac{dr_i}{R} - \frac{r_i}{R}\sum_j \frac{dr_j}{R} - \frac{1}{2}\sum_k \frac{1}{R^2}dr_i dr_k - \frac{1}{2}\sum_j \frac{1}{R^2}dr_i dr_j + \frac{1}{2}\sum_{j,k} 2r_i\frac{1}{R^3}dr_j dr_k \\
&= \frac{dr_i}{R} - \frac{r_i}{R}\sum_j \frac{dr_j}{R} - \sum_k \frac{1}{R^2}dr_i dr_k + \frac{r_i}{R}\sum_{j,k}\frac{1}{R^2}dr_j dr_k
\end{aligned}$$

By Itô's lemma,we have $dr_j dr_k = r_j^2\sigma_j^2 dt$ if $j = k$, and 0 otherwise [12]. Hence,

$$dx_i = \frac{dr_i}{R} - \frac{r_i}{R}\sum_j \frac{dr_j}{R} - \frac{dr_i^2}{R^2} + \frac{r_i}{R}\sum_k \frac{dr_k^2}{R^2}$$

[12] Indeed, Itô's lemma says that $dW_i dW_j = \delta_{ij}dt$ and that $dt dB_i = dB_i dt = 0$. Then, we have $dr_i dr_j = 0$ for all $i \neq j$ and $dr_i^2 = r_i^2(u_i^2 dt^2 + \sigma_i^2 dW_i^2) = r_i^2\sigma_i^2 dt$ as stated.

Finally,

$$dx_i = \frac{dr_i}{R} - \frac{r_i}{R}\sum_j \frac{dr_j}{R} - \frac{r_i^2}{R^2}\sigma_i^2 dt + \frac{r_i}{R}\sum_k \frac{r_k^2}{R^2}\sigma_k^2 dt \tag{4.3}$$

By (3), we have

$$dr_i(t) = r_i(t)\Big(\big(B[t, r(t)] - D[t, r(t)]\big)dt + u_i[x(t)]dt + \sigma_i dW_i(t)\Big) \tag{3}$$

Whose expression will be simplified by:

$$dr_i = r_i\Big(\big(B_t[r] - D_t[r]\big)dt + u_i[x]dt + \sigma_i dW_i\Big) \tag{3'}$$

We integrate (3') in the expression (4.3):

$$\begin{aligned}
dx_i = {} & \frac{r_i}{R}\Big(\big(B_t[r] - D_t[r]\big)dt + u_i[x]dt + \sigma_i dW_i\Big) - \frac{r_i}{R}\sum_j \frac{r_j}{R}\Big(\big(B_t[r] - D_t[r]\big)dt + u_j[x]dt + \sigma_j dW_j\Big) \\
& - \frac{r_i^2}{R^2}\sigma_i^2 dt + \frac{r_i}{R}\sum_k \frac{r_k^2}{R^2}\sigma_k^2 dt \qquad (4.4) \\
= {} & x_i\big(B_t[r] - D_t[r]\big)dt + x_i\Big(u_i[x]dt + \sigma_i dW_i\Big) \\
& - x_i\sum_j x_j\big(B_t[r] - D_t[r]\big)dt - x_i\sum_j x_j\Big(u_j[x]dt + \sigma_j dW_j\Big) \\
& - x_i^2\sigma_i^2 dt + x_i\sum_k x_k^2\sigma_k^2 dt \qquad (4.5) \\
= {} & x_i\Big(\big(B_t[r] - D_t[r]\big)dt - \sum_j x_j \cdot \big(B_t[r] - D_t[r]\big)dt\Big) \\
& + x_i\Bigg(\big(u_i[x]dt + \sigma_i dW_i\big) - \sum_j x_j\big(u_j[x]dt + \sigma_j dW_j\big)\Bigg) \\
& - x_i^2\sigma_i^2 dt + x_i\sum_k x_k^2\sigma_k^2 dt \qquad (4.6) \\
= {} & x_i\Bigg(u_i[x]dt + \sigma_i dW_i - \sum_j x_j\big(u_j[x]dt + \sigma_j dW_j\big) - x_i\sigma_i^2 dt + \sum_k x_k^2\sigma_k^2 dt\Bigg) \qquad (4.7)
\end{aligned}$$

Remarks: To get (4.5), we use $x_i = r_i/R$. From (4.6) to (4.7) we use the fact that $\sum_j x_j = 1$.

(4.7)=(4), thus we have computed the stochastic version of replicator dynamics.

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