A CHARACTERIZATION OF SETS OF EQUILIBRIUM PAYOFFS OF FINITE GAMES WITH AT LEAST 3 PLAYERS

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Abstract. The set of equilibrium payoffs of any finite game with N players is a nonempty, compact and semi-algebraic subset of $\mathbb{R}^N$. We establish the converse as long as $N \geq 3$: for any nonempty, compact and semialgebraic set E in $\mathbb{R}^N$, there exists a game with N players such that E is the set of equilibrium payoffs of this game. In addition, if the semialgebraic set is defined by polynomials with integer coefficients, the game can be constructed with integer payoffs. Related results hold when one considers sets of equilibria instead of sets of equilibrium payoffs. The proofs are constructive and hence have implications on the complexity and computability of some decision problems on 3-player games.

1. Introduction

It is well known that the set of (mixed) Nash equilibria or of Nash equilibrium payoffs of any finite game is nonempty, compact and semi-algebraic. In the particular case of two players, more is known: the set of Nash equilibria is a finite union of convex polytopes [8], and a subset F of $\mathbb{R}^2$ is the set of Nash equilibrium payoffs of a bimatrix game if and only if [9] it is of the form: $F = \cup_{1 \leq i \leq K}[a_i, b_i] \times [c_i, d_i]$, where $K \in \mathbb{N}$.

For 3 players or more, Datta [5] showed that any real algebraic variety is isomorphic to the set of completely mixed Nash equilibria of a 3-player game, and also to the set of completely mixed equilibria of an N-player game in which each player has two strategies. Balkenborg and Vermeulen [1] showed that any nonempty connected compact semi-algebraic set is homeomorphic to a connected component of the set of Nash equilibria of a finite binary game. Independantly, Levy [10], and Vigeral and Viossat [14] established that any nonempty compact semi-algebraic set in $\mathbb{R}^N$ is the projection on some coordinates of the set of equilibria, or of equilibrium payoffs, of a game with $n > N$ players.

These results show that all nonempty compact semi-algebraic sets may be encoded as sets of Nash equilibria, up to some isomorphisms, homeomorphisms, or projections, In the case of equilibrium payoffs we give a definitive answer to this line of research by proving that if $N \geq 3$ any nonempty, compact and semialgebraic subset of $\mathbb{R}^N$ is the set of equilibrium payoff of some N player game. Thus there is no need for any isomorphism, homeomorphism, or projection. Our proof is based on a related result on sets of equilibria: any nonempty compact and semialgebraic subset of $[0, 1[^N$ is the projection of the set of equilibria of some N-player game on the first component of each player. Moreover, if the polynomials involved in the definition of the set are with coefficients in $\mathbb{Z}$, the game has payoffs in $\mathbb{Z}$ as well.

All our proofs are constructive and elementary in the sense that we do not use any result from real algebraic geometry. As a first consequence, while decision problems on sets of equilibria of 2-player games are typically NP-complete [7], we prove that for three players or more the same type of problems are exactly as hard as deciding whether or not a $\mathbb{Z}$-semi algebraic set is nonempty. As a second consequence, decision problems on equilibria involving integers might be undecidable, because of the negative answer to Hilbert tenth problem on Diophantine equations [11].

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This article is organized as follows: we introduce some definitions and notations in Section 2, which allows us to state precisely our main results in Section 3. Section 4 is devoted to the proof of the main results in some particular cases which allow for a simpler construction ; in Section 5 the additional elements needed in the general case are given. In Section 6 we verify that our constructions are polynomial in the size of the imput, which is important for later complexity results. Section 7 is devoted to important generalizations of our results in two directions: to games with integer payoffs, and to projections on several actions per player. In sections 8 and 9 we apply our constructions to investigate, respectively, the complexity and computability of some decison problems on 3-player games.

2. Definitions and notations

2.1. **Finite games.** A finite game with N players will typically be denoted as Γ, its sets of pure actions as $\mathcal{A}^i$ and its payoff functions as g^i (we will write g^i both for the function defined on pure strategies and its multilinear extension). An action of Player i will be denoted with uppercase letters, most often in $\{A, B, W, X, Y\}$. A superscript will identify the player: for example A_3^2 is a particular pure action of Player 2. For any subset S of the players, we denote $\mathcal{A}^S$ the set $\prod_{i\in S}\mathcal{A}^i$. If A is in $\mathcal{A}^N$ (the set of all pure strategy profiles), then for any S we denote as A^S the corresponding profile in $\mathcal{A}^S$; and we write simply A^{-i} for $A^{N\setminus\{i\}}$. A mixed action profile will typically be denoted as σ, $\sigma^i(A^i)$ (or simply $\sigma(A^i)$) being the probability that player i plays its pure action A^i. The set of (mixed) Nash equilibria (resp. of Nash equilibrium payoffs) of Γ is denoted $\mathrm{NE}(\Gamma)$ (resp. $\mathrm{NEP}(\Gamma)$).

Recall that two N-player games Γ and Γ' are strategically equivalent if they share the same action sets $\mathcal{A}^i$ and if their payoff functions g and g' satisfy

$$g'^i(a^i, a^{-i}) = \alpha^i g^i(a^i, a^{-i}) + h^i(a^{-i}),\ \forall i \in N,\ \forall a \in \mathcal{A}^N,$$

for some positive α^i and some functions $h^i : \mathcal{A}^{-i} \to \mathbb{R}$. It is immediate that

Lemma 1. *If Γ and Γ' are strategically equivalent , then* $\mathrm{NE}(\Gamma) = \mathrm{NE}(\Gamma')$.

It turns out that writing games in normal form is not convenient in our framework and we thus use another way of defining games. We say that a map f from $\Pi_{j\neq i}\Delta(\mathcal{A}^i)$ to $\mathbb{R}$ is multiaffine if it can be written as

$$f(\sigma^{-i}) = \sum_{S\subset N\setminus\{i\}} \sum_{A_S\in\mathcal{A}^S} \lambda_{A_S} \prod_{j\in S} \sigma^j(A_S^j)$$

where the λ_{A_S} are reals. We then apply (most often implicitely) the following easy lemma

Lemma 2. *For every game Γ, each player i and each pure action A^i, the multilinear extension of $g^i(A^i,\cdot)$ is multiaffine from $\Pi_{j\neq i}\Delta(\mathcal{A}^i)$ to $\mathbb{R}$. Conversely, if we are given a collection of maps $f^i_{A^i}$ such that each $f^i_{A^i}$ is multiaffine from $\Pi_{j\neq i}\Delta(\mathcal{A}^i)$ to $\mathbb{R}$, there exists a unique game Γ whose multilinear extensions of the payoff functions satisfy $g^i(A^i,\sigma^{-i}) = f^i_{A^i}(\sigma^{-i})$. If in addition all coefficients λ involved in the definition of all $f^i_{A^i}$ are integers, then all pure payoffs in Γ are integers as well.*

Proof. The first part is clear. For the second part, if

$$f^i_{A^i}(\sigma^{-i}) = \sum_{S\subset N\setminus\{i\}} \sum_{A_S\in\mathcal{A}^S} \lambda_{A^i,A_S} \prod_{j\in S} \sigma^j(A_S^j)$$

then the only possible candidate Γ is the one with $g^i(A) = f^i_{A^i}(A^{-i}) = \sum_{S\subset N\setminus\{i\}} \lambda_{A^i,A^S}$ and by multilinearity we then also have $g^i(A^i,\sigma^{-i}) = f^i_{A^i}(\sigma^{-i})$ for every σ^{-i}. □

Hence to define a game it is enough to stipulate all functions $g^i(A^i,\sigma^{-i})$, as long as they are multiaffine.

To ease the reading we will use the corresponding lowercase letter to represent the probability that some pure action is played: for example we will write $a_3^2(\sigma)$ instead of $\sigma(A_3^2)$. Also, we will most often drop all references to σ, for example we will just write a_3^2 instead of $a_3^2(\sigma)$, and $g^i(A^i)$ instead of $g^i(A^i,\sigma^{-i})$. As an example, the three player game written in normal form as

$$\begin{array}{cc} & \begin{array}{cc} A_1^2 & A_2^2 \end{array} \\ \begin{array}{c} A_1^1 \\ A_2^1 \end{array} & \begin{pmatrix} (0,0,0) & (0,0,0) \\ (0,0,0) & (0,0,0) \end{pmatrix} \\ & A_1^3 \end{array} \qquad \begin{array}{c} \begin{array}{cc} A_1^2 & A_2^2 \end{array} \\ \begin{pmatrix} (0,0,0) & (0,1,1) \\ (0,0,1) & (1,1,1) \end{pmatrix} \\ A_2^3 \end{array}$$

may be defined as

$$\begin{aligned} g^1(A_1^1) &= g^2(A_1^2) = g^3(A_1^3) = 0 \\ g^1(A_2^1) &= a_2^2 a_2^3 \\ g^2(A_2^2) &= a_2^3 \\ g^3(A_2^3) &= 1 - a_1^1 a_1^2 \end{aligned}$$

2.2. **Polynomials.** A (multivariate) polynomial will be denoted with an uppercase letter, usually P or Q ; for the (multidimensional) unknown we will use the letter z. The i-th coordinate of z is denoted z_i. For any integers N and D define $\mathbb{N}_D^N := \mathbb{N}^N \cap [0,D]^N \setminus \{0\}$ the set of nonzero N-uples of integers between 0 and D . For any $d = (d_1, \cdots, d_N) \in \mathbb{N}_D^N$ one writes z^d for $\prod_k (z_k)^{d_k}$. Hence any polynomial in N variables and maximum degree D in each variable can be written as

$$P(z) = c + \sum_{d \in \mathbb{N}_D^N} c_d z^d.$$

We will denote $e^i \in \mathbb{N}_D^N$ the N-uple with ith coordinate equal to 1 and all others to 0, hence $z^{e^i} = z_i$. Finally, when we need to write some power of the probability that a player plays some action, we will reserve the superscript for the numbering of the player. For example $(a_3^2)^5$ is the probability that Player 2 plays his action A_3^2, to the power 5.

2.3. **Semi algebraic sets.** In this section we recall some facts about semi algebraic sets that will be used in the paper. The reader interested in proofs of these results is refered to the literature on the subject, for example [4]. Let us first recall the definition of a semi algebraic set.

Definition 3. A set $F \subset \mathbb{R}^n$ is a semi algebraic set (resp. a basic semi algebraic set) if it can be written as a finite union and interection (resp. as a finite intersection) of sets of the form $\{x \in \mathbb{R}^n, P_k(x) \leq 0\}$ and $\{x \in \mathbb{R}^n, P_k(x) < 0\}$, where the P_k are polynomials.

A fundamental result, due to Tarski and Seidenberg is the following:

Theorem 4. *Let $F \subset \mathbb{R}^n$ be a semi algebraic set, and $\pi : \mathbb{R}^n \to \mathbb{R}^{n-1}$ be the projection on the first $n-1$ coordinates. Then $\pi(F)$ is a semi algebraic set.*

An easy corollary, that we will also call Tarski-Seidenberg theorem for conveniance, is

Corollary 5. *Let $F \subset \mathbb{R}^n$ be a semi algebraic set, and $f : \mathbb{R}^n \to \mathbb{R}^m$ a polynomial mapping. Then $f(F)$ is semi algebraic.*

We also recall the following nontrivial consequence of Tarski-Seidenberg theorem, known in the literature as the finiteness theorem:

Proposition 6. *Any closed semi algebraic set in $\mathbb{R}^n$ can be written as a finite union and intersection of sets of the form $\{x \in \mathbb{R}^n, P_k(x) \leq 0\}$.*

For a fixed N-player finite game in which player i action set is $\mathcal{A}^i$, and for any $\mathcal{B} \subset \cup_{i=1}^N \mathcal{A}^i$, denote as Proj_B the map that send a mixed strategy $\sigma = (\sigma(a))_{a \in \cup_{i=1}^N \mathcal{A}^i}$ to $\mathrm{Proj}_B(\sigma) := (\sigma(b))_{b \in \mathcal{B}}$. The function Proj_B is a projection hence a polynomial mapping.

We can now state the following proposition, relating games and semi algebraic sets.

Proposition 7. *Let Γ be an N-player finite game in which player i action set is $\mathcal{A}^i$. Then all these sets are nonempty, compact and semi algebraic:*

- *The set* $\mathrm{NE}(\Gamma)$ *of Nash equilibria of* Γ.
- *The set* $\mathrm{NEP}(\Gamma)$ *of Nash equilibrium payoffs of* Γ.
- *For any* $\mathcal{B} \subset \cup_{i=1}^{N} \mathcal{A}^i$*, the set* $\mathrm{Proj}_B(\mathrm{NE}(\Gamma))$.

This proposition is well known but let us give a short proof for completness:

Proof. $\sigma \in \mathrm{NE}(\Gamma)$ if and only if it satisfies the following polynomial inequalities:

- $\sigma^i(a^i) \geq 0$, for all i and $a^i \in \mathcal{A}^i$
- $\sum_{a^i \in \mathcal{A}^i} \sigma^i(a^i) - 1 \leq 0$ and $-\sum_{a^i \in \mathcal{A}^i} \sigma^i(a^i) + 1 \leq 0$ for all i.
- $\sigma^i(a^i)[g^i(b^i, \sigma^{-i}) - g^i(a^i, \sigma^{-i})] \leq 0$ for all i and a^i, b^i in $\mathcal{A}^i$, where g^i is the payoff function of player i.

Hence $\mathrm{NE}(\Gamma)$ is a closed (basic) semi algebraic set. It is also clearly bounded, and nonempty by Nash's theorem. To conclude, one remarks that the image of a nonempty, compact and semi algebraic set by a polynomial mapping is nonempty, compact and, by Tarski-Seidenberg theorem, semi algebraic. □

3. Statement of the first main results

Our main result is

Theorem 8. *Let $N \geq 3$ be an integer. A set $F \subset \mathbb{R}^N$ is the set of equilibrium payoffs of some finite N-player game if and only if F is nonempty, compact, and semi algebraic.*

It turns out that the following proposition is the key ingredient in the proof of the previous Theorem. Moreover, it will have implications in itself on the complexity on some decision problems on 3-player games, see Section 8.

Proposition 9. *Let $N \geq 3$, and $F \subset [0,1[^N$ be a nonempty closed semi algebraic set. Then there exists an N-player finite game Γ, and a particular pure action profile $X_* = (X_*^1, \cdots, X_*^N)$ such that*

a) $\mathrm{Proj}_{X_*}(\mathrm{NE}(\Gamma)) = F$
b) $\mathrm{NEP}(\Gamma) = \{0\}$.

Proposition 9 tells us that one can transform a semialgebraic set to the set of equilibria of a game, up to the addition of some actions for each player. In [?, ?] the fact that one can do so by instead adding some players was shown. While this may seems similar, there a two important differences between those two types of results. First, adding actions does not change the dimension of the vector payoffs, which allows to deduce Theorem 8 from Proposition 9. In fact, Theorem 8 might also be viewed as a universal result involving a projection where one uses the "canonical" projection given by the payoff function of the game. Second, and unfortunately, adding actions turns out to be more burdensome than adding players for reasons that will become clear in the constructions below.

We claim that Theorem 8 follows easily from Proposition 9 :

Proof of Theorem 8. Let F be a nonempty, compact, and semi algebraic subset of $\mathbb{R}^N$, and first assume that $F \subset [0,1[^N$. Let Γ be a finite game given by the conclusion of Proposition 9. Let Γ' be defined from Γ by adding 1 to the payoff of each player i iff player[1] $i-1$ plays X_*^{i-1}. Hence Γ and Γ' are strategically equivalent thus have the same set of equilibria. Because of properties a) and b), the set of equilibrium payoffs of Γ is $\{(e_N, e_1, \cdots, e_{N-1}) | (e_1, \cdots, e_N) \in F\}$. By relabeling the players one get a game Γ'', in which Player i plays the role of Player $i+1$ in Γ', whose set of equilibrium payoffs is F.

If F is not a subset of $[0,1[^N$, F being bounded one can choose $\alpha \in \mathbb{R}$ and $\beta > 0$ such that $F' := \alpha + \beta F$ is in $[0,1[^N$. By the previous argument, there is a finite game Γ'' whose set of

[1]every labelling of the players in the paper has to be understood modulo N

equilibrium payoffs is F'. Then $\Gamma''' := \frac{1}{\beta}\Gamma'' - \frac{\alpha}{\beta}$ is strategically equivalent to Γ'', and thus its set of equilibrium payoffs is F.

$\square$

For the ease of reading we will first, in the next section, prove Proposition 9 with stronger assumptions on F.

Proposition 10. *Let $F = \cap_{k=1}^{K}\{z \in \mathbb{R}^3,\ P_k(z) \leq 0\}$ for K polynomials $P_1, \cdots, P_K$ in three variables. Assume that $F \subset]0, 1/10[^3$ and is nonempty. Then there exists a 3-player finite game Γ, and a particular pure action profile $X_* = (X_*^1, X_*^2, X_*^3)$ such that*

a) $\mathrm{Proj}_{X_*}(\mathrm{NE}(\Gamma)) = F$
b) $\mathrm{NEP}(\Gamma) = \{0\}$.

Hence N is assumed to be exactly 3, $[0, 1[$ has been replaced by $]0, 1/10[$, and F is a basic semi algebraic set. We will prove in Section 5 the more general Proposition 9.

4. Proof of Proposition 10

The basic idea is that if some action X^i is played in some equilibrium while another action Y^i is not, then $g^i(X^i) \geq g^i(Y^i)$, which gives an inequality satisfied by the probabilities of actions of other players in this equilibrium. If we could decide that all equilibria have the same fixed support, we would have a family of inequalities (and thus also equalities) involving the probabilities of all actions. Since there are three players, there is enough space to ensure inequalities such as $x_1^1 = x_1^2$ or $x_2^1 = x_1^1 x_1^2$ (by defining adequatly the payoff of some strategies of Player 3 not played at equilibrium). Remark that combining those two equalities yield $x_2^1 = (x_1^1)^2$. Hence we could ensure that some probabilities are polynomials in other ones, and, again using inequalities, that these polynomials take nonpositive values and hence that the desired tuple of probabilities is in a prescribed basic semi algebraic set F. These techniques are quite similar to the one used in [5] to link equilibria with full support and algebraic sets.

The problem is that one cannot hope that this works so easily. It won't for empty semi algebraic sets (since any finite game has a Nash equilibrium), and empty semi algebraic sets look like[2] nonempty ones. So there will be other equilibria to deal with. Since the number of actions used in the previous paragraph to construct various inequalities may be large (depending of the degrees of the polynomials in the definition of F), there could be a large number of other equilibria, and ensuring that each of them has the desired property may be cumbersome. To avoid this we construct the game such that, in addition to the previous "nice" equilibria with fixed support, there is only one other "bad" equilibrium. Basically, we do this by giving large payoffs, outside of nice equilibria, only to two specific actions of each player. We also define the payoffs of these two specific actions for each player so that each player wants to play the first one only if the next player plays the second one with large probability. By a circular argument ($N = 3$ is odd) there will then be only one bad equilibrium. By constructing the payoffs according to the coordinates of some given element $\widehat{z}$ of the nonempty F, one ensures that in this bad equilibrium the desired tuple of probabilities equals $\widehat{z} \in F$. Some small adjustments are then needed to ensure part b) of the proposition.

Let us now define more precisely the general architecture of the construction and its three different steps. The set $\mathcal{A}^i$ of actions of each Player i consists of two families of actions: $\mathcal{A}^i = \mathcal{X}^i \cup \mathcal{Y}^i$. The elements of $\mathcal{X}^i$ are called unknowns and denoted with the letter X (with subscripts) ; the elements of $\mathcal{Y}^i$ are called constraints and denoted with the letter Y (with subscripts). Typically, the unknowns will have a payoff of 0 (irrespective of what other players play) and will be played at equilibrium, whereas the constraints will not be played at equilibrium (except the unique bad one), and will have a payoff depending on which unknowns the other players are

[2] meaning that it is computationaly hard [2] to decide if a semi algebraic set is empty or not just looking at the polynomials involved in its definition

playing. Equilibria with support contained in $\mathcal{X} := \Pi_{i=1}^3 \mathcal{X}^i$ are called nice equilibria[3], the set of such equilibria (resp. equilibrium payoffs) for a game Γ is denoted by $\mathrm{NNE}(\Gamma)$ (resp. $\mathrm{NNEP}(\Gamma)$). Other equilibria are called bad equilibria. The construction is done in three steps.

- Step 1 : we construct a Game Γ_1 for which
  a) $\mathrm{Proj}_{X_*}(\mathrm{NNE}(\Gamma_1)) = F$
  b) $\mathrm{NNEP}(\Gamma_1) = \{0\}$.
  That is, Proposition 10 is satisfied when one only considers nice equilibria.
- Step 2 : The game Γ_1 will have many bad equilibria. In this step, one modifies the game into another one called Γ_2, so that the set of nice equilibrium is the same, and with only one bad equilibrium Z, such that $\mathrm{Proj}_{X_*}(Z) \in F$. Hence $\mathrm{Proj}_{X_*}(\mathrm{NE}(\Gamma_2)) = F$
- Step 3 : In this step we make another modification to ensure that Part b) of Proposition 10 is satisfied also for the unique bad equilibrium.

Step 1: dealing with nice equilibria.

Denote by D a bound on the maximal degree in each variable of each polynomial P_k. Each polynomial P_k is given as $P_k(z) = c_k + \sum_{d\in\mathbb{N}_D^3} c_{k,d} z^d$, and c is a bound on all c_k and $c_{k,d}$.

We now define the action set of each player in Γ_1. We start by the unknows, which have each a payoff function being identically equal to 0. The set $\mathcal{X}^i$ of each action of Player i consists of

- A special action X_*^i, called the base unknown, which is the one on which the projection is made.
- $((D+1)^3 - 1)$ monomial unknowns denoted as X_d^i for each $d = (d_1, d_2, d_3) \in \mathbb{N}_D^3$. Their role is to represent monomials in the x_*^i. More precisely, the game is constructed such that in any nice equilibrium, one has $x_d^i = (x_*)^d := (x_*^1)^{d_1}(x_*^2)^{d_2}(x_*^3)^{d_3}$.
- An additional action X_0^i called the dump unknown. For any nice equilibrium the probability that Player i plays in $\mathcal{X}^i$ has to be 1, the role of X_0^i is thus to ensure this by giving to each player some action in which to dump the remaining probability.

We now define the constraints. The set $\mathcal{Y}^i$ of each Player i consists of

- 8 Initialization constraints. For each $j \neq i$, $j' \neq i$, and $s \in \{+,-\}$, let $Y_{j,j',s}^i$ be an action with payoff
$$g^i\left(Y_{j,j',s}^i\right) = s\left(x_*^j - x_{e^j}^{j'}\right).$$
- $6((D+1)^3 - 1)$ induction constraints. For each $j \in 1,2,3$ and each $d \in \mathbb{N}_D^3$, define $d + e^j$ by $(d+e^j)_j = d_j + 1$ and $(d+e^j)_m = d_m$ for $m \neq j$. Then for $s \in \{+,-\}$, let $Y_{j,d,s}^i$ be an action with payoff
$$g^i\left(Y_{j,d,s}^i\right) = s\left(x_{d+e^j}^{i+1} - x_d^{i+1} x_{e^j}^{i+2}\right).$$
- K semi algebraic constraints. For every k, let Y_k^i be an action with payoff
$$g^i\left(Y_k^i\right) = c_k + \sum_{d\in\mathbb{N}_D^3} c_{k,d} x_d^{i+1}.$$

Clearly, any nice equilibrium gives a payoff of 0 to each player : $\mathrm{NNEP}(\Gamma) = \{0\}$. We claim that $\mathrm{Proj}_{X_*}(\mathrm{NNE}(\Gamma)) = F$.

Let us first prove that $\mathrm{Proj}_{X_*}(\mathrm{NNE}(\Gamma)) \subset F$ by considering a nice equilibrium. Recall that, by definition, all constraints are played with probability 0 in this equilibrium. Since the payoff of any nice equilibrium is zero, we get that for any Player i and any constraint Y^i, $g^i(Y^i) \leq 0$. Using this we prove that in any nice equilibrium, one has
$$x_d^i = (x_*)^d \tag{1}$$

[3]Note that we only require that no constraints are played for an equilibria to be nice, and not that every unknown is played with positive probability. It turns out that this stronger property will in fact be satisfied by every nice equilibrium in this section, but this is no longer the case in the more general framework of Section 5.

for each Player i and $d \in \mathbb{N}_D^3$. This is done by induction on $\delta = d_1 + d_2 + d_3$. The case $\delta = 1$ (that is, $d = e^j$ for some j) is settled using the initialization constraints. For any j and $j' \in \{1,2,3\}$, there exists at least an $i \in \{1,2,3\}$ such that $i \neq j, j'$. Since $g^i\left(Y^i_{j,j',s}\right) \leq 0$ one gets

$$\pm\left(x^j_* - x^{j'}_{e^j}\right) \leq 0$$

hence $x^{j'}_{d^j} = x^j_*$ as claimed. Assume next that the induction hypothesis is true for δ. Fix any player i, any $d \in \mathbb{N}_D^3$ with $\delta = d_1 + d_2 + d_3$, and let $j \in \{1,2,3\}$. Since $g^i\left(Y^i_{j,d,s}\right) \leq 0$ one gets

$$\pm\left(x^{i+1}_{d+e^j} - x^{i+1}_d x^{i+2}_{e^j}\right) \leq 0$$

hence

$$\begin{aligned} x^{i+1}_{d+e^j} &= x^{i+1}_d x^{i+2}_{e^j} \\ &= (x_*)^d x^j_* \text{ by the induction hypothesis and the case } \delta = 1 \\ &= (x_*)^{d+e^j}. \end{aligned}$$

Since this is true for every choice of i, d and j, the induction hypothesis is proven for $\delta + 1$ and (1) is established. Using now the semialgebraic constraints, one has for any k and i

$$\begin{aligned} 0 &\geq g^i\left(Y^i_k\right) \\ &= c_k + \sum_{d \in \mathbb{N}_D^3} c_{k,d} x^{i+1}_d \\ &= c_k + \sum_{d \in \mathbb{N}_D^3} c_{k,d} x^d_* \text{ by (1)} \\ &= P_k(x_*) \end{aligned}$$

and $x_* \in F$. Thus $\mathrm{Proj}_{X_*}(\mathrm{NNE}(\Gamma)) \subset F$.

Conversely, let $z \in F$. Define a strategy profile by $x^i_* = z_i$, $x^i_d = z^d$, $x^i_0 = 1 - z_i - \sum_{d \in \mathbb{N}_D^3} z^d$, and $y^i = 0$ for every constraint. Then it is clear by construction that all initialization and induction constraints give a payoff of zero, while all semi algebraic constraints give a nonpositive payoff since $z \in F$. Hence there is no profitable deviation and to show that $z \in \mathrm{Proj}_{X_*}(\mathrm{NNE}(\Gamma))$ we only need to verify that $x^i_0 \geq 0$ for all i. Indeed,

$$\begin{aligned} x^i_0 &= 1 - z_i - \sum_{d \in \mathbb{N}_D^3} z^d \\ &\geq 1 - z_i - \sum_{d \in \mathbb{N}^3 \setminus \{(0,0,0)\}} z^d \\ &\geq 1 - \frac{1}{10} - \left(\left(\frac{1}{1 - 1/10}\right)^3 - 1\right) \\ &> \frac{1}{2} \end{aligned} \tag{2}$$

and we have proved that $F \subset \mathrm{Proj}_{X_*}(\mathrm{NNE}(\Gamma_1))$. This concludes Step 1 of the contruction.

Remark 11. In fact we established that Proj_{X_*} is a bijection between F and $\mathrm{NNE}(\Gamma_1)$.

Step 2 : dealing with bad equilibria.

Let $C > \max\{1, 2c\}$ and $\widehat{z} \in F$ (by nonemptiness). The assumptions on F imply that $\widehat{z}_i > 0$ for every i. We define Γ_2 by modifying the game Γ_1 constructed in the last step in two ways :

- The payoff of the unknown X_*^i of each player is no longer 0 but

$$g^i(X_*^i) = \frac{C}{1-\widehat{z}_{i+1}}\left(1 - \sum_{x^{i+1}\in\mathcal{X}^{i+1}} x^{i+1}\right) \tag{3}$$

- Each player has an additional constraint Y_*^i with payoff

$$g^i(Y_*^i) = C(1-2x_0^{i+1}) \tag{4}$$

We now establish a sequence of claims.

Claim 1 : Γ_2 satisfies the same properties as Γ_1,

a) $\mathrm{Proj}_{X_*}(\mathrm{NNE}(\Gamma_2)) = F$
b) $\mathrm{NNEP}(\Gamma_2) = \{0\}$.

Abusing notations, we identify any profile in Γ_1 to the corresponding profile in Γ_2 with each Y_*^i played with probability 0. Observe first that in any nice equilibrium of Γ_2, $g^i(X_*^i)$ equals 0 for each player. Because of this and since we only added potential deviations for each player, $\mathrm{NNE}(\Gamma_2) \subset \mathrm{NNE}(\Gamma_1)$. Moreover, in any nice equilibrium of $\mathrm{NNE}(\Gamma_1)$, (2) implies that the corresponding profile in Γ_2 satisfies $g^i(Y_*^i) \leq 0$ for all i, thus $\mathrm{NNE}(\Gamma_1) \subset \mathrm{NNE}(\Gamma_2)$. Hence Γ_2 still satisfies the two properties of Γ_1.

Claim 2 : in any bad equilibrium, at least one player has a positive payoff.
If not, this would imply that $g^i(X_*^i) \leq 0$ for all i and the equilibrium would be nice.

Claim 3 : in any bad equilibrium, $x_0^i = 0$ for all player i.
By the previous claim, there is i such that $x_0^i = 0$, and thus $g^{i-1}(Y_*^{i-1}) = C > 0 = g^{i-1}(X_0^{i-1})$. Hence $x_0^{i-1} = 0$ and iterating this yields the claim.

Claim 4 : no other unknown that the X_*^i may be in the support of any bad equilibrium.
This is an immediate consequence of the previous claim, which implies that the payoff of each player is at least C, and is thus positive, in any bad equilibrium.

Claim 5 : no other constraints that the Y_*^i may be in the support of any bad equilibrium.
Claim 3 implies that the payoff of each player is at least C in any bad equilibrium. The initialization and induction constraints give a payoff less than 1, and Claim 4 implies that the payoff of the semi algebraic constraint Y_k^i is $c_k < C$.

Claim 6 : there is a unique bad equilibrium, for which $x_*^i = \widehat{z}_i$ for each player.
We have just proved that only X_*^i and Y_*^i might be played with positive probability in a bad equilibrium. Their respective payoff is thus, in any bad equilibrium, $g^i(X_*^i) = C\frac{1-x_*^{i+1}}{1-\widehat{z}_{i+1}}$ and $g^i(Y_*^i) = C$. Now, $x_*^{i+1} = 1$ would imply $g^i(X_*^i) < g^i(Y_*^i)$ and hence $x_*^i = 0$, while $x_*^{i+1} = 0$ would imply that $g^i(X_*^i) > g^i(Y_*^i)$ (recall that $\widehat{z}_i > 0$) and hence $x_*^i = 1$. Hence a circular argument establishes that there is no bad equilibrium in which some player plays a pure action. Thus $g^i(X_*^i) = g^i(Y_*^i) = C$ for every i, implying the claim.

Claim 7 $\mathrm{Proj}_{X_*}(\mathrm{NE}(\Gamma_2)) = F$.
This is an immediate consequence of Claim 1 and 6 since $\widehat{z} \in F$.

Step 3 : translating the payoff in the bad equilibrium.

Note that in Γ_2 the payoff of the bad equilibrium is C for each player and not the desired 0. We fix this by constructing a game Γ_3, adding to every payoff of each player i in Γ_2 the quantity $-C\frac{y_*^{i+1}}{1-\widehat{z}_{i+1}}$. Γ_2 and Γ_3 are strategically equivalent hence $\mathrm{Proj}_{X_*}(\mathrm{NE}(\Gamma_3)) = F$. Since $y_*^{i+1} = 0$ in any nice equilibrium, the payoff of every player in any nice equilibrium of Γ_3 is still 0. Since $y_*^{i+1} = 1 - x_*^{i+1} = 1 - \widehat{z}_{i+1}$ in the bad equilibrium, the payoff of every player in the bad equilibrium of Γ_3 is $C - C = 0$. Hence $\mathrm{NEP}(\Gamma_3) = \{0\}$ and Γ_3 satisfies both properties of Proposition 10.

5. Proof of Proposition 9

We now explain how to adapt the proof of Proposition 10 to the more general framework of Proposition 9. There are 4 issues : N may be larger than 3, F may contain elements with coordinates close to 1, F may contain elements with zero coordinates, and F may be a general semi algebraic set.

5.1. **More than 3 players.** For a general N, choose $\epsilon_N > 0$ small enough such that

$$1 - \epsilon_N - \left(\left(\frac{1}{1-\epsilon_N} \right)^N - 1 \right) > \frac{1}{2}. \tag{5}$$

Step 1 is easily adapted for any basic semi algebraic set $F \subset]0, \epsilon_N[^N$. The unknows are X_*^i, X_0^i, and X_d^i for $d \in \mathbb{N}_D^N := \mathbb{N}^N \cap [0, D]^N \setminus \{0\}$. The constraints are defined in a similar way than for $N = 3$.

- Initialization constraints. For each $j \neq i$, $j' \neq i$, and $s \in \{+, -\}$, let $Y_{j,j',s}^i$ be an action with payoff
$$g^i\left(Y_{j,j',s}^i\right) = s\left(x_*^j - x_{e^j}^{j'}\right).$$
- Induction constraints. For each $j \in N$ and each $d \in \mathbb{N}_D^N$, define $d + e^j$ by $(d + e^j)_j = d_j + 1$ and $(d + e^j)_m = d_m$ for $m \neq j$. Then for $s \in \{+, -\}$, let $Y_{j,d,s}^i$ be an action with payoff
$$g^i\left(Y_{j,d,s}^i\right) = s\left(x_{d+e^j}^{i+1} - x_d^{i+1} x_{e^j}^{i+2}\right).$$
- K semi algebraic constraints. For every k, let Y_k^i be an action with payoff
$$g^i\left(Y_k^i\right) = c_k + \sum_{d \in \mathbb{N}_D^N} c_{k,d} x_d^{i+1}.$$

Equation (5) ensures that $x_0^i \geq 1/2$ in any nice equilibrium, as in the case $N = 3$.

In Step 2 one choose similarly $\widehat{z} \in F$. For odd N the construction is then the same that for $N = 3$, and the same arguments apply. Unfortunately for an even N the circular argument in Claim 6 to prove that there is no pure bad equilibrium does not work as is, and we need to adapt a bit the construction. For $N \geq 6$ there is an easy fix: cut the set of players in two parts of odd cardinality. For example let us call type 1 players those in the set $\{1, 2, 3\}$ and type 2 players those in $\{4, \cdots, N\}$. The payoff of X_*^i is then defined as in (3), except that $i + 1$ is replaced by the number of the next player *of the same type* (so the payoff of X_*^3 depends on the x^1, and the payoff of X_*^N depends on the x^4). The payoff of Y_*^i is defined exactly as[4] in (3) (so the payoff of X_*^3 depends on the x^4, and the payoff of X_*^N depends on the x^1). Then Claim 1 to 5 follow as in Section 4. For Claim 6 we treat separately players of type 1 and 2, and since both 3 and $N - 3$ are odd the same circular argument implies that there is a unique bad equilibrium with $x_* = \widehat{z}$.

For $N = 4$ we need to modify more deeply the construction in Step 2. Fix $C > \max\{1, 2c\}$ and let $g^i(Y_*^i) = C(1 - 2x_0^{i+1})$ as in Section 4. For $i \geq 3$ the payoff of X_*^i is still defined as

$$g^i(X_*^i) = \frac{C}{1 - \widehat{z}_{i+1}} \left(1 - \sum_{x^{i+1} \in \mathcal{X}^{i+1}} x^{i+1} \right)$$

but for X_*^1 we now define

$$g^1(X_*^1) = \alpha \frac{C}{1 - \widehat{z}_2} \left(1 - \sum_{x^2 \in \mathcal{X}^2} x^2 \right) + (1 - \alpha) \frac{C}{1 - \widehat{z}_3} \left(1 - \sum_{x^3 \in \mathcal{X}^3} x^3 \right)$$

where α is any positive real less than $\min(1 - \widehat{z}_2, \widehat{z}_3)$. For X_*^2 we define

$$g^2(X_*^2) = 2 \frac{C}{1 - \widehat{z}_1} \left(1 - \sum_{x^1 \in \mathcal{X}^1} x^1 \right) - \frac{C}{1 - \widehat{z}_3} \left(1 - \sum_{x^3 \in \mathcal{X}^3} x^3 \right).$$

[4] It is important that these payoffs do not depend only on the players of the same type, to avoid situations where one type of players would play a bad equilibria and the others a nice one between themselves.

Claim 1 of Step 2 is then the same as in any nice equilibrium $g^i(X^i_*)$ is equal to 0 and $g^i(Y^i_*) \leq 0$.

Claim 2 is similar: if $g^i(X^i_*) \leq 0$ for all i then for $i = 1$ it yields

$$\sum_{x^2 \in \mathcal{X}^2} x^2 = \sum_{x^3 \in \mathcal{X}^3} x^3 = 1,$$

and for $i = 3$ and 4 we get

$$\sum_{x^4 \in \mathcal{X}^4} x^4 = \sum_{x^1 \in \mathcal{X}^1} x^1 = 1,$$

hence the equilibrium is nice, a contradiction. Claim 3 and 4 and 5 works in the exact same way as in Section 4. So only X^i_* and Y^i_* may be played with positive probability in a bad equilibrium. To prove claim 6, remark that for any $i \neq 1, 2$ we have that $x^{i+1}_* = 0$ implies $x^i_* = 1$ while $x^{i+1}_* = 1$ implies $x^i_* = 0$. Considering now the payoff of X^1_*, and since α has been chosen small enough, one sees that $x^3_* = 0$ implies that $x^1_* = 1$, while $x^3_* = 1$ implies that $x^1_* = 0$. Since $N - 1 = 3$ is odd a circular argument establishes that there is no bad equilibrium in which some player in $\{1, 3, 4\}$ plays a pure action. Thus $g^i(X^i_*) = g^i(Y^i_*) = C$ for all $i \in \{1, 3, 4\}$. This immediately yields $x^i_* = \widehat{z}_i$ for $i = 1$ and 4. Since $x^1_* = \widehat{z}_1$, the payoff of Player 2 if he plays X^2_* is

$$g^2(X^2_*) = C\left(2 - \frac{1 - x^3_*}{1 - \widehat{z}_3}\right). \tag{6}$$

while $g^1(X^1_*) = C$ yields

$$\alpha\frac{1 - x^2_*}{1 - \widehat{z}_2} + (1 - \alpha)\frac{1 - x^3_*}{1 - \widehat{z}_3} = 1. \tag{7}$$

We now see that if $x^2_* = 1$ then equation (7) implies $x^3_* < \widehat{z}_3$ and then (6) gives $g^2(X^2_*) < C$, a contradiction. Similarly if $x^2_* = 0$ then equation (7) implies $x^3_* > \widehat{z}_3$ and then (6) gives $g^2(X^2_*) > C$, a contradiction. Hence Player 2 also plays both X^2_* and Y^2_* with positive probability, thus $g^2(X^2_*) = C$ and (6) gives $x^3_* = \widehat{z}_3$. Then equation (7) implies $x^2_* = \widehat{z}_2$ and there is a unique bad equilibrium, for which $x_* = \widehat{z}$. Claim 7 follows immediately.

Step 3 works in the same exact way for a general N than for $N = 3$.

5.2. $\boldsymbol{F \subset]0,1[^N}$. The only reason why we assumed that $F \subset]0, 1/10[^3$ in Section 4 (or $F \subset]0, \epsilon_N[^N$ for general N) instead of $F \subset]0, 1[^N$ is to ensure that all profiles stay in the simplex. Indeed, it is not possible for unknows to represent many monomials in x_* if F has elements with coordinates close to 1. The idea is then to modify the construction so that unknow X^i_d is, in any nice equilibrium, played with probability $\eta(x_*)^d$ instead of $(x_*)^d$, where η is a very small positive number.

Precisely, if $F \subset]0, 1[^N$ then $F \subset]0, 1 - 2\epsilon[^N$ for some $\epsilon > 0$ since F is closed. Fix $\eta > 0$ such that

$$1 - (1 - 2\epsilon) - \eta\left(\left(\frac{1}{1 - (1 - 2\epsilon)}\right)^N - 1\right) > \epsilon. \tag{8}$$

Step 1 of the construction is then adapted so that in any nice equilibrium,

$$x^i_d = \eta(x_*)^d \tag{9}$$

for every player i and $d \in \mathbb{N}^N_D$. To do this we just modify the payoff of the initialization constraints to be

$$\pm\left(\eta x^j_* - x^{j'}_{e^j}\right),$$

and the payoff of the induction constraints to be

$$\pm\left(\eta x^{i+1}_{d+e^j} - x^{i+1}_d x^{i+2}_{e^j}\right).$$

The same induction argument as in Section 4 establishes (9). To ensure that $x_* \in F$ in any nice equilibrium, one then only need to slightly modify the payoff of the semi algebraic constraints :

$$g^i\left(Y^i_k\right) = \eta c_k + \sum_{d\in\mathbb{N}^N_D} c_{k,d} x^{i+1}_d.$$

The definition of η ensures that all actions are played with nonnegative probabilities. In fact (8) implies that in any nice equilibrium $x^i_0 > \epsilon > 0$. The only thing left to modify is the payoff of Y^i_* in Step 2 to

$$g^i(Y^i_*) = C\left(1 - \frac{x^{i+1}_0}{\epsilon}\right)$$

so that $g^i(Y^i_*) \leq 0$ in any nice equilibrium. The rest of the construction and proof is unchanged.

5.3. $\boldsymbol{F \subset [0,1[^N}$. We assumed $F \subset]0,1[^N$ in the previous sections only to ensure that there exists $\widehat{z} \in F$ with positive coordinates, which is essential in the end of Step 2. If this is not the case, let $\widehat{z} \in F$ with some zero coordinates and define a fictitious $\widehat{z}'$ by $\widehat{z}'_i = \widehat{z}_i$ when $\widehat{z}_i > 0$ and $\widehat{z}'_i = \frac{1}{2}$ when $\widehat{z}_i = 0$. We now do Step 2 with $\widehat{z}'$ (which has positive coordinates) replacing $\widehat{z}$. Also, for every i such that $\widehat{z}_i = 0$, we fix

$$g^i(X^i_*) = 0.$$

For the construction to still work we choose, for all such i, an arbitrary unknown X^i other than X^i_* and X^i_0 and let it play the role of X^i_* in Step 2 of Section 4. That is, denoting this arbitrary unknown by ${X'}^i_*$, we define

$$g^i({X'}^i_*) = \frac{C}{1-\widehat{z}'_{i+1}}\left(1 - \sum_{x^{i+1}\in\mathcal{X}^{i+1}} x^{i+1}\right)$$

For any player i such that $\widehat{z}_i \neq 0$ the construction is unchanged : denote ${X'}^i_* = X^i_*$ and let

$$g^i({X'}^i_*) = \frac{C}{1-\widehat{z}'_{i+1}}\left(1 - \sum_{x^{i+1}\in\mathcal{X}^{i+1}} x^{i+1}\right)$$

Now the same argument[5] as in Section 4 shows that no unknown other than ${X'}^i_*$ may be played in any bad equilibrium. ${X'}^i_*$ may be either ${X_*}^i$ or a monomial unknown, but in any case the payoff of each other constraint than Y^i_* is less than $\max(1, 2c) < C$. Hence Y^i_*, with a payoff of C (recall that ${X'}^i_* \neq X^i_0$ for all i), is the only constraint that may be played in any bad equilibrium. The arguments at the end of Step 2 of Section 4 or Section 7.1 then ensures that there is a unique bad equilibrium in which ${x'}^i_* = \widehat{z}'_i$ for all i. Hence $x^i_* = \widehat{z}_i$ for all i such that $\widehat{z}_i > 0$, and for the other players we have $x^i_* = 0 = \widehat{z}_i$. Thus $x_* = \widehat{z}$ in the only bad equilibrium as required.

Step 3 is as in Section 4, replacing $\widehat{z}$ by $\widehat{z}'$.

5.4. **General semi algebraic sets.** Assume now that F is a general semi algebraic set. Without loss of generality (repeating several times a polynomial if needed) we write

$$F = \bigcap_{k=1}^{K}\bigcup_{l=1}^{L}\{z \in \mathbb{R}^n,\ P_{k,l}(z) \leq 0\}, \tag{10}$$

with $P_{k,l}(z) = c_{kl} + \sum_{d\in\mathbb{N}^N_D} c_{k,l,d} z^d$. The idea is now to write "$P_1(z) \leq 0$ or $P_2(z) \leq 0'$" as the equivalent formula

$$\exists u_1, u_2 \geq 0,\ u_1, u_2 \geq 0, u_1 + u_2 = 1,\ u_1 P_1(z) + u_2 P_2(z) \leq 0.$$

and to use unknowns and constraints to ensure that these equalities and inequalities are satisfied in any nice equilibrium.

[5]Of course if N is even one needs additionaly to change the payoff of some ${X'}^i_*$ as in Section 7.1

We thus modify Step 1 of the construction in the following way. As in Section 5.2 let ϵ and $\eta > 0$ be such that $F \subset]0, 1-2\epsilon[^N$ and

$$1-(1-2\epsilon)-\eta\left(\left(\frac{1}{1-(1-2\epsilon)}\right)^N - 1\right) > \epsilon,$$

and fix $\tau > 0$ small enough such that

$$1-(1-2\epsilon)-\eta\left(\left(\frac{1}{1-(1-2\epsilon)}\right)^N - 1\right) - KL\tau > \epsilon. \tag{11}$$

We now add KL addtional actions, called boolean unknowns, for each player i. They are denoted by $X^i_{k,l}$ and with payoff 0. We also add additional constraints with payoff depending only of the boolean unknowns : for each player i and each k a constraint with payoff $\tau - \sum_{l=1}^{L} x^{i+1}_{k,l}$.

These constraints imply that, in any nice equilibrium, for any i and k, there is at least one l such that $x^i_{k,l} > 0$. Recall that in any nice equilibrium the monomial unknowns satisfy $x^i_d = \eta(x_*)^d$ for all i.

Finally, we modify the payoff of the semi algebraic contraints:

$$g^i(Y^i_k) = \sum_{l=1}^{L} x^{i+1}_{kl}\left(\eta c_{k,l} + \sum_{d\in\mathbb{N}^N_D} c_{k,l,d}x^{i+2}_d\right).$$

In any nice equilibrium all x^{i+1}_{kl} are nonnegative and at least one is positive for any fixed k, so $g^i(Y^i_k) \leq 0$ implies that there exists at least an l such that $P_{kl}(x_*) \leq 0$. Since this is true for all k, $x_* \in F$ in any nice equilibrium and $\mathrm{NNE}(\Gamma_1) \subset F$. To prove the reverse inclusion, let $z \in F$ and fix, for each i, $x^i_* = z_i$, $x^i_d = \eta(x_*)^d$, $x^i_{k,l} = \tau$ if $P_{k,l}(z) \leq 0$, and $x^i_{k,l} = 0$ if $P_{k,l}(z) > 0$. Then, since $z \in F$, all constraints give a nonpositive payoff and we only need to check that $x^i_0 \geq \epsilon$, which is true by inequality (11). Steps 2 and 3 are unchanged.

Remark 12. Any closed semialgebraic set can be written as in (10) but it might be computionally costly to do so. One can however adapt the construction above to any set given by unions and intersections in any order. As an example, if F is given as a union of intersections

$$F = \bigcup_{l=1}^{L}\bigcap_{k=1}^{K}\{z \in \mathbb{R}^n,\ P_{k,l}(z) \leq 0\},$$

then for any $\tau > 0$, $z \in F$ if and only if there exists nonnegative real numbers $x_1, \cdots, x_L$ such that

$$\begin{aligned} \tau - \sum_{l=1}^{L} x_l &\leq 0, \\ x_l x_{l'} &\leq 0\ \forall l < l', \\ \sum_{l=1}^{L} x_l P_{k,l}(z) &\leq 0\ \forall k. \end{aligned}$$

Hence we add L boolean unknowns $X^i_1, \cdots, X^i_L$ for each player, and constraints ensuring that the inequalities above hold in any nice equilibrium.

6. Comments

6.1. **Tightness of the result.** Proposition 9 cannot be generalized to $F \subset [0,1]^N$ instead of $F \subset [0,1[^N$, even for $N = 3$ and a basic semi algebraic set. Consider the basic semi algebraic set $F = \{(z_1, z_2, z_3),\ z_1 \geq 0,\ z_1 \leq 1,\ z_2 = z_3 = (2z_1-1)^2\} \subset [0,1]^3$ and assume by contradiction that $\mathrm{Proj}_{X_*}(\mathrm{NE}(\Gamma)) = F$ for some finite game Γ. Since $(1,1,1) \in F$, the pure profile X_* is a

Nash equilibrium of Γ. Since $(0,1,1) \in F$ there exists a mixed action σ^1 of Player 1 such that $\sigma^1(X^1_*) = 0$ and (σ^1, X^2_*, X^3_*) is another Nash equilibrium. Define $\tilde{\sigma}^1 = \frac{1}{2}\sigma^1 + \frac{1}{2}X^1_*$. Then $\tilde{\sigma}^1$ is a best reply to (X^2_*, X^3_*), since both σ^1 and X^1_* are. Also, since X^2_* is a best reply to both (X^1_*, X^3_*) and (σ^1, X^3_*) it has to be a best reply to $(\tilde{\sigma}^1, X^3_*)$ by linearity. Similarly X^3_* is a best reply to $(\tilde{\sigma}^1, X^2_*)$. Hence $(\tilde{\sigma}^1, X^2_*, X^3_*)$ is a Nash equilibrium of Γ but $\mathrm{Proj}_{X_*}(\tilde{\sigma}^1, X^2_*, X^3_*) = \left(\frac{1}{2}, 1, 1\right) \notin F$, a contradiction.

6.2. **Number of actions.** In Section 4 one checks that the construction use $2 + (D+1)^3 - 1$ unknowns and $9 + 6((D+1)^3 - 1) + K$ contraints for each player. Hence the number of actions needed is $O(D^3 + K)$ for each player. For the more general Proposition 9, when $N = 3$ one only needs to take into account Section 5.4 in which KL constraints and $2KL + K(L-1) + 2K$ are added. Hence the number of actions needed is $O(D^3 + KL)$ for each player.

For N players, the approach in Section 7.1 gives $O(ND^N + KL)$ actions per player and is thus exponential in N. However the construction was very inefficient and one can improve it in two ways. First, many constraints are redondant. For example the fact that $x^1_{d^1} = x^1_*$ is ensured by constraints of the other $N-1$ players with the same payoff $\pm(x^1_{d^1} - x^1_*)$. Similarly there are many redundant induction constraints. Second, and more importantly, we should use the fact that the semi algebraic constraints of one player may depend on the unknowns of all the other players.

In fact, assuming $N \geq 4$, it turns out it is enough for each player i to have $(D+1)^2 - 1$ monomial unknowns of the form $x^i_{ae^{i-1}+be^i}$, for $0 \leq a, b \leq D$ and $a + b > 0$, such that in any nice equilibrium

$$x^i_{ae^{i-1}+be^i} = \eta (x_*)^{ae^{i-1}+be^i}. \tag{12}$$

To ensure this, we only need for each Player i:

- 4 initialization constraints with payoff $\pm(x^{i+2}_{e^{i+2}} - \eta x^{i+2}_*)$ and $\pm(x^{i+2}_{e^{i+1}} - \eta x^{i+1}_*)$. This gives (12) for $i+2$ and $a + b = 1$.
- $2(D-1)$ constraints with payoff $\pm(\eta x^{i+2}_{(a+1)e^{i+1}} - x^{i+2}_{ae^{i+1}} x^{i+1}_*)$ for $1 \leq a \leq D-1$. This gives (12) for $i+2$, $b = 0$ and $2 \leq a \leq D$.
- $2(D-1)$ constraints with payoff $\pm(\eta x^{i+2}_{(b+1)e^{i+2}} - x^{i+2}_{be^{i+2}} x^{i+3}_{e^{i+2}})$ (possible since $i + 3 \neq i$ as $N \geq 4$) for $1 \leq b \leq D-1$. This gives (12) for $i+2$, $a = 0$ and $2 \leq b \leq D$.
- $2D^2$ constraints with payoff $\pm(\eta x^{i+2}_{ae^{i+1}+be^{i+2}} - x^{i+2}_{be^{i+2}} x^{i+1}_{ae^{i+1}})$ for $1 \leq a, b \leq D$. This gives (12) for $i+2$ and $1 \leq a, b \leq D$.

As in Section 5.4, write

$$F = \bigcap_{k=1}^{K} \bigcup_{l=1}^{L} \{z \in \mathbb{R}^n,\ P_{k,l}(z) \leq 0\},$$

with $P_{k,l}(z) = c_{kl} + \sum_{d \in \mathbb{N}^N_D} c_{k,l,d} z^d$, and define boolean constraints $x^i_{k,l}$. For any $d = (d_1, \cdots, d_n) \in \mathbb{N}^N_D$, to ease the reading we denote $d_{i,j} = d_i e^i + d_j e^j$ and, abusing notations, we write d_j for $d_j e^j$. The semi algebraic constraints are now with payoff

$$g^i(Y^i_k) = \sum_{l=1}^{L} x^{i+2}_{kl} \left(\eta c_{k,l} \sum_{d \in \mathbb{N}^N_D} c_{k,l,d} x^{i+1}_{d_{i,i+1}} x^{i+3}_{d_{i+2,i+3}} \prod_{j \neq i, i+1, i+2, i+3} x^j_{d_j} \right).$$

They are well defined since $N \geq 4$ and thus $i + 3 \neq i$. In any nice equilibrium, (12) implies that

$$g^i(Y^i_k) = \eta \sum_{l=1}^{L} x^{i+2}_{kl} P_{k,l}(x_*)$$

as in Section 5.4. Hence this construction shows that, for $N \geq 4$, one can construct a game satisfying Proposition 9 with $O(D^2 + KL)$ actions for each player. In particular the number of actions per player does not depend on N and is polynomial in D.

For specific semialgebraic sets the dependance may even be logarithmic in D. The monomial x^{2^f} may be represented using only $O(f)$ inequalities, iterating $x^{2^f} = \left(x^{2^{f-1}}\right)^2$. Then, using the binary representation of any integer $q = \sum_{p=0}^{f} \lambda_p 2^p$, one can write

$$x^q = \prod_{0\le p\le f,\ \lambda_p=1} x^{2^{\lambda_p}}.$$

So a fixed monomial in one variable can be represented using $O(\log(D))$ constraints. Hence if all polynomials in the definition of F are $S-$sparse, meaning that at most S monomials in N variables are with nonzero coefficient, then the number of actions of each player is $O(S\log(D) + KL)$.

7. Generalizations

7.1. **A useful lemma.** The aim of this section is to generalize the "circular argument" of Claim 6 of Step 2 of the construction, to more complex frameworks in which more than two actions per player are played with positive probabilities. This will be important to generalize our results further.

Denote Δ_T the T dimensional simplex $\{(a_0, \cdots, a_T),\ a_t \ge 0, \sum a_t = 1\}$.

Definition 13. Let $N = 3$ or $N \ge 5$ and $T \ge 2$ be two integers and for every $1 \le i \le N$ and $1 \le t \le T$ let f_t^i be a function from $(\Delta_T)^N$ to $\mathbb{R}$. Denote the coordinates of an element of $(\Delta_T)^N$ as $(a_0^1, \cdots, a_T^1, a_0^2, \cdots a_0^N, \cdots, a_T^N)$. For $0 < \delta < 1$ we say that the family of functions $\{f_t^i\}$ is δ-circular if

1) For all i and t, f_t^i does not depend on the a_s^i, $s = 0$ to T, and is multiaffine in the other coordinates. Moreover, for every i and every $t \ne 0$, f_t^i is bounded by 1.
2) There exists some $\bar{a}_t^i > 0$ for all $i = 1$ to N and $t = 0$ to $T-1$, and a family κ_t of permutations of $\{1, \cdots, N\}$ for $t = 0$ to $T-1$ such that
   i) $\sum_{t=0}^{T-1} \bar{a}_t^i < 1$ for every i.
   ii) All the cycles of κ_t have odd length, for every t.
   iii) For every i and every $t_0 = 0$ to $T-1$, if $a \in \Delta_T^N$ satisfies
      - $a_{t_0}^i \ge \bar{a}_{t_0}^i$
      - $f_t^j(a) = 0$ for every j and every $1 \le t < t_0$

      then $f_{t_0+1}^{\kappa_{t_0}(i)}(a) > \delta$.
   iv) For every i and every $t_0 = 0$ to $T-1$, if $a \in \Delta_T^N$ satisfies
      - $a_{t_0}^i = 0$
      - $f_t^j(a) = 0$ for every j and every $1 \le t < t_0$

      then $f_{t_0+1}^{\kappa_{t_0}(i)}(a) < -\delta$.

Lemma 14. *Let $N \ge 3$ and $T \ge 2$ be two integers, and $\{f_t^i\}$ a δ-circular family of functions. Consider a game Γ with sets of actions $\mathcal{A}^i = \{A_0^i, \cdots, A_T^i\}$, with arbitrary payoffs for the actions A_0^i, and such that*

$$g^i(A_t^i, \sigma^{-i}) := g^i(A_{t-1}^i, \sigma^{-i}) + \left(\frac{\delta}{2}\right)^{t-1} f_t^i(\sigma^{-i})$$

for every i , every $1 \le t \le T$ and every pure profile σ^{-i}. Then $\mathrm{NE}(\Gamma_\delta) = \{a \in (\Delta_T)^N, f_t^i(a) = 0\ \forall i \in N,\ \forall 1 \le t \le T\}$.

Proof. Since each f_t^i does not depend on the a_s^i, $s = 0$ to T, and is multiaffine in the other coordinates, the game Γ is well defined. Clearly $\{a \in (\Delta_T)^N, f_t^i(a) = 0\ \forall i \in N,\ \forall 1 \le t \le T\} \subset \mathrm{NE}(\Gamma_\delta)$ since each player is then indifferent between all his actions, and conversely the set of completely mixed equilibria of Γ is a subset of $\{a \in (\Delta_T)^N, f_t^i(a) = 0\ \forall i \in N,\ \forall 1 \le t \le T\}$. Hence we just have to prove that any equilibrium is completely mixed.

Fix an equilibrium a of Γ, we first prove on induction on $0 \le t_0 \le T-1$, that for every i we have $0 < a_{t_0}^i < \bar{a}_{t_0}^i$. Let us start with $t_0 = 0$. By iii) , if $a_0^i \ge \bar{a}_0^i$ for some i, then $f_1^{\kappa_0(i)}(a) > \delta > 0$

hence $g^{\kappa_0(i)}\left(A_1^{\kappa_0(i)}\right) > g^{\kappa_0(i)}\left(A_0^{\kappa_0(i)}\right)$ and $a_0^{\kappa_0(i)} = 0$. On the other hand, if $a_0^i = 0$ for some i then by iv) $f_1^{\kappa_0(i)}(a) < -\delta$. Hence for every $t \geq 1$,

$$\begin{aligned} g^{\kappa_0(i)}(A_t^{\kappa_0(i)}) &\leq g^{\kappa_0(i)}(A_1^{\kappa_0(i)}) + \sum_{t'=2}^{t} \left(\frac{\delta}{2}\right)^{t'-1} |f_t^{\kappa_0(i)}(a)| \\ &< g^{\kappa_0(i)}(A_0^{\kappa_0(i)}) - \delta + \sum_{t'=2}^{t} \left(\frac{\delta}{2}\right)^{t'-1} \\ &< g^{\kappa_0(i)}(A_0^{\kappa_0(i)}) - \delta + \frac{\delta/2}{1-\delta/2} \\ &< g^{\kappa_0(i)}(A_0^{\kappa_0(i)}) \text{ since } \delta < 1. \end{aligned}$$

Thus $a_t^{\kappa_0(i)} = 0$ for all $t \geq 1$ and $a_0^{\kappa_0(i)} = 1 \geq \bar{a}_0^i$.

Repeating these arguments, and since by ii) there is an odd k such that $(\kappa_0)^k(i) = i$, one sees that $a_0^i \geq \bar{a}_0^i$ iff $a_0^i = 0$. Hence $0 < a_0^i < \bar{a}_0^i$ as claimed.

Assume now that $1 \leq t_0 \leq T-1$ and that the induction hypothesis holds for every $t < t_0$. Then in particular $0 < a_t^i$ for all $t < t_0$ hence by the indifference principle $f_t^j(a) = 0$ for every j and every $1 \leq t < t_0$. Using iii) exactly as for $t_0 = 0$, one then proves that $a_{t_0}^i \geq \bar{a}_{t_0}^i$ implies $a_{t_0}^{\kappa_{t_0}(i)} = 0$. On the other hand, using iv) exactly as for $t_0 = 0$, one proves that $a_{t_0}^i = 0$ implies $a_{t'}^{\kappa_{t_0}(i)} = 0$ for all $t' > t_0$ and thus that $a_{t_0}^{\kappa_{t_0}(i)} = 1 - \sum_{t=0}^{t_0-1} a_t^{\kappa_{t_0}(i)} > 1 - \sum_{t=0}^{t_0-1} \bar{a}_t^{\kappa_{t_0}(i)} > \bar{a}_{t_0}^{\kappa_{t_0}(i)}$ by property i). Then , using assumption ii), $a_{t_0}^i \geq \bar{a}_{t_0}^i$ iff $a_{t_0}^i = 0$. Hence $0 < a_{t_0}^i < \bar{a}_{t_0}^i$ as claimed.

We proved that $0 < a_t^i$ for every i and for all $t \leq T-1$. Since $a_T^i = 1 - \sum_{t=0}^{T-1} a_t^i > 1 - \sum_{t=0}^{T-1} \bar{a}_t^i > 0$, once again by property i), the proposition is established. $\square$

One may remark that for any circular family of functions, the permutations κ_t cannot have any fixed point: if this was the case then properties 2)iii and 2)iv would imply that $f_{t+1}^i(a)$ depends on a_t^i, which would contradict property 1). In the particular case of $N = 4$, observe that there does not exist any permutation of a set with four elements with no fixed points and no cycles of even length, so Definition would be void and one needs to adapt it. The natural idea is to take κ_t as a circular permutation of $\{1,2,3,4\}$ (say $\kappa(t) = t-1$), and to change the sign in the definition in the payoff

$$g^i(A_t^i, \sigma^{-i}) = g^i(A_{t-1}^i, \sigma^{-i}) - \left(\frac{\delta}{2}\right)^{t-1} f_t^i(\sigma^{-i})$$

for some particular player i (say $i = 2$), so that the circular argument becomes

$$a^4 = 0 \implies a^3 \geq \bar{a}^3 \implies a^2 \geq \bar{a}^2 \implies a^1 = 0 \implies a^4 \geq \bar{a}^4$$

and the desired contradiction. This direct adaptation would however cause issues with Claim 2 of Step 2 of our construction so we need to make additional adjustments.

Definition 15. Let $N = 4$ and $T \geq 2$, and for every $1 \leq i \leq N$ and $1 \leq t \leq T$ let f_t^i be a function from $(\Delta_T)^N$ to $\mathbb{R}$. Denote the coordinates of an element of $(\Delta_T)^N$ as $(a_0^1, \cdots, a_T^1, a_0^2, \cdots a_0^N, \cdots, a_T^N)$. For $0 < \delta < 1$ we say that the family of functions $\{f_t^i\}$ is δ-circular if

1) For all i and t, f_t^i does not depend on the a_s^i, $s = 0$ to T, and is multiaffine in the other coordinates. Also, f_0^2 does not depend on the a_s^1, $s = 0$ to T, and f_0^4 does not depend on the a_s^2, $s = 0$ to T. Moreover, for every i and every $t \neq 0$, f_t^i is bounded by 1.
2) There exists some $\bar{a}_t^i > 0$ for all $i = 1$ to N and $t = 0$ to $T-1$, and a family κ_t of permutations of $\{1, \cdots, 4\}$ for $t = 0$ to $T-1$ such that
   i) $\sum_{t=0}^{T-1} \bar{a}_t^i < 1$ for every i.
   ii) $\kappa_0(i) = i-1$ for all i, and for each $t \geq 1$, either $\kappa_t(i) = i+1$ for all i or $\kappa_t(i) = i-1$ for all i.
   iii) For every i and every $t_0 = 0$ to $T-1$, if $a \in \Delta_T^N$ satisfies

- $a^i_{t_0} \geq \bar{a}^i_{t_0}$
- $f^j_t(a) = 0$ for every j and every $1 \leq t < t_0$

then $f^{\kappa_{t_0}(i)}_{t_0+1}(a) > \delta$.

iv) For every i and every $t_0 = 0$ to $T-1$, if $a \in \Delta^N_T$ satisfies
- $a^i_{t_0} = 0$
- $f^j_t(a) = 0$ for every j and every $1 \leq t < t_0$

then $f^{\kappa_{t_0}(i)}_{t_0+1}(a) < -\delta$.

Lemma 16. *Let $N = 4$ and $T \geq 2$, and $\{f^i_t\}$ a δ-circular family of functions. Let $M := \sup_{a\in(\Delta_T)^N} |f^1_1(a)|$ and α be any positive real number less than* $\max\left(1, \frac{\delta}{2(1+M)}\right)$. *Consider a game Γ with sets of actions $\mathcal{A}^i = \{A^i_0, \cdots, A^i_T\}$, with arbitrary payoffs for the actions A^i_0, and such that*

$$\begin{aligned}
g^1(A^1_1, \sigma^{-1}) &:= g^1(A^1_0, \sigma^{-1}) + \alpha f^1_1(\sigma^{-i}) + (1-\alpha) f^2_1(\sigma^{-i}) && (13)\\
g^2(A^2_1, \sigma^{-1}) &:= g^2(A^2_0, \sigma^{-1}) + 2f^4_1(\sigma^{-i}) - f^2_1(\sigma^{-i}) && (14)\\
g^3(A^3_1, \sigma^{-1}) &:= g^3(A^3_0, \sigma^{-1}) + f^3_1(\sigma^{-i}) && (15)\\
g^4(A^4_1, \sigma^{-1}) &:= g^4(A^4_0, \sigma^{-1}) + f^4_1(\sigma^{-i}) && (16)\\
g^i(A^i_t, \sigma^{-i}) &:= g^i(A^i_{t-1}, \sigma^{-i}) + \left(\frac{\alpha\delta}{4}\right)^{t-1} f^i_t(\sigma^{-i}) \text{ for every } i \neq 2 \text{ and } 2 \leq t \leq T && (17)\\
g^2(A^i_t, \sigma^{-i}) &:= g^2(A^i_{t-1}, \sigma^{-i}) - \left(\frac{\alpha\delta}{4}\right)^{t-1} f^i_t(\sigma^{-i}) \text{ for every } 2 \leq t \leq T. && (18)
\end{aligned}$$

Then $\mathrm{NE}(\Gamma_\delta) = \{a \in (\Delta_T)^N, f^i_t(a) = 0 \ \forall i \in N, \ \forall 1 \leq t \leq T\}$.

Proof. Since each f^i_t does not depend on the a^i_s, $s = 0$ to T, is multiaffine in the other coordinates, and because of the additional assumptions on f^2_0 and f^4_0, the game Γ is well defined. Clearly the set $\{a \in (\Delta_T)^N, f^i_t(a) = 0 \ \forall i \in N, \ \forall 1 \leq t \leq T\} \subset \mathrm{NE}(\Gamma_\delta)$ since each player is then indifferent between all his actions. Conversely, since $\alpha > 0$, the set of completely mixed equilibria of Γ is a subset of $\{a \in (\Delta_T)^N, f^i_t(a) = 0 \ \forall i \in N, \ \forall 1 \leq t \leq T\}$. Hence we just have to prove that any equilibrium is completely mixed.

Fix an equilibrium a of Γ, we first prove on induction on $0 \leq t_0 \leq T-1$, that for every i we have $0 < a^i_{t_0} < \bar{a}^i_{t_0}$. Let us start with $t_0 = 0$. If $a^3_0 \geq \bar{a}^3_0$, then

$$\begin{aligned}
g^1(A^1_1) &= g^1(A^1_0) + \alpha f^1_1(a) + (1-\alpha) f^2_1(a)\\
&> g^1(A^1_0) + \alpha f^1_1(a) + (1-\alpha)\delta\\
&> g^1(A^1_0) \text{ by definition of } \alpha
\end{aligned}$$

and $a^1_0 = 0$. On the other hand, if $a^3_0 = 0$, then

$$\begin{aligned}
g^1(A^1_1) &= g^1(A^1_0) + \alpha f^1_1(a) + (1-\alpha) f^2_1(a)\\
&< g^1(A^1_0) + \alpha f^1_1(a) - (1-\alpha)\delta\\
&< g^1(A^1_0) - \frac{\delta}{2} \text{ by definition of } \alpha.
\end{aligned}$$

Hence for $t \geq 1$,

$$\begin{aligned}
g^1(A^1_t) &\leq g^1(A^1_1) + \sum_{t'=2}^{t} \left(\frac{\alpha\delta}{4}\right)^{t'-1} |f^1_t(a)|\\
&< g^1(A^1_0) - \frac{\delta}{2} + \sum_{t'=2}^{t} \left(\frac{\delta}{4}\right)^{t'-1}\\
&< g^1(A^1_0)
\end{aligned}$$

and $a_0^1 = 1 \geq \bar{a}_0^1$. Hence we have the sequence of implications

$$a_0^3 \geq \bar{a}_0^3 \implies a_0^1 = 0 \implies a_0^4 \geq \bar{a}_0^4 \implies a_0^3 = 0 \implies a_0^1 \geq \bar{a}_0^1 \implies a_0^4 = 0 \implies a_0^3$$

and thus $0 < a_0^i < \bar{a}_0^i$ for every $i \neq 2$. In particular this implies that the payoff of player 1 and 4 in this equilibrium is $g^1(A_0^1)$ and $g^1(A_0^4)$ respectively, that A_1^1 and A_1^4 are not profitable deviations, and that at least one A_t^1 and $A_{t'}^4$ are played with positive probability, with $t, t' \geq 1$. Hence

$$-\frac{\alpha\delta}{2} < -\frac{\alpha\delta/4}{1-\alpha\delta/4} \leq \alpha f_1^1(a)+(1-\alpha)f_1^2(a) \leq 0 \tag{19}$$

$$-\frac{\alpha\delta}{2} < -\frac{\alpha\delta/4}{1-\alpha\delta/4} \leq \qquad f_4^1(a) \qquad \leq 0 \tag{20}$$

Assume first that $a_0^2 = 0$. Then $f_1^1(a) < -\delta$ and inequality (19) implies $(1-\alpha)f_1^2(a) > \frac{\alpha\delta}{2}$. Then

$$\begin{aligned} g^2(A_1^2) &= g^2(A_0^2) + 2f_1^4(a) - f_1^2(a) \\ &< g^2(A_0^2) - \frac{\alpha\delta}{2} \text{ by (20)} \\ &< g^2(A_0^2) - \frac{\alpha\delta/4}{1-\alpha\delta/4} \end{aligned}$$

which implies that $g^2(A_1^2) < g^2(A_0^2)$ for every $t \geq 1$, a contradiction. Assume on the other hand that $a_0^2 \geq \bar{a}_0^2$. Then $f_1^1(a) > \delta$ and inequality (19) implies $(1-\alpha)f_1^2(a) < -\alpha\delta$. Hence

$$\begin{aligned} g^2(A_1^2) &= g^2(A_0^2) + 2f_1^4(a) - f_1^2(a) \\ &> g^2(A_0^2) - 2\frac{\alpha\delta}{2} + \alpha\delta \text{ by (20)} \\ &= g^2(A_0^2) \end{aligned}$$

and $a_0^2 = 0$, a contradiction. We have thus established that $0 < a_0^i < \bar{a}_0^i$ for $i = 2$ as well.

The rest of the proof is exactly as in the proof of Lemma 14, except that the cycles of κ have an even length of 4, and the contradiction arise since now $a_{t_0}^{\kappa^{-1}(2)} \geq \bar{a}_{t_0}^{\kappa^{-1}(2)}$ implies $a_{t_0}^2 \geq \bar{a}_{t_0}^2$ and $a_{t_0}^{\kappa^{-1}(2)} = 0$ implies $a_{t_0}^2 = 0$. □

7.2. **Games with pure payoffs in $\mathbb{Z}$.** A natural question is to characterize the sets of equilibrium payoffs, or the sets of equilibria up to projection of the first action of each player, of all N-player games with pure payoffs in Z. We say that a set F is $\mathbb{Z}$-semi algebraic is it is the finite union and intersection of sets of the form $\{z \in \mathbb{R}^n,\ P(z) \leq 0\}$ or $\{z \in \mathbb{R}^n,\ P(z) < 0\}$ with $P \in \mathbb{Z}[X]$. The necessary conditions are then the same:

Proposition 17. *Let Γ be a finite game with pure payoffs in $\mathbb{Z}$, and denote by X_*^i the first action of each player. Then the three sets* $\mathrm{NE}(\Gamma)$, $\mathrm{NEP}(\Gamma)$ *and* $\mathrm{Proj}_{X_*}(\mathrm{NE}(\Gamma))$ *are non empty, compact, and $\mathbb{Z}$-semi algebraic.*

Using the technical lemmas in the previous section we can establish the result in the opposite direction.

Proof. $\mathrm{NE}(\Gamma)$ is clearly compact, non empty by Nash's theorem, and $\mathbb{Z}$-semi algebraic since all pure payoffs are integers. $\mathrm{Proj}_{X_*}(\mathrm{NE}(\Gamma))$ is thus also compact and non empty, and it is $\mathbb{Z}$-semi algebraic by, for example, Theorem 2.92 in [2]. The set $\mathrm{NEP}(\Gamma)$ is clearly compact and nonempty. Remark that it is a projection of the set $\{(x, y_1, \cdots, y_n),\ x \in \mathrm{NE}(\Gamma),\ y_i = g^i(x)\}$. This last set is $\mathbb{Z}$-semi algebraic since $\mathrm{NE}(\Gamma)$ is and the g^i are polynomials in $Z[X]$; so once again Theorem 2.92 in [2] gives the result. □

Proposition 18. *Let $N \geq 3$, and $F \subset [0,1[^N$ be a nonempty closed $\mathbb{Z}$-semi algebraic set. Then there exists an N-player finite game Γ with pure payoffs in $\mathbb{Z}$, and a particular pure action profile $X^* = (X_*^1, \cdots, X_*^N)$ such that*

a) $\mathrm{Proj}_{X_*}(\mathrm{NE}(\Gamma)) = F$

b) $\mathrm{NEP}(\Gamma) = \{0\}$.

Moreover, if $F \subset [0, 1 - \frac{1}{M}[$ *for some integer* M*,* $F = \bigcap_{k=1}^{K} \bigcup_{l=1}^{L} \{z \in \mathbb{R}^n,\ P_{k,l}(z) \leq 0\}$*,* c *is a bound on the coefficients of the polynomials,* D *is a bound on the degree of the polynomials, and* $\widehat{z} \in F$ *is a tuple of algebraic numbers, each of which is given as the unique solution of a polynomial of degree less than some* D' *in an interval whose endpoints are two rational numbers, and* S *is a bound for the denominators of all these rational numbers, then*
- the number of actions of each player in Γ *is bounded by a polynomial in* D*,* D'*,* K *and* L*.*
- the bitsizes of the pure payoffs of Γ *are bounded by a polynomial in* $\log M$*,* $\log c$*,* $\log S$*,* K*,* L *and* D'*.*

Proof. First of all remark that it is enough to construct a game Γ with rational pure payoffs, as one can then multiply all payoffs by a suitable integer without changing properties a) and b).

In our constructions in sections 4 and 5, the payoffs depends on the coefficients of the polynomials, on some parameters chosen small or large enough (hence rationals if one wishes so), and of the coordinates of a particular $\widehat{z} \in F$. The proposition is thus established as soon as $F \cap \mathbb{Q}^N$ is nonempty. In general unfortunately there is no reason for F to contain a point with rational coordinates ; however Tarski-Seidenberg theorem guarantees that F contains a point with algebraic coordinates. Let $\widehat{z}$ be such a point. The idea is to adapt Step 2 of the proof so that there is a unique bad equilibrium, which projects to $\widehat{z}$, with using payoffs with rational coefficients. Step 1 and 3 are not modified. Recall that there were already adjustments made to Step 2 in the case of an even number of players (in Section 7.1) and of a $\widehat{z}$ with some zero coordinates (in Section 5.3). For the sake of simplicity we first assume here that N is odd and that all coordinates of $\widehat{z}$ are positive ; we will deal with the general case at the end of the proof.

For every i let $P_i \in \mathbb{Z}[X]$ be a polynomial with only single roots and such that $P_i(\widehat{z_i}) = 0$ (take for example the minimal polynomial of $\widehat{z_i}$). Changing the sign of P_i if necessary, one may assume that $P_i(z) > 0$ for small positive $z - \widehat{z_i}$, while $P_i(z) < 0$ for small negative $z - \widehat{z_i}$. Let D' be strictly larger than the degree of all P_i.

The construction will now use in Step 2, in addition to Y_*^i, $D' + 2$ new constraints for each player denoted by the letter W ; denote by $\mathcal{W}^i$ the set of these constraints. The basic idea is to ensure that in all bad equilibria $P_i(x_*^i) = 0$. Let us first briefly explain the role of each new constraints:

- a new constraint is denoted W_v^i, v meaning variable. As there is no reason for the P_i to have a single zero in $[0, 1]$, proving $P_i(x_*^i) = 0$ does not imply that $x_*^i = \widehat{z_i}$. The role of W_v^i is thus to "translate" x_*^i: we will prove that in any bad equilibrium one has $x_*^i = \beta_i + (\gamma_i - \beta_i) w_v^{i-2}$ (for constants β_i and γ_i to be determined), ensuring that x_*^i lies in some interval $[\beta_i, \gamma_i]$ in which the only zero of P_i is $\widehat{z_i}$.
- D' new constraints W_t^i, $1 \leq t \leq D'$ will represent powers of w_v^{i-2}. For technical reasons we will use "translated" powers: we will prove that in any bad equilibria
$$w_t^i - \theta t = \theta (w_v^{i-2})^t.$$
for some θ to be determined.
- The last constraint is denoted W_P^i (P meaning polynomial). Its payoff will depend on the w_t^{i+1}, and hence, by the previous relations, on powers of x_*^{i+1}, in such a way that $P_{i+1}(x_*^{i+1}) = 0$ in any bad equilibrium.

Before giving the precise payoffs we first define some new constants. For each i, let $0 < \beta_i$ and $\gamma_i < 1$ be two rationals such that $\widehat{z_i}$ is the only zero of P_i in $]\beta_i, \gamma_i[$. Increasing β_i if needed, we assume that $\widehat{z_i}$ is in the very left of the interval in the following sense:
$$\frac{\widehat{z_i} - \beta_i}{\gamma_i - \beta_i} < 1 - \gamma_i.$$
Because of our assumptions on P_i, $P_i(z)$ is negative on $[\beta_i, \widehat{z_i}[$ and positive on $]\widehat{z_i}, \gamma_i]$. Denote by Q_i the polynomial $Q_i(z) = P_i(\beta_i + (\gamma_i - \beta_i) z)$. By definition of D' one can write $Q_i(z) = \sum_{t=0}^{D'-1} b_{i,t} z^t$, and the $b_{i,t}$ are rationals, with $b_{i,0} < 0$. Let b be a rational majorant of all the $|b_{i,t}|$.

Fix also two positive rationals θ and $\widehat{y}$ such that

$$\widehat{y} + (D'+2)^2\theta + \gamma_i + \frac{\widehat{z}_i - \beta_i}{\gamma_i - \beta_i} < 1 \tag{21}$$

for all i. Finally chose a positive rational $\delta < \frac{1}{2}$ such that

$$\begin{aligned} \frac{\delta}{1-\delta} &< \frac{\theta}{4} && (22)\\ \frac{\delta}{1-\delta} &< \beta_i \text{ for all } i. && (23)\\ \frac{\delta}{1-\delta} &< -\frac{\theta b_{i,0}}{2D'b} \text{ for all } i. && (24) \end{aligned}$$

We are now ready to give the definitions of the payoffs. Step 1 is exactly as in the previous sections, with the constants ϵ, η, τ chosen in $\mathbb{Q}$. At the end of this step all pure payoffs are thus rationals and properties a) and b) are satisfied for nice equilibria. In Step 2 one choses a rational $C > \max(1, 2c)$ and define, as in Section 5.2,

$$g^i(Y^i_*) = C\left(1 - \frac{x_0^{i+1}}{\epsilon}\right). \tag{25}$$

One modifies the payoff of X^i_* in the following way:

$$g^i(X^i_*) = \frac{C}{\widehat{y}}\left(1 - \sum_{x^{i+1}\in\mathcal{X}^{i+1}} x^{i+1} - \sum_{w^{i+1}\in\mathcal{W}^{i+1}} w^{i+1}\right). \tag{26}$$

One now defines the payoff of the constraints in $\mathcal{W}^i$:

$$\begin{aligned} g^i(W^i_1) &= g^i(X^i_*) + \delta(x_*^{i+1} - \beta_{i+1} - (\gamma_{i+1} - \beta_{i+1})w_v^{i-1}) - \frac{2x_0^{i+1}}{\epsilon} && (27)\\ g^i(W^i_2) &= g^i(W^i_1) + \delta^2(w_1^{i+1} - \theta - \theta w_v^{i-1}) && (28)\\ g^i(W^i_t) &= g^i(W^i_{t-1}) + \frac{\delta^t}{4}\left(w_{t-1}^{i+1} - (t-1)\theta - w_v^{i-1}(w_{t-2}^{i+1} - (t-2)\theta)\right) \text{ for } 3 \le t \le D' && (29)\\ g^i(W^i_v) &= g^i(W^i_{D'}) + \frac{\delta^{D'+1}}{4}\left(w_{D'}^{i+1} - D'\theta - w_v^{i-1}(w_{D'-1}^{i+1} - (D'-1)\theta)\right) && (30)\\ g^i(W^i_P) &= g^i(W^i_v) + \delta^{D'+2}\frac{\theta b_{i+1,0} + \sum_{t=1}^{D'-1} b_{i+1,t}(w_t^{i+1} - t\theta)}{2D'b}. && (31) \end{aligned}$$

Note that all of these payoffs are rational when players choose pure strategies. We now prove the same seven claims that in Section 4.

Claim 1 : adding these actions does not change the set of nice equilibria. Indeed, in any nice equilibria $g^i(X^i_*)$ is still 0, and $g^i(Y^i_*)$ is nonpositive since $x_0^{i+1} \ge \epsilon$.

One checks easily that in any profile,

$$\begin{aligned} |g^i(W^i_{t+1}) - g^i(W^i_t)| &\le \delta^{t+1} \text{ for all } 1 \le t < D'\\ |g^i(W^i_v) - g^i(W^i_{D'})| &\le \delta^{D'+1}\\ |g^i(W^i_P) - g^i(W^i_v)| &\le \delta^{D'+2} \end{aligned}$$

Hence one has for any $W^i \in \mathcal{W}$ in any nice equilibrium:

$$
\begin{aligned}
g^i(W^i) &\leq g^i(X^i_*) + \delta - \delta\frac{2x_0^{i+1}}{\epsilon} + \sum_{t=2}^{D'+2}\delta^t \\
&\leq \frac{\delta}{1-\delta} - 2\delta \\
&< 0 \text{ since } \delta < \frac{1}{2}
\end{aligned} \tag{32}
$$

and the claim is verified.

Claim 2 : if the payoff in a bad equilibrium was 0 for each player, it would imply that $g^i(Y^i_*) \leq 0$ and $x_0^{i+1} \geq \epsilon$. As in (32) this immediately implies that $g^i(W^i) < g^i(X^i_*)$ for all $W^i \in \mathcal{W}$ and these constraints are not played. Hence for all i,

$$0 \geq g^i(X^i_*) = \frac{C}{\widehat{y}}\left(1 - \sum_{x^{i+1}\in\mathcal{X}^{i+1}} x^{i+1}\right)$$

and the equilibrium is nice, a contradiction. □

Claim 3, 4 and 5 follow exactly as in Section 4. Hence, in any bad equilibrium, the only strategies that may be played by player i with positive probability are X^i_*, Y^i_*, and the W^i_t. This implies that, in bad equilibria, the following actions have simplified payoffs:

$$
\begin{aligned}
g^i(Y^i_*) &= C \\
g^i(X^i_*) &= C\frac{y^i_*}{\widehat{y}} \\
g^i(W^i_1) &= g^i(X^i_*) + \delta(x^{i+1}_* - \beta_{i+1} - (\gamma_{i+1} - \beta_{i+1})w_v^{i-1})
\end{aligned}
$$

We now come to the bulk of the proof, Claim 6, that is the fact that there is a unique bad equilibrium which projects to $\widehat{z}$. For this we use Lemma 14 with $T = D'+3$ and $A^i_0, \cdots, A^i_{D'+3}$ being named $Y^i_*, X^i_*, W^i_1, \cdots, W^i_{D'}, W^i_v, W^i_P$ in that order. For $a \in \Delta := \Pi_i\Delta(Y^i_*, X^i_*, W^i_1, \cdots, W^i_{D'}, W^i_v, W^i_P)$ define

$$
\begin{aligned}
f^i_*(a) &= \frac{C}{\widehat{y}}(y^{i+1}_* - \hat{y}) \\
f^i_1(a) &= x^{i+1}_* - \beta_{i+1} - (\gamma_{i+1} - \beta_{i+1})w_v^{i-1} \\
f^i_2(a) &= w_1^{i+1} - \theta - \theta w_v^{i-1} \\
f^i_t(a) &= \frac{1}{4}\left[w^{i+1}_{t-1} - (t-1)\theta - w_v^{i-1}(w^{i+1}_{t-2} - (t-2)\theta)\right] \text{ for } 3 \leq t \leq D' \\
f^i_v(a) &= \frac{1}{4}\left[w^{i+1}_{D'} - D'\theta - w_v^{i-1}(w^{i+1}_{D'-1} - (D'-1)\theta)\right] \\
f^i_P(a) &= \frac{\theta b_{i+1,0} + \sum_{t=1}^{D'-1} b_{i+1,t}(w_t^{i+1} - t\theta)}{2D'b}.
\end{aligned}
$$

Assume for a moment that the family of functions f is δ-circular, then by Lemma 14 in any bad equilibrium all functions are equal to 0, which implies that for every i and t,

$$
\begin{aligned}
x^i_* &= \beta_i + (\gamma_i - \beta_i)w_v^{i-2} && (33) \\
w^i_t &= \theta(t + (w_v^{i-2})^t). && (34)
\end{aligned}
$$

Now looking at f^i_P this implies $0 = Q_{i+1}(w_v^{i-1}) = P_{i+1}(x^{i+1}_*)$, and since $x^{i+1}_* \in [\beta_{i+1}, \gamma_{i+1}]$ by (33) one gets $x_* = \widehat{z}$. Hence one has just to verify that the family of functions f is δ-circular.

Assumption 1) of Definition 7.1 clearly holds. Define for each player i

$$\begin{aligned}\bar{y}^i_* &= \widehat{y}+\theta\\ \bar{x}^i_* &= \gamma_i+\theta\\ \bar{w}^i_t &= (t+2)\theta \text{ for all } 1\le t\le D'\\ \bar{w}^i_v &= \frac{\widehat{z}_i-\beta_i}{\gamma_i-\beta_i}+\theta\end{aligned}$$

Then by (21), assumption 2)i) of Definition 7.1 holds as well. Also define $\kappa_t(i)=i-1$ for every t except $\kappa_v(i)=i+1$. Since N is odd assumption 2)ii) of Definition 7.1 is satisfied. It remains to check assumptions 2)iii and 2)iv)

- If $y^i_*\ge\bar{y}^i_*$, then $f^{i-1}_*(a)\ge\frac{C\theta}{\widehat{y}}>0$.
- If $y^i_*=0$ then $f^{i-1}_*(a)=-C\le-1<-\frac{\delta}{1-\delta}$.
- If $x^i_*\ge\bar{x}^i_*$, then $f^{i-1}_1(a)\ge\theta>0$.
- If $x^i_*=0$ then $f^{i-1}_1(a)\le-\beta_{i+1}<-\frac{\delta}{1-\delta}$ by (23).
- If $w^i_1\ge\bar{w}^i_1$ then $f^{i-1}_2(a)\ge\theta>0$.
- If $w^i_1=0$ then $f^{i-1}_2(a)\le-\theta<-\frac{\delta}{1-\delta}$.
- Let $2\le t_0\le D'$, and assume that $f^j_t(a)=0$ for every j and every $1\le t<t_0$, and $f^j_{t_0}(a)\le 0$ for every j. Then one proves easily by induction that $w^j_t=\theta(t+(w^{j-2}_v)^t)$ for all j and $1\le t\le t_0-2$, and that $w^j_{t_0-1}\le\theta(t_0-1+(w^{j-2}_v)^{t_0-1})$ for all j. Hence $w^i_{t_0}\ge\bar{w}^i_{t_0}$ implies[6]

$$\begin{aligned}f^{i-1}_{t_0+1}(a) &\ge \frac{1}{4}\left[2\theta-(w^i_{t_0-1}-(t_0-1)\theta)\right]\\ &\ge \frac{1}{4}\left[2\theta-\theta(w^{i-2}_v)^{t_0-1}\right]\\ &> 0.\end{aligned}$$

- Let $2\le t_0\le D'$, and assume that $w^i_{t_0}=0$. Then[7] $f^{i-1}_{t_0+1}(a)\le-\frac{\theta}{4}<-\frac{\delta}{1-\delta}$.
- Assume finally that $f^j_*(a)=0$ for every j and that $f^j_t(a)=0$ for every j and every $1\le t\le D'$. This implies on one hand that $x^j_*=\beta_j+(\gamma_j-\beta_j)w^{j-2}_v$ for all j, and on the other hand that $w^j_t=\theta(t+(w^{j-2}_v)^t)$ for all j and $1\le t\le D'-1$. Hence

$$\begin{aligned}f^{i+1}_P(a) &= \frac{Q_{i+2}(w^i_v)}{2D'b}\\ &= \frac{P_{i+2}(x^{i+2}_*)}{2D'b}.\end{aligned}$$

  Now if $w^i_v\ge\bar{w}^i_v$ then $x^{i+2}_*>\widehat{z_{i+2}}$ and, since $x^{i+2}_*\le\gamma_{i+2}$, $f^{i+1}_P(a)=\frac{P_{i+2}(x^{i+2}_*)}{2D'b}\ge 0$. If $w^i_v=0$, $f^{i+1}_P(a)=\frac{Q_{i+2}(0)}{2D'b}=\frac{\theta b_{i,0}}{2b}<-\frac{\delta}{1-\delta}$.

We have thus verified that f is δ-circular which ends the proof of Step 2. Step 3 is the same as in Section 4.

It remains to explain the adjustments to the case of a $\widehat{z}$ with some zero coordinates, and of an even number of players. If $\widehat{z}$ has some zero coordinates, we just have to use a fictitious $\widehat{z}'$ as in Section 5.3, the details being left to the reader. If N is even and greater than 6, as in Section 7.1 we cut N in two sets of odd cardinality and with at least three elements each. Call type 1 and type 2 elements of these respective sets. The payoff of the Y^i_* is defined as in (25) ; the payoffs of the X^i_* and the W^i is defined as in equation (26) to (31) where $i+1$ and $i-1$ are replaced by "the next player of the same type" and "the previous player of the same type" respectively. Claim 1 to 5 follows similarly. For Claim 6 we use Lemma 14 in which $\kappa_t(i)$ is the previous (or

[6] When $t_0=D'$ we write t_0+1 for v

[7] See previous footnote

next if $t = v$) player of the same type as i. Since there are an odd number of players of each type assumption 2)ii) is satisfied and the rest of the proof is the same.

When $N = 4$ one needs to adapt the construction, using Lemma 16 instead of Lemma 14

Proposition 19. *Let $N \geq 3$, and F be a nonempty compact $\mathbb{Z}$-semi algebraic set. Then F is the set of equilibrium payoffs of some finite N-player game with pure payoffs in $\mathbb{Z}$.*

Proof. Let B be an integer such that $F \subset]-B, B[^N$. Define $F' = \frac{B+F}{4B}$, thus $F' \subset]0, \frac{1}{2}[$ and one may apply the previous proposition to F' to get a game Γ, with integer pure payoffs, such that

a) $\mathrm{Proj}_{X_*}(\mathrm{NE}(\Gamma)) = F'$
b) $\mathrm{NEP}(\Gamma) = \{0\}$.

As in the proof of Theorem 8, by adding 1 to the payoff of each player i iff player $i-1$ plays X_*^{i-1}, and then relabelling the players, we get a game Γ' with integer pure payoffs and whose set of equilibrium payoffs is F'. Then $4B\Gamma - B$ is a game with integer pure payoffs and whose set of equilibrium payoffs is F. $\square$

7.3. **Projection on more than one action per player.**

Definition 20. Let $N \in \mathbb{N}^*$ and $(T_1, \cdots, T_N) \in (\mathbb{N}^*)^N$. Let $F \subset \mathbb{R}^{T_1+\cdots+T_N}$ and denote the coordinates of any element $z \in F$ as $z_{i,t}$ for $i = 1$ to N and $t = 1$ to T_i. F is $(T_1, \cdots, T_N)$-admissible if the following properties are satisfied for all $z \in F$:

- $z_{i,t} \geq 0$ for all i and t
- for all i, $\sum_{t=1}^{T_i} z_{i,t} \leq 1$.

F is strongly $(T_1, \cdots, T_N)$-admissible if the second property is replaced for all $z \in F$ by

- for all i, $\sum_{t=1}^{T_i} z_{i,t} < 1$.

We now generalize Proposition 9 to projection of the set of Nash equilibria on the T_i first actions of each player i. Clearly such a projection is always nonempty, closed, semialgebraic and $(T_1, \cdots, T_N)$-admissible. We now prove a reciprocal:

Proposition 21. *Let $N \geq 3$ and $(T_1, \cdots, T_N) \in (\mathbb{N}^*)^N$. Let $F \subset \mathbb{R}^{T_1+\cdots+T_N}$ be a nonempty closed semi algebraic set and assume it is strongly $(T_1, \cdots, T_N)$-admissible. Then there exists an N-player game Γ, and T_i special actions $X_{*,1}^i, \cdots, X_{*,T_i}^i$ for each player i, such that*

a) $\mathrm{Proj}_{\{X_{*,t}^i\}}(\mathrm{NE}(\Gamma)) = F$
b) $\mathrm{NEP}(\Gamma) = \{0\}$.

Proof. The general architecture of the construction follows the one for Proposition 9. We do not detail Step 1 as it is very similar : one define unknows and constraints so that every monomial in the $x_{*,t}^i$ is represented in nice equilibria, and semi algebraic constraints such that $x_* \in F$ in any nice equilibrum. The assumption $N \geq 3$ ensures that one can make products in the definition of the payoff functions, and thus that this construction is possible. As in the original construction define for each player a dump unknown X_0^i such that $x_0^i \geq \epsilon$ in every nice equilibrium, for some positive ϵ.

The difficulty lies in generalizing Step 2. First of all we remark that without loss of generality we may assume that $T_i = T$ for all i. Indeed if this is not the case, let T be the largest T_i and define $F' \in \mathbb{R}^{NT}$ as any nonempty closed semi algebraic and strongly $(T, \cdots, T)$-admissible set, such that $\mathrm{Proj}_{\{X_{*,t}^i\}}(F') = F$ (one may for example take $F' = F \times \{0\}^{NT-T_1-\cdots-T_N}$). Now apply the result to F' and then project on $\{X_{*,j}^i\}$.

Let now $\widehat{z} = (\widehat{z_{1,1}}, \cdots, \widehat{z_{N,T_N}}) \in F$. We assume for the moment that $\widehat{z_{i,t}} > 0$ for all i and t, the other case being explained briefly in the end of the proof. As usual the contruction will be different for odd and even N.

Asume first that N is odd. Let C be large enough, and define for all player i the constraint Y_*^i by

$$g^i(Y_*^i) := C\left(1 - \frac{x_0^{i+1}}{\epsilon}\right).$$

One now need to define the payoffs of all $X^i_{*,t}$. For this let $\delta > 0$ be such that $\frac{\delta}{1-\delta} < \widehat{z_{i,t}}$ for all i and t, and denote $\widehat{y_i} := 1 - \sum_{t=1}^{T} \widehat{z_{i,t}}$ which is positive for all i by strong admissibility. Now

$$g^i(X^i_{*,t}) := \frac{C}{\widehat{y_{i+1}}} \left(1 - \sum_{x^{i+1} \in \mathcal{X}^{i+1}} x^{i+1} \right) \left(1 + \sum_{t'=1}^{t-1} \delta^{t'} (x^{i-1}_{*,t'} - \widehat{z_{i-1,t'}}) \right). \tag{35}$$

We remark that for all i and t, $g^i(X^i_{*,t}) = 0$ if and only if Player $i+1$ only plays unknowns with positive probability. Claim 1 to 5 follows as in the original contruction, and ensure that all nice equilibria have a payoff of zero, that the projection of the set of nice equilibria is F, and that the only actions that may be played in any bad equilibrium are the $X^i_{*,t}$ and Y^i_*. We now prove that there is a unique bad equilibrium, which projects on $\widehat{z}$. Fix a bad equilibrium, and remark that, since unknowns other than the $X^i_{*,t}$ are not played, payoff of action $X^i_{*,t}$ can be written as

$$g^i(X^i_{*,t}) := \frac{C y^{i+1}_*}{\widehat{y_{i+1}}} \left(1 + \sum_{t'=1}^{t-1} \delta^{t'} (x^{i-1}_{*,t'} - \widehat{z_{i-1,t'}}) \right)$$

while $g^i(Y^i_*) = C$. We first claim that $y^i_* \in]0,1[$ for all i. Indeed, $y^{i+1}_* = 0$ would imply $g^i(X^i_{*,t}) = 0 < C = g^i(Y^i_*)$ for all t so $y^i_* = 1$; while $y^{i+1}_* = 1$ would imply that $g^i(X^i_{*,1}) > C = g^i(Y^i_*)$ so $y^i_* = 0$. Since N is odd the usual circular argument apply. Hence Y^i_* is played with positive probability and the payoff of each player is C. Hence $g^i(X^i_{*,t}) \leq C$ and

$$y^i_* \leq \widehat{y_i} \tag{36}$$

for all i.

We now prove by induction on t that for every t from 1 to $T-1$, $x^i_{*,t'} > 0$ for all i. Start with $t = 1$ and assume by contradiction that $x^i_{*,1} = 0$ for some player i. Looking now at the payoff of Player $i+1$ for any action $X^{i+1}_{*,t}$ with $t > 1$ we see that

$$\begin{aligned}
g^i(X^{i+1}_{*,t}) &= \frac{C y^{i+2}_*}{\widehat{y_{i+2}}} \left(1 + \sum_{t'=1}^{t-1} \delta^{t'} (x^i_{*,t'} - \widehat{z_{i,t'}}) \right) \\
&\leq \frac{C y^{i+2}_*}{\widehat{y_{i+2}}} \left(1 - \delta \widehat{z_{i,1}} + \sum_{t'=2}^{t-1} \delta^{t'} \right) \\
&< \frac{C y^{i+2}_*}{\widehat{y_{i+2}}} \text{ by definition of } \delta \\
&= g^i(X^{i+1}_{*,1}).
\end{aligned}$$

Hence $x^{i+1}_{*,t} = 0$ for all $t > 1$ and (36) implies $x^{i+1}_{*,1} = 1 - y^{i+1}_* > z_{i+1,1}$. Now looking at player $i+2$ one gets

$$\begin{aligned}
g^i(X^{i+2}_{*,2}) &= \frac{C y^{i+3}_*}{\widehat{y_{i+3}}} \left(1 + \delta (x^{i+1}_{*,1} - \widehat{z_{i+1,1}}) \right) \\
&> \frac{C y^{i+3}_*}{\widehat{y_{i+3}}} \\
&= g^i(X^{i+2}_{*,1})
\end{aligned}$$

and $x^{i+2}_{*,1} = 0$. Since N is odd a circular argument implies that all $x^i_{*,1}$ are both equal to 0 and larger than $z_{i,1}$, a contradiction which establishes the induction hypothesis for $t = 1$.

Let now $1 < t \leq T-1$ and assume that the induction hypothesis is true for every $t' < t$. In particular $x^i_{*,1} > 0$ for every i hence $g^i(X^i_{*,1}) = C$ and

$$y^i_* = \widehat{y_i} \tag{37}$$

for every i. Since $x^i_{*,t'} > 0$ for every $2 \le t' < t$, one gets $g^i(X^i_{*,t'}) = C$ and

$$x^i_{*,t'} = \widehat{z_{i,t'}} \tag{38}$$

for every i and $t' \le t-2$. Also, $g^i(X^i_{*,t}) \le C$ implies

$$x^i_{*,t-1} \le \widehat{z_{i,t-1}} \tag{39}$$

for every i. Assume now by contradiction that $x^i_{*,t} = 0$ for some player i. Looking at Player $i+1$ payoff for $X^{i+1}_{*,t''}$ for $t'' > t$ one gets

$$\begin{aligned} g^i(X^{i+1}_{*,t''}) &= C\left(1 + \sum_{t'=t}^{t''-1} \delta^{t'}(x^i_{*,t'} - \widehat{z_{i,t'}})\right) \\ &\le C\left(1 - \delta\widehat{z_{i,t}} + \sum_{t'=t+1}^{t''} \delta^{t'}\right) \\ &< C \text{ by definition of } \delta. \end{aligned}$$

Hence $x^{i+1}_{*,t''} = 0$ for all $t'' > t$. Equations (37) to (39) then imply that

$$x^{i+1}_{*,t} \ge 1 - \widehat{y_i} - \sum_{t'=1}^{t-1} \widehat{z_{i,t'}} = \sum_{t''=t}^{T} \widehat{z_{i,t''}}$$

and $x^{i+1}_{*,t} > \widehat{z_{i,t}}$. Now looking at player $i+2$ one gets

$$\begin{aligned} g^i(X^{i+2}_{*,t+1}) &= C\left(1 + \delta^{t-1}(x^{i+1}_{*,t-1} - \widehat{z_{i+1,t-1}}) + \delta^t(x^{i+1}_{*,t} - \widehat{z_{i+1,t}})\right) \\ &> C\left(1 + \delta^{t-1}(x^{i+1}_{*,t-1} - \widehat{z_{i+1,t-1}})\right) \\ &= g^i(X^{i+2}_{*,t}) \end{aligned}$$

and $x^{i+2}_{*,t} = 0$. Once again a circular argument yields the desired contradiction.

We have thus proved that $x^i_{*,t'} > 0$ for all i and $1 \le t < T$. Hence (38) applies for all i and $1 \le t < T-1$, while (39) reads as

$$x^i_{*,T-1} \le \widehat{z_{i,T-1}} \tag{40}$$

for all i. Combining all this with (37) we get

$$x^i_{*,T} \ge 1 - \widehat{y_i} - \sum_{t'=1}^{T-1} \widehat{z_{i,t'}} = \widehat{z_{i,T}} \tag{41}$$

hence $x^i_{*,T} > 0$ as well. This imply that (40), and then (41), are equalities for all i. Finally $x^i_{*,t} = \widehat{z_{i,t}}$ for all i and t, hence there is a unique bad equilibrium and its projection is $\widehat{z} \in F$. This concludes Step 2, and Step 3 is the same as in the proof of Proposition 9.

If N is even and $N \ge 6$, we do as in Section 7.1: we separate the players in two types depending of whether they belong to $\{1, 2, 3\}$ or $\{4, \cdots, N\}$, and in (35) we replace $i+1$ and $i-1$ by, respectively, "the next player of the same type" and "the previous player of the same type". Everything follows then exactly as in the odd case since both 3 and $N-3$ are odd.

Once again the more problematic case is $N = 4$. In that case we define, as in the odd case,

$$g^i(Y^i_*) := C\left(1 - \frac{x^{i+1}_0}{\epsilon}\right).$$

The payoffs of the $X^i_{*,t}$ are more complex. Denote $\widehat{y_i} := 1 - \sum_{t=1}^{T} \widehat{z_{i,t}}$, which is positive for all i by strong admissibility. Also fix $0 < \alpha < \frac{1}{2}$ such that

$$\begin{aligned}
\alpha &< \frac{\widehat{y_2}}{2} && (42)\\
\alpha &< \frac{1-\widehat{y_2}}{2} && (43)\\
\alpha &< \frac{\widehat{z_{i,t}}}{2} \text{ for all } i \text{ and } t && (44)\\
\frac{\alpha}{1-\alpha} &< \widehat{z_{i,t}} \text{ for all } i \text{ and } t && (45)
\end{aligned}$$

and then $0 < \delta$ such that

$$\begin{aligned}
\frac{1+2\delta}{(1-\delta)^2} &< \frac{1}{1-\alpha} && (46)\\
\frac{\delta}{(1-\delta)} &< \frac{\widehat{z_{i,t}}}{2} - \alpha \text{ for all } i \text{ and } t && (47)\\
\frac{\delta}{(1-\delta)} &< \frac{\alpha\widehat{z_{i,t}}}{2} \text{ for all } i \text{ and } t. && (48)
\end{aligned}$$

Then

$$\begin{aligned}
g^1(X^1_{*,t}) &:= \left(\alpha\frac{C}{\widehat{y_2}}\left(1-\sum_{x^2\in\mathcal{X}^2} x^2\right) + (1-\alpha)\frac{C}{\widehat{y_3}}\left(1-\sum_{x^3\in\mathcal{X}^3} x^3\right)\right)\left(1+\sum_{t'=1}^{t-1}\delta^{t'}(x^4_{*,t'}-\widehat{z_{4,t'}})\right)\\
g^2(X^2_{*,t}) &:= \frac{2C}{\widehat{y_1}}\left(1-\sum_{x^1\in\mathcal{X}^1} x^1\right) - \frac{C}{\widehat{y_3}}\left(1-\sum_{x^3\in\mathcal{X}^3} x^3\right)\\
&\quad +\frac{C}{\widehat{y_4}}\left(1-\sum_{x^4\in\mathcal{X}^4} x^4\right)\left(2\sum_{t'=1}^{t-1}\delta^{t'}(x^3_{*,t'}-\widehat{z_{3,t'}}) - \sum_{t'=1}^{t-1}\delta^{t'}(x^1_{*,t'}-\widehat{z_{1,t'}})\right)\\
g^3(X^3_{*,t}) &:= \frac{C}{\widehat{y_4}}\left(1-\sum_{x^4\in\mathcal{X}^4} x^4\right)\left(1+\sum_{t'=1}^{t-1}\delta^{t'}\left(\alpha(x^2_{*,t'}-\widehat{z_{2,t'}})+(1-\alpha)(x^1_{*,t'}-\widehat{z_{1,t'}})\right)\right)\\
g^4(X^4_{*,t}) &:= \frac{C}{\widehat{y_1}}\left(1-\sum_{x^1\in\mathcal{X}^1} x^1\right)\left(1+\sum_{t'=1}^{t-1}\delta^{t'}(x^3_{*,t'}-\widehat{z_{3,t'}})\right).
\end{aligned}$$

It is immediate that all $g^i(X^i_{*,t})$ equal 0 in any nice equilibrium. In any bad equilibrium, either $g^1(X^1_{*,1}) > 0$ (if $\sum_{x^2\in\mathcal{X}^2} x^2 < 1$ or $\sum_{x^3\in\mathcal{X}^3} x^3 < 1$), $g^3(X^3_{*,1}) > 0$ (if $\sum_{x^4\in\mathcal{X}^4} x^4 < 1$) or $g^4(X^4_{*,1}) > 0$ (if $\sum_{x^1\in\mathcal{X}^1} x^1 < 1$). Claim 1 to 5 of the original proof then follow easily: all nice equilibria have a payoff of zero, the projection of the set of nice equilibria is F, and the only actions that may be played in any bad equilibrium are the $X^i_{*,t}$ and Y^i_*. We now prove that there is a unique bad equilibrium, which projects on $\widehat{z}$. Fix a bad equilibrium, and remark that, since unknowns other than the $X^i_{*,t}$ are not played, payoffs of actions $X^i_{*,t}$ can be written as

$$\begin{aligned}
g^1(X^1_{*,t}) &:= \left(\alpha\frac{Cy^2_*}{\widehat{y_2}} + (1-\alpha)\frac{Cy^3_*}{\widehat{y_3}}\right)\left(1+\sum_{t'=1}^{t-1}\delta^{t'}(x^4_{*,t'}-\widehat{z_{4,t'}})\right)\\
g^2(X^2_{*,t}) &:= \frac{2Cy^1_*}{\widehat{y_1}} - \frac{Cy^3_*}{\widehat{y_3}} + \frac{Cy^4_*}{\widehat{y_4}}\left(2\sum_{t'=1}^{t-1}\delta^{t'}(x^3_{*,t'}-\widehat{z_{3,t'}}) - \sum_{t'=1}^{t-1}\delta^{t'}(x^1_{*,t'}-\widehat{z_{1,t'}})\right)\\
g^3(X^3_{*,t}) &:= \frac{Cy^4_*}{\widehat{y_4}}\left(1+\sum_{t'=1}^{t-1}\delta^{t'}\left(\alpha(x^2_{*,t'}-\widehat{z_{2,t'}})+(1-\alpha)(x^1_{*,t'}-\widehat{z_{1,t'}})\right)\right)\\
g^4(X^4_{*,t}) &:= \frac{Cy^1_*}{\widehat{y_1}}\left(1+\sum_{t'=1}^{t-1}\delta^{t'}(x^3_{*,t'}-\widehat{z_{3,t'}})\right).
\end{aligned}$$

while $g^i(Y^i_*) = C$.

We first claim that $y^i_* > 0$ for all i. As in the construction for odd N, $y^4_* = 0$ would imply $g^3(X^i_{*,t}) = 0 < C = g^3(Y^i_*)$ for all t so $y^3_* = 1$; while $y^4_* = 1$ would imply that $g^3(X^3_{*,1}) > C = g^3(Y^3_*)$ so $y^3_* = 0$. Similarly $y^1_* = 0$ would imply $y^4_* = 1$ while $y^1_* = 1$ would imply $y^4_* = 0$. Finally, $y^3_* = 0$ would imply $g^1(X^1_{*,t}) \leq \frac{2\alpha C}{\widehat{y_2}} < C = g^1(Y^i_*)$ for all t so $y^1_* = 1$, while $y^3_* = 1$ would imply that $g^1(X^1_{*,1}) \geq \frac{(1-\alpha)C}{\widehat{y_2}} > C = g^1(Y^3_*)$ so $y^1_* = 0$. Hence $y^i_* \in]0,1[$ for all $i \in \{1,3,4\}$. In particular the payoff at equilibrium of these players is C. Thus $g^4(X^4_{*,t}) \leq C$ and $g^3(X^3_{*,t}) \leq C$ so $y^1_* \leq \widehat{y_1}$ and $y^4_* \leq \widehat{y_4}$. Assume now by contradiction that $y^2_* = 0$. Recall that $y^1_* < 1$ (or else $y^4_* = 0$), so there must exists t such that

$$\begin{aligned}
C &= g^1(X^1_{*,t})\\
&= (1-\alpha)\frac{Cy^3_*}{\widehat{y_3}}\left(1+\sum_{t'=1}^{t-1}\delta^{t'}(x^4_{*,t'}-\widehat{z_{4,t'}})\right)\\
&\leq (1-\alpha)\frac{Cy^3_*}{(1-\delta)\widehat{y_3}}
\end{aligned}$$

hence $\frac{y^3_*}{\widehat{y_3}} \geq \frac{1-\delta}{1-\alpha}$. Plugging all this in the definition of $g^2(X^2_{*,t})$ yields

$$\begin{aligned}
g^2(X^2_{*,t}) &\leq C\left(2-\frac{1-\delta}{1-\alpha}+2\sum_{t'=1}^{t-1}\delta^{t'}(x^3_{*,t'}-\widehat{z_{3,t'}}) - \sum_{t'=1}^{t-1}\delta^{t'}(x^1_{*,t'}-\widehat{z_{1,t'}})\right)\\
&\leq C\left(2-\frac{1-\delta}{1-\alpha}+3\frac{\delta}{1-\delta}\right)\\
&< C\left(2-\frac{1+2\delta}{1-\delta}+3\frac{\delta}{1-\delta}\right) \text{ by (46)}\\
&= C
\end{aligned}$$

for all t, hence $y^2_* = 1$ a contradiction. We have thus proved that $y^i_* > 0$ for all i.

We now prove by induction on t that for every t from 1 to $T-1$, $x^i_{*,t'} > 0$ for all i. Start with $t = 1$. Since $g^i(X^i_{*,t}) \leq C$ for $i = 1$, 3 and 4 repectively we have

$$y^3_{*,1} \leq \frac{\widehat{y_3}}{1-\alpha} \tag{49}$$
$$y^4_{*,1} \leq \widehat{y_4} \tag{50}$$
$$y^1_{*,1} \leq \widehat{y_1} \tag{51}$$

We then establish the following chain of implications:

$$\begin{aligned}
x^3_{*,1}=0 &\implies x^4_{*,1}=1-y^4_* \text{ exactly as in the proof for odd } N\\
&\implies x^4_{*,1}\geq 1-\widehat{y_4}>\widehat{z_{4,1}} \text{ by (50)}\\
&\implies x^1_{*,1}=0 \text{ exactly as in the proof for odd } N\\
&\implies g^3(X^3_{*,t})-g^3(X^3_{*,1})<\alpha\frac{\delta}{1-\delta}-(1-\alpha)\delta\widehat{z_{1,1}}+(1-\alpha)\frac{\delta^2}{1-\delta} \text{ for all } t>1\\
&\implies g^3(X^3_{*,t})-g^3(X^3_{*,1})<\delta\left(\frac{\delta+\alpha-\alpha\delta}{1-\delta}-\frac{\widehat{z_{1,1}}}{2}\right)<0 \text{ for all } t, \text{ by (47)}\\
&\implies x^3_{*,1}=1-y^3_*\geq 1-\frac{\widehat{y_3}}{1-\alpha} \text{ by (49)}\\
&\implies x^3_{*,1}>1-\frac{\widehat{y_3}}{1-\widehat{z_{3,2}}} \text{ by (44)}\\
&\implies x^3_{*,1}>1-\frac{1-\widehat{z_{3,1}}-\widehat{z_{3,2}}}{1-\widehat{z_{3,2}}}=\frac{\widehat{z_{3,1}}}{1-\widehat{z_{3,2}}}>\widehat{z_{3,1}}\\
&\implies x^4_{*,1}=0 \text{ exactly as in the proof for odd } N\\
&\implies x^1_{*,1}=1-y^1_* \text{ exactly as in the proof for odd } N\\
&\implies x^1_{*,1}\geq 1-\widehat{y_1} \text{ by (51)}\\
&\implies x^1_{*,1}\geq \widehat{z_{1,1}}+\widehat{z_{1,2}}>\widehat{z_{1,1}}+\frac{\alpha}{1-\alpha}\widehat{z_{1,2}}\\
&\implies g^3(X^3_{*,2})-g^3(X^3_{*,1})>\delta(-\alpha+(1-\alpha)\frac{\alpha}{1-\alpha}\widehat{z_{1,2}})=0\\
&\implies x^3_{*,1}=0.
\end{aligned}$$

Since there are contradictions in this chain (for example both $x^3_{*,1}=0$ and $x^3_{*,1}>\widehat{z_{3,1}}$), all the propositions have to be false and in particular $x^i_{*,1}>0$ for $i\in\{1,3,4\}$. Hence $g^i(X^i_{*,t})=C$ for those i and

$$\alpha\frac{y^2_*}{\widehat{y_2}}+(1-\alpha)\frac{y^3_*}{\widehat{y_3}} = 1 \tag{52}$$
$$y^4_* = \widehat{y_4} \tag{53}$$
$$y^1_* = \widehat{y_1}. \tag{54}$$

Also $x^3_{*,1}\leq\widehat{z_{3,1}}$, hence

$$\begin{aligned}
\sum_{t=2}^{T}x^3_{*,t} &= 1-x^3_{*,1}-y^3_*\\
&\geq 1-\widehat{z_{3,1}}-\frac{\widehat{y_3}}{1-\alpha} \text{ by (49)}\\
&> 1-\widehat{z_{3,1}}-\frac{\widehat{y_3}}{1-\widehat{z_{3,2}}} \text{ by (44)}\\
&\geq 1-\widehat{z_{3,1}}-\frac{1-\widehat{z_{3,1}}-\widehat{z_{3,2}}}{1-\widehat{z_{3,2}}}\\
&> 0
\end{aligned}$$

and at least one $X^3_{*,t}$ is played with positive probability for $t\geq 2$. It implies that there exists $t_0\geq 2$ such that $g^3(X^3_{*,t_0})=C$ and thus

$$\sum_{t'=1}^{t_0-1}\delta^{t'}\left(\alpha(x^2_{*,t'}-\widehat{z_{2,t'}})+(1-\alpha)(x^1_{*,t'}-\widehat{z_{1,t'}})\right)=0 \tag{55}$$

Also, $g^3(X^3_{*,t}) \leq C$ for this particular t_0 implies

$$\sum_{t'=1}^{t_0-1} \delta^{t'}(x^3_{*,t'} - \widehat{z_{3,t'}}) \leq 0 \tag{56}$$

Assume now by contradiction that $x^2_{*,1} = 0$. Then (55) implies

$$\begin{aligned}
0 &= -\alpha\delta\widehat{z_{2,t'}} + \sum_{t'=2}^{t_0-1} \delta^{t'} \left(\alpha(x^2_{*,t'} - \widehat{z_{2,t'}}) + (1-\alpha)(x^1_{*,t'} - \widehat{z_{1,t'}})\right) \\
&\leq -\alpha\delta\widehat{z_{2,t'}} + (1-\alpha)\delta(x^1_{*,1} - \widehat{z_{1,1}}) + \frac{\delta^2}{1-\delta} \\
&< \delta\left((1-\alpha)(x^1_{*,1} - \widehat{z_{1,1}}) - \frac{\delta}{1-\delta}\right) \text{ by (48)}
\end{aligned}$$

hence

$$x^1_{*,1} - \widehat{z_{1,1}} > \frac{\delta}{(1-\delta)(1-\alpha)} > \frac{\delta}{(1-\delta)}. \tag{57}$$

Now using (57), (56) and (53) one gets for all $t \geq 2$

$$g^2(X^2_{*,t}) - g^2(X^2_{*,1}) \leq C\left(-\delta(x^1_{*,1} - \widehat{z_{1,1}}) + \frac{\delta^2}{1-\delta}\right) < 0$$

and $x^2_{*,t} = 0$ for all $t \geq 2$. Since, by assumption, $x^2_{*,1} = 0$ as well, we have $y^2_* = 1$, hence $y^3_* < \widehat{y_3}$ by (52). And finally, using (54), we get

$$g^2(X^2_{*,1}) = 2C - C\frac{y^3_*}{\widehat{y_3}} > C,$$

a contradiction. Hence $x^2_{*,1} > 0$ as well, and the induction hypothesis is proved for $t = 1$.

Now $g^2(X^2_{*,1}) = C$, hence $y^3_* = \widehat{y_3}$, and then $y^2_* = \widehat{y_2}$ by (52). Hence, in any bad equilibrium, the payoffs can now be written in the simpler form:

$$\begin{aligned}
g^1(X^1_{*,t}) &:= C\left(1 + \sum_{t'=1}^{t-1} \delta^{t'}(x^4_{*,t'} - \widehat{z_{4,t'}})\right) \\
g^2(X^2_{*,t}) &:= C\left(1 + 2\sum_{t'=1}^{t-1} \delta^{t'}(x^3_{*,t'} - \widehat{z_{3,t'}}) - \sum_{t'=1}^{t-1} \delta^{t'}(x^1_{*,t'} - \widehat{z_{1,t'}})\right) \\
g^3(X^3_{*,t}) &:= C\left(1 + \sum_{t'=1}^{t-1} \delta^{t'}\left(\alpha(x^2_{*,t'} - \widehat{z_{2,t'}}) + (1-\alpha)(x^1_{*,t'} - \widehat{z_{1,t'}})\right)\right) \\
g^4(X^4_{*,t}) &:= C\left(1 + \sum_{t'=1}^{t-1} \delta^{t'}(x^3_{*,t'} - \widehat{z_{3,t'}})\right).
\end{aligned}$$

Let now $1 < t \leq T-1$; we assume that the induction hypothesis is true for every $t' < t$ and establish it for t. The proof is largely similar to the case $t = 1$ so we omit most computations and just explain the general scheme. First of all, the induction hypothesis up to $t-1$ implies that $x^i_{i,t'} = \widehat{z_{i,t'}}$ for all i and $t \leq t-2$ (use first that $g^1(X^1_{*,t'}) = g^4(X^4_{*,t'}) = C$, then $g^2(X^2_{*,t'}) = C$ and finally $g^3(X^3_{*,t'}) = C$). Also $g^i(X^i_{*,t}) \leq C$ hence

$$\begin{aligned}
x^1_{*,t-1} &\leq \widehat{z_{1,t-1}} + \frac{\alpha}{1-\alpha} && (58)\\
x^3_{*,t-1} &\leq \widehat{z_{3,t-1}} && (59)\\
x^4_{*,t-1} &\leq \widehat{z_{4,t-1}} && (60)
\end{aligned}$$

Using these three inequalities along with inequalities (42) to (48) establishes that

$$x^3_{*,t} = 0 \implies x^4_{*,t} > \widehat{z_{4,t}} \implies x^1_{*,t} = 0 \implies x^3_{*,t} > \widehat{z_{3,t}} \implies x^4_{*,t} = 0 \implies x^1_{*,t} \geq \widehat{z_{1,t}} + \frac{\alpha \widehat{z_{1,t+1}}}{1-\alpha} \implies x^3_{*,t} = 0,$$

the only significant difference with the case $t = 1$ being the implication $x^4_{*,t} = 0 \implies x^1_{*,t} \geq \widehat{z_{1,t}} + \frac{\alpha \widehat{z_{1,t+1}}}{1-\alpha}$ for which one has to use (48), since (51) is replaced by the weaker inequality (58). Hence $x^i_{*,t} > 0$ for $i \in \{1,3,4\}$, $g^i(X^i_{*,t}) = C$ for those i and

$$\begin{aligned} \alpha(x^2_{*,t-1} - \widehat{z_{2,t-1}}) + (1-\alpha)(x^1_{*,t-1} - \widehat{z_{1,t-1}}) &= 0 \\ x^3_{*,t-1} &= \widehat{z_{3,t-1}} \\ x^4_{*,t-1} &= \widehat{z_{4,t-1}} \end{aligned}$$

Using these three equalities, as well as $x^3_{*,t} \leq \widehat{z_{3,t}}$ and inequalities (42) to (48) we see that (once again it is very similar to the case $t = 1$ so we leave the computations to the reader)

$$x^2_{*,t} = 0 \implies x^1_{*,t} - \widehat{z_{1,t}} > \frac{\delta}{(1-\delta)} \implies x^2_{*,t''} = 0 \, \forall t'' > t \implies x^2_{*,t-1} > z_{2,t-1} \implies x^1_{*,t-1} < z_{1,t-1}$$

which implies $g^2(X^i_{*,t}) > C$, a contradiction and thus $x^2_{*,t} > 0$, and the induction hypothesis is proved for t.

Hence $x^i_{*,t} > 0$ for all i and $t \leq T-1$. As we already remarked it immediately impies that $x^i_{*,t} = \widehat{z_{i,t}}$ for all i and $t \leq T-2$. Now $g^1(X^1_{*,T}) \leq C$ hence $x^4_{*,T-1} \leq \widehat{z_{4,T-1}}$ and thus $x^4_{*,T} \geq 1 - \widehat{y_4} - \sum_{t=1}^{T-1} \widehat{z_{4,t}} = \widehat{z_{4,T}} > 0$. Hence $g^4(X^1_{*,T}) = C$ which immediately implies

$$\begin{aligned} x^3_{*,T-1} &= \widehat{z_{3,T-1}} \\ x^3_{*,T} &= 1 - \widehat{y_3} - \sum_{t=1}^{T-1} \widehat{z_{3,t}} \\ &= \widehat{z_{3,T}}. \end{aligned}$$

Now $g^2(X^2_{*,T}) \leq C$ and $x^3_{*,T-1} = \widehat{z_{3,T-1}}$ implies $x^1_{*,T-1} \geq \widehat{z_{1,T-1}}$; and $x^3_{*,T} > 0$ yields

$$\begin{aligned} 0 &= \alpha(x^2_{*,T-1} - \widehat{z_{2,T-1}}) + (1-\alpha)(x^1_{*,T-1} - \widehat{z_{1,T-1}}) \\ &\geq \alpha(x^2_{*,T-1} - \widehat{z_{2,T-1}}) \end{aligned} \tag{61}$$

hence $x^2_{*,T-1} \leq \widehat{z_{2,T-1}}$ and $x^2_{*,T} \geq 1 - \widehat{y_2} - \sum_{t=1}^{T-1} \widehat{z_{2,t}} = \widehat{z_{2,T}} > 0$. So $g^2(X^2_{*,T}) = C$ hence

$$\begin{aligned} x^1_{*,T-1} &= \widehat{z_{1,T-1}} \\ x^1_{*,T} &= 1 - \widehat{y_1} - \sum_{t=1}^{T-1} \widehat{z_{1,t}} \\ &= \widehat{z_{1,T}}. \end{aligned}$$

In particular $x^1_{*,T} > 0$ so $g^1(X^1_{*,T}) = C$ hence

$$\begin{aligned} x^4_{*,T-1} &= \widehat{z_{4,T-1}} \\ x^4_{*,T} &= 1 - \widehat{y_4} - \sum_{t=1}^{T-1} \widehat{z_{4,t}} \\ &= \widehat{z_{4,T}}. \end{aligned}$$

Finally, putting $x^1_{*,T-1} = \widehat{z_{1,T-1}}$ in (61) we get

$$\begin{aligned} x^2_{*,T-1} &= \widehat{z_{2,T-1}} \\ x^2_{*,T} &= 1 - \widehat{y_2} - \sum_{t=1}^{T-1} \widehat{z_{2,t}} \\ &= \widehat{z_{2,T}} \end{aligned}$$

and we have proven that there is a unique bad equilibrium for which $x^i_{*,t} = \widehat{z_{i,t}}$ for all i and t. This concludes Step 2, and Step 3 is the same as in the proof of Proposition 9.

Finally we have to settle the case when some $\widehat{z_{i,t}} = 0$. In that case, either for an odd or an even N, we do the same trick that in Section 5.3: we replace $\widehat{z}$ by a fictitious $\widehat{z'}$, with $\widehat{z'_{i,t}} = \widehat{z_{i,t}}$ when $\widehat{z_{i,t}} > 0$ and $\widehat{z'_{i,t}} > 0$ arbitrary (but sufficiently small so that $\sum_{t=1}^{T} \widehat{z'_{i,t}} < 1$) when $\widehat{z_{i,t}} = 0$. Also, when $\widehat{z_{i,t}} = 0$ the payoff of X^i_{*t} is instead given to an arbitrary monomial unknown of player i. Then the previous proofs imply that there is a unique bad equilibrium in which $x^i_{*t} = \widehat{z_{i,t}}$ when $\widehat{z_{i,t}} > 0$, while when $\widehat{z_{i,t}} = 0$ the action $X^i_{*,t}$ has a payoff identically equal to 0 and is not played in the bad equilibrium, giving $x^i_{*t} = 0$ as required. □

Proposition 22. *Let $N \geq 3$ and $(T_1, \cdots, T_N) \in (\mathbb{N}^*)^N$. Let $F \subset \mathbb{R}^{T_1+\cdots+T_N}$ be a nonempty closed $\mathbb{Z}$-semi algebraic set and assume it is strongly $(T_1, \cdots, T_N)$-admissible. Then there exists an N-player game Γ with integer pure payoffs, and T_i special actions $X^i_{*,1}, \cdots, X^i_{*,T_i}$ for each player i, such that*

a) $\mathrm{Proj}_{\{X^i_{*,t}\}}(\mathrm{NE}(\Gamma)) = F$
b) $\mathrm{NEP}(\Gamma) = \{0\}$.

Proof. Combine ideas in the proof of Prop 18 and Prop 22. □

8. Applications to the complexity of some problems

8.1. **Links with the existential theory of the reals.** We now apply the constructions of the previous sections. First, it implies that certain problems on equilibrium are computationally hard, since they are at least as hard as some problems on semi-algebraic sets. Recall that for 2 player games many problems involving equilibrium sets are already known to be NP-Hard [7]. We prove that for three players the same type of problems are exactly as hard as deciding whether or not a $\mathbb{Z}$-semi algebraic set is nonempty (we say that the problems are $\exists\mathbb{R}$-complete) ; this complexity class being known to lie somewhere between NP and $PSPACE$ [12].

Since sets arising in our settings are naturally compact, we will need the following lemma. While it is very close from results in [13] we could not find it explicitely in the literature so we give a short proof.

Lemma 23. *Deciding whether a compact $\mathbb{Z}$-semi algebraic set is nonempty is $\exists\mathbb{R}$-complete.*

Proof. Let E be a (non necessarily compact) $\mathbb{Z}$-semi algebraic set. In its definition replace any strict inequality $P_i(\cdot) < 0$ by $a_i^2 P_i(\cdot) + 1 = 0$ for an additional variable a_i. This gives an (higher dimensional) set E' that is closed, which is nonempty if and only if E is, and which size is polynomial in the size of E. Let L be the size of E' and n its dimension. By Corollary 3.4 in [13], E' is nonempty if only the compact set $E'' := E' \cap [-B, B]^n$ is, where $B = 2^{L^{8n}} \leq 2^{2^{8nL}} \leq 2^{2^{8L^2}}$. We thus just have to prove that the size of E'' is polynomial in L. To define E'', to the definition of E' we add $8L^2 + 1$ variables b_j and equalities $b_0 = 2$, $b_{j+1} = (b_j)^2$ for all j so that $b_{8L^2} = 2^{2^{8L^2}}$. We then just add to add inequalies $-b_{8L^2} \leq x_k \leq b_{8L^2}$ for every variable of E'.

Hence for any given $\mathbb{Z}$-semi algebraic set E we constructed a compact $\mathbb{Z}$-semi algebraic set E'' which is nonempty iff E is, and is of a size polynomial in the size of E. The result follows immediately. □

The following proposition generalizes some results from [3, 13]

Proposition 24. *The following problems are $\exists\mathbb{R}$-complete:*

a) *For any fixed positive integer k, given a 3-player game with integer pure payoffs, to determine if there is more than k equilibria.*
b) *For any fixed nonempty $\mathbb{Z}$-semi algebraic set E strictly included in $\mathbb{R}^3$, and given a 3-player game with integer pure payoffs, to determine whether there is one equilibrium payoff in E.*
c) *In particular, given a 3-player game with integer pure payoffs, to determine whether there is one equilibrium with positive (or negative, or 0, or greater than any fixed algebraic number) payoff for one (or several) players.*

Proof. Let us first prove b) : let E be such a set and assume we have an algorithm to determine whether a 3-player game has at least an equilibrium payoff in E. Let $e \in E$ and $e' \notin E$ two tuples of algebraic numbers. These can be found using for example Tarski Seidenberg theorem, in a time that depends only on the size of E. Let now $F \in \mathbb{R}^N$ be a compact (see Lemma 23) $\mathbb{Z}$-semi algebraic set and consider the set $(\{e\} \times F) \cup (\{e'\} \times \{0\}^N) \subset \mathbb{R}^{N+3}$. It is compact, nonempty and semi algebraic, hence up to some rescalling one can write it as a strongly admissible (T_1, T_2, T_3) subset for some $T_1 + T_2 + T_3 = N + 3$. By Proposition[8] 22 one can construct in polynomial time (in the size[9] of F) a game Γ such that this set is the projection on some actions of the 3 players of the set of equilibria of Γ, and such that all equilibria give a payoff of 0 to all players. In particular the projection of the set of equilibria of Γ on some action of each player is, up to some rescaling, $\{e, e'\}$ if F is nonempty and $\{e'\}$ if F is empty. Reasoning as in the proof of Theorem 9 one gets a game Γ' such that it has an equilibrium payoff in E if and only if F is nonempty. Hence one can decide whether or not F is nonempty by running the algorithm to decide if there is an equilibrium payoff of Γ' in E.

The other direction is straightforward: assume we have an algorithm to determine whether a semi-algebraic set is nonempty, let E be a fixed $\mathbb{Z}$-semi algebraic set in $\mathbb{R}^3$, and Γ a 3-player game with integer pure payoffs and m actions in total. Consider the subset of $\mathbb{R}^{m+3}$ defined by $F := \{(x, z) | x \in NE(\Gamma) \text{ and } e = g(x)\} \bigcap (\mathbb{R}^m \times E)$. F is semialgebraic, and one verifies that its size is polynomial in the size of Γ. Clearly F is nonempty if and only there is an equilibrium payoff of Γ in E. Hence one can decide whether or not there is an equilibrium payoff of Γ' in E by running the algorithm to decide if F is nonempty.

Let us now prove a) : first assume we have an algorithm to determine whether a 3-player game has more than k equilibria and let F be a compact $\mathbb{Z}$-semi algebraic set of dimension n. The set $F' = (\{1, 2, \cdots, k\} \times F) \bigcup 0^{n+1}$ is clearly compact and $\mathbb{Z}$-semi algebraic. Moreover, it has a unique element if F is empty, and at least $k+1$ if F is not. As in the proof of point b) we can then use Proposition 21 to conclude. In the other direction, if Γ is a three-player game, it is enough to consider the $\mathbb{Z}$-semi algebraic set $\{(x_1, \cdots, x_{k+1} | x_j \neq x_l \text{ for } j \neq l,\ x_j \in NE(\Gamma) \text{ for all } j\}$ which is nonempty if and only if Γ has more than k equilibria. □

8.2. ***NP*-hardness of some problems.** We can also prove that for 3 players the results of NP-hardness cited above in the two player case hold even if the pure payoff are restricted to be in some fixed finite set, for example

Proposition 25. *All the following problems are NP-Hard:*

- *Given a 3-player game with pure payoffs in $\{-1, 0, 1\}$, to determine if there is more than one equilibrium.*
- *Given a 3-player game with pure payoffs in $\{-1, 0, 1\}$, to determine if there is an infinite number of equilibria.*
- *Given a 3-player game with pure payoffs in $\{-1, 0, 1\}$, to determine if the number of mixed equilibria is not odd (meaning either even of infinite).*
- *Given a 3-player game with pure payoffs in $\{-1, 0, 1\}$, to determine if there is an equilibrium in which the payoff of the first player is 0.*

[8]In fact if one can find a e' with rational coordinates (for example if E is closed) then Proposition 21 is enough
[9]Recall that e and e' are independent of F

- *Given a* 3*-player game with pure payoffs in* $\{-1,0,1\}$*, to determine if there is an equilibrium in which the first player plays is first action with positive probability.*
- *Given a* 3*-player game with pure payoffs in* $\{-1,0,1\}$*, to determine if there is an equilibrium in which the first player plays is first action with probability* $\frac{1}{2}$*.*
- *Given a* 3*-player game with pure payoffs in* $\{-1,0,1\}$*, to determine if there is an equilibrium in which the first player plays is first action with probability less than* 1*.*

Proof. We show 3-SAT can be reduced in polynomial time to any of these problems ; this will imply the result since 3-SAT is NP-complete.

Consider a 3-SAT instance

$$\bigwedge_{k=1}^{K}(a_{1,k}\vee a_{2,k}\vee a_{3,k})$$

where each $a_{j,k}$ is in $\{b_1,\cdots b_T,\neg b_1,\cdots \neg b_T\}$. Without loss of generality all disjonctive clauses are distinct hence $K\leq\binom{2T}{3}=O(T^3)$.

We first construct a semialgebraic set F in $\mathbb{R}^{2T}$ such that F is empty if and only if the 3-SAT instance is not satisfiable. To each of the K disjonctive clause one associate the monomial

$$P_k(z_1,\cdots,z_{2T})=z_{1,k}z_{2,k}z_{3,k},$$

where $z_{j,k}=z_t$ if $a_{j,k}=b_t$, and $z_{j,k}=z_{T+t}$ if $a_{i,k}=\neg b_t$. For example the polynomial associated to $(b_1\vee\neg b_2\vee b_4)$ is $z_1z_{T+2}z_4$.

Let $0<\phi_1<\phi_2$ be an arbitrary real, and define

$$\begin{aligned} F &= \left(\bigcap_{k=1}^{K}\{P_k(z)\leq 0\}\right)\bigcap\left(\bigcap_{t=1}^{T}\{z_tz_{T+t}\leq 0\})\right)\bigcap\left(\bigcap_{t=1}^{2T}\{-z_t\leq 0\})\right)\\ &\quad\bigcap\left(\bigcap_{t=1}^{2T}\{z_t\leq\phi_2\})\right)\bigcap\left(\bigcap_{t=1}^{T}\{\phi_1-z_t-z_{T+t}\leq 0\})\right)\end{aligned}$$

We claim that if the 3-SAT instance is not satisfiable, then F is empty, while if the 3-SAT instance is satisfiable then F is infinite. Assume first that $z\in F$. Then for all $1\leq t\leq T$ exactly one of z_t and z_{t+T} equals 0. Define b_t as TRUE if z_t equals 0 and b_t as FALSE if z_{t+T} equals 0. Since all z_t are nonnegative, $P_k(z)\leq 0$ if and only if it is zero, and then the associated disjonctive clause is true. Since this is true for all k the 3-SAT instance is satisfiable. Assuming now that the 3-SAT instance is satisfiable, let $(b_1,\cdots,b_T)$ satisfying it. Fix $\phi_1<\phi<\phi_2 i$ and define, for every $1\leq t\leq T$, $z_t=0$ if b_t is TRUE, $z_t=\phi$ if b_t is FALSE, and $z_{T+t}=\phi-z_t$. Then $z\in F$ for all such ϕ, so F is infinite.

Define $F'=\{(z,z,z),\ z\in F\}\subset\mathbb{R}^{6T}$. We will now use the idea of the previous sections to construct a 3-player game wich has one bad equilibrium, and a set of nice equilibria whose projection on the first $2T$ actions of each player is F' (for a good choice of ϕ_1 and ϕ_2). Since we want the pure payoffs of this game to be in $\{-1,0,1\}$ we unfortunately cannot apply directly Proposition 21. On the bright side, we do not care about the precise coordinates of the bad equilibrium so we do not need such a precise machinery than in the proof of Proposition 21 (in fact there will essentialy be no Step 2 and 3).

As always each of the three players has a set of actions $\mathcal{A}^i=\mathcal{X}^i\cup\mathcal{Y}^i$ where the actions in $\mathcal{X}^i$ are unknowns with a payoff of 0 and actions in $\mathcal{X}^i$ are constraints. We change a little bit the terminology : an equilibrium is nice iff every player has a payoff of 0, even if some contraints are played with positive probability. This will ease the construction a bit, and won't be an issue since we do not care about the projection of nice equilibria but only their cardinality.

Unknowns of player i will consist of

- $2T$ unknowns $X^i_{*,t}$ that corresponds to the variable z_t
- Monomial unknowns to represent every monomial in the $x^i_{*,t}$ of total degree between 1 and 3 in , and degree 0 or 1 in each variable.

- U positive unknowns $X^i_{+,u}$, $u = 1$ to U, for a U to be fixed later on. Their role will be to represent small positive numbers while only unsing contraints with pure payoffs in $\{-1, 0, 1\}$.

There is no need for a dump variable as we allow players to put positive probabilities on constraints in nice equilibria.

The constraints of player i will consist in

- For each $1 \leq t \leq 2T$, constraints with payoff $\pm(x^{i+1}_{*,t} - x^{i+2}_{*,t})$, ensuring that $x^i_{*,t}$ is independant of i for each t in any nice equilibrium. Denote by $x_{*,t}$ the common value.
- Initialization and induction constraints such that each monomial unknown represent the correct monomial in the $x_{*,t}$.
- K semi algebraic contraints such that in any nice equilibrium $P_k(x^*) \leq 0$ for each k.
- Two constraints $Y^i_{*,-}$ and $Y^i_{*,+}$ with payoff respectively $2x^{i+1}_{+,1} - 1$ and $1 - 2x^{i+1}_{+,1}$.
- For each $1 \leq u \leq U - 1$, two constraints with payoff $\pm(x^{i+1}_{+,u+1} - x^{i+1}_{+,u} x^{i+2}_{+,u})$.
- For each $1 \leq t \leq 2T$, a constraint with payoff $x^{i+1}_{+,U-1} - x^{i+1}_{*,t}$.
- For each $1 \leq t \leq T$, two constraints with payoff $x^{i+1}_{*,t} x^{i+2}_{*,T+t}$ and $x^{i+1}_{+,U} - x^{i+1}_{*,t} - x^{i+1}_{*,T+t}$.

One verifies that each of these constraints have pure payoffs in $\{-1, 0, 1\}$. In any nice equilibrium each constraint gives a nonpositive payoff. It implies that $x^i_{+,1} = \frac{1}{2}$, and then by induction that $x^{i+1}_{+,u} = \frac{1}{2^{2^{u-1}}} > 0$ for each i and u. Hence $(x_*, x_*, x_*) \in F'$ for $\phi_1 = \frac{1}{2^{2^{U-1}}}$ and $\phi_2 = \frac{1}{2^{2^{U-2}}}$. One the other hand, let $(z, z, z) \in F'$ for this choice of ϕ_1 and ϕ_2, fix $x^i_{*,t} = z_t$, $x^{i+1}_{+,u} = \frac{1}{2^{2^{u-1}}} > 0$. Also fix the probability of all monomial unknowns to be the correct monomial in the $x_{*,t}$, and dump the remaining probability on $Y^i_{*,-}$, which has a payoff of 0. Then as usual the only thing to verify is that we stay in the simplex. For each player,

$$\sum_{u=1}^{U} x^i_{+,u} = \sum_{u=1}^{U} \frac{1}{2^{2^{u-1}}} < \sum_{u=1}^{+\infty} \frac{1}{2^{2^{u-1}}} < \frac{7}{8}.$$

Each player has $2T \leq 2T^3$ original unknows, and $\binom{2T}{3} + \binom{2T}{3} + \binom{2T}{3} \leq 6T^3$ monomial unknows. Each of these has a probability less than $\phi_2 = \frac{1}{2^{2^{U-2}}}$. Hence it is enough that $\frac{8T^3}{2^{2^{U-2}}} \leq \frac{1}{8}$, that is $2^{2^{U-2}} \geq 64T^3$. This is clearly the case if one takes, for example $U = 2 + 64T^3 = O(T^3)$. Remark that this U being fixed, the number of actions of each player is clearly a $O(T^3 + K + U) = O(T^3)$ and is thus polynomial in T.

We have thus proved that the projection of the set of nice equilibria on the first $2T$ actions of each player is F'. Consider now the bad equilibria. There is no action to add as $Y^i_{*,+}$ essentially plays the role of Y^i_* in the usual contruction. In a bad equilibrium, there is a player i with a positive payoff. This implies that all of his unknowns are played with zero probability, and hence that the payoff of $i-1$ is at least $g^{i-1}(Y^i_{*,+}) = 1 > 0$. Hence in any bad equilibrium the payoff of each player is positive, and all unknowns are played with 0 probability. Then one checks that all contraints give a payoff of 0, except $Y^i_{*,+}$ with a payoff of 1 and $Y^i_{*,-}$ with a payoff of -1. Hence the only bad equilibria is the pure profile $Y_{*,+}$. To conclude one only needs to check that, by construction:

- The game has a unique equilibrium if the instance is not satisfiable, and infinitely many if the instance is satisfiable
- The game has a nice equilibrium (with payoff 0) iff the instance is satisfiable
- The game has an equilibrium with $y^i_{*,+} < 1$ iff the instance is satisfiable
- The game has an equilibrium with $x^i_{+,1} = \frac{1}{2}$ iff the instance is satisfiable
- The game has an equilibrium with $x^i_{+,1} > 0$ iff the instance is satisfiable.

□

Remark 26. It would be interesting to know if the answers are the same if the set of pure payoffs is restricted to $\{0, 1\}$ instead of $\{-1, 0, 1\}$.

9. Applications to the computability of some problems

9.1. Undecidability of problems involving integers. Denote by $\frac{1}{\mathbb{N}^*\setminus\{1\}}$ the set $\left\{\frac{1}{n}, n \in \mathbb{N}^* \setminus \{1\}\right\}$.

Proposition 27. *There exists no algorithm which solve the following decision problems, given a finite game with integer payoffs:*

a) *Is there an equilibrium in which all players play there first action with a probability in* $\frac{1}{\mathbb{N}^*\setminus\{1\}}$.

b) *Is there an equilibrium in which the payoff of all players is in* $\frac{1}{\mathbb{N}^*\setminus\{1\}}$.

This is true even for a fixed number of players and actions, provided they are larger than some explicit constants.

Proof. Let $P \in \mathbb{Z}[z_1, \cdots, z_N]$ be a polynomial in N variables with integer coefficients and degree at most d in each variable. The function

$$(z_1, \cdots, z_N) \longrightarrow z_1^d \cdots z_N^d P\left(\frac{1-z_1}{z_1}, \cdots, \frac{1-z_N}{z_N}\right)$$

can be continuously extended to a polynomial Q in $\mathbb{Z}[z_1, \cdots, z_N]$ with degree at most d in each variable. Since $z \to \frac{1-z}{z}$ is a bijection mapping $\frac{1}{\mathbb{N}^*\setminus\{1\}}$ onto $\mathbb{N}^*$, Q has a zero in $\left(\frac{1}{\mathbb{N}^*\setminus\{1\}}\right)^N$ if and only if P has a zero in $(\mathbb{N}^*)^N$.

Let $\widehat{z} = \left(\frac{2}{5}, \cdots, \frac{2}{5}\right)$ and

$$F = (\{z,\ P(z) = 0\} \cup \{\widehat{z}\}) \cap \left[0, \frac{1}{2}\right]^N .$$

$F \subset [0,1[^N$ is nonempty, closed and $\mathbb{Z}$-semi algebraic, and $\widehat{z} \in \mathbb{Q}^N \bigcap F$. Also, $F \cap \left(\frac{1}{\mathbb{N}^*\setminus\{1\}}\right)^N = \emptyset$ if and only if P has no zero in $(\mathbb{N}^*)^N$. By Proposition 18 and 19, one can explicitely construct two games Γ_1 and Γ_2 with integer coefficients such that $\mathrm{Proj}_{X_*}(\mathrm{NE}(\Gamma_1)) = F$, and $\mathrm{NEP}(\Gamma_2) = F$. Hence if an algorithm existed to answer a) or b), then one would be able to solve Hilbert's tenth problem, which is impossible [6, 11]. Since Hilbert's tenth problem is undecidable even for fixed N and d, provided they are larger than some explicit constant, it is impossible to answer to a) and b) even for a fixed number of players and actions, provided they are large enough. □

9.2. Undecidability of problems involving rational numbers.

Proposition 28. *The following problems are either all decidable or all undecidable:*

a) *Hilbert's tenth problem on $\mathbb{Q}$: deciding, for every N and every polynomial with integer coefficients and N variables, whether P has a zero in $\mathbb{Q}^N$.*

b) *Deciding if a finite game with integer pure payoffs has an equilibrium in which for each player, his first action is played with probability in $\mathbb{Q}^*$.*

c) *Deciding if a finite game with integer pure payoffs has an equilibrium in which all players get a payoff in $\mathbb{Q}^*$.*

Proof. Assume first that Hilbert's tenth problem is undecidable on $\mathbb{Q}$, that is that there is no algorithm which decides if a polynomial P with integer coefficients has a zero with rational coordinates. Let $P \in \mathbb{Z}[z_1, \cdots, z_N]$ be any polynomial in N variables with integer coefficients and degree at most d in each variable. Define the intervals $I_1 =]-\infty, 0]$ and $I_2 = [0, +\infty[$. Then $f_1(x) := \frac{2x-1}{x}$ is a bijection mapping $]0, 1/2] \cap \mathbb{Q}$ onto $I_1 \cap \mathbb{Q}$; and $f_1(x) := \frac{1-2x}{x}$ is a bijection mapping $]0, 1/2] \cap \mathbb{Q}$ onto $I_2 \cap \mathbb{Q}$. Hence for any $e = (e_1, \cdots, e_N) \in \{1, 2\}^N$, the function

$$z \in \mathbb{R}^N \longrightarrow z_1^d \cdots z_N^d P\left(f_{e_1}(z_1), \cdots, f_{e_n}(z_n)\right)$$

can be continuously extended to a polynomial P_e in $\mathbb{Z}[z_1, \cdots, z_N]$ with degree at most d in each variable, and P_e has a zero in $([0, 1/2] \cap \mathbb{Q}^*)^N$ iff P has a zero in $\prod_{n=1}^N (I_{e_n} \cap \mathbb{Q})$.

Define $\widehat{z} = (0, \cdots, 0)$ and

$$F = \left(\{\widehat{z}\} \cup \bigcup_{e \in \{1,2\}^N} \{z \in \mathbb{R}^N, P_e(z) = 0\} \right) \cap \left[0, \frac{1}{2}\right]^N$$

Then $F \subset [0,1[^N$ is nonempty, closed and $\mathbb{Z}$-semi algebraic, and $\widehat{z} \in \mathbb{Q}^N \bigcap F$. Also, $F \cap (\mathbb{Q}^*)^N = \emptyset$ iff and only if P has no zero in $\mathbb{Q}^N$. By Proposition 18 and 19, one can explicitely construct two games Γ_1 and Γ_2 with integer coefficients such that $\mathrm{Proj}_{X_*}(\mathrm{NE}(\Gamma_1)) = F$, and $\mathrm{NEP}(\Gamma_2) = F$. Hence if an algorithm existed to answer b) or c), then one would be able to solve Hilbert's tenth problem on $\mathbb{Q}$, a contradiction.

Assume now that Hilbert's tenth problem is decidable on $\mathbb{Q}$. We first prove that this implies that there exists an algorithm that decides if any $\mathbb{Z}$-semi algebraic set contains a point with rationals coordinates. Let F be such a semi algebraic set in $\mathbb{R}^N$, written as union and intersection of sets of the form $\{z \in \mathbb{R}^N,\ P(z) < 0\}$ or $\{z \in \mathbb{R}^N,\ P(z) \leq 0\}$ (all polynomials having coefficients in $\mathbb{Z}$). Recall that by Lagrange four squares theorem, any positive integer is the sum of the squares of four integers. It implies that any rational $\frac{p}{q} = \frac{pq}{q^2}$ can be written as $\frac{p_1^2+p_2^2+p_3^2+p_4^2}{q^2}$, and thus that a rational is nonnegative iff it is the sum of the squares of four rationals. Hence, any set $\mathbb{Q}^N \cap \{z \in \mathbb{R}^N,\ P(z) \leq 0\}$ is the projection on the first N coordinates of the set $\mathbb{Q}^{N+4} \cap \{z \in \mathbb{R}^{N+4},\ P(z_1, \cdots, z_N) - z_{T+1}^2 - z_{T+2}^2 - z_{T+3}^2 - z_{T+4}^2 = 0\}$. Similarly, any set $\mathbb{Q}^N \cap \{z \in \mathbb{R}^N,\ P(z) < 0\}$ is the projection on the first N coordinates of the set $\mathbb{Q}^{N+4} \cap \{z \in \mathbb{R}^{N+4},\ P(z_1, \cdots, z_N)(z_{T+1}^2 + z_{T+2}^2 + z_{T+3}^2 + z_{T+4}^2) + 1 = 0\}$. Hence, adding 4 new variables for each polynomial, one can construct a semi algebraic set $F' \subset \mathbb{R}^{N'}$ for some $N' > N$, such that F' is defined by union and intersection of sets of the form $\{z \in \mathbb{R}^{N'},\ P(z) = 0\}$, and such that $F \cap \mathbb{Q}^N$ is the projection of $F' \cap \mathbb{Q}^{N'}$ on its first N coordinates. In particular, F contains a a point with rationals coordinates iff F' does. Now, using that

$$P_1(z) = 0 \text{ and } P_2(z) = 0 \quad \Leftrightarrow \quad P_1^2(z) + P_2^2(z) = 0$$
$$P_1(z) = 0 \text{ or } P_2(z) = 0 \quad \Leftrightarrow \quad P_1(z)P_2(z) = 0$$

one can rewrite F' as the set of zeroes of a single polynomial with integer coefficients. Since we assumed Hilbert's tenth problem is decidable on $\mathbb{Q}$, there is an algorithm to decide if F' contains a point with rational coordinates and thus to decide if F contains a point with rationals coordinates.

Now for any finite game Γ, $\mathrm{NE}(\Gamma)$ can be explicitely written as union and intersection of sets of the form $\{z,\ P(z) \leq 0\}$ for polynomials with integer coefficients. Since there are constructive proofs of Tarki-Seidenberg theorem, one can also write $\mathrm{Proj}_{X_*}(\mathrm{NE}(\Gamma))$ as union and intersection of sets of this form. Hence $F := (\mathbb{R}^*)^N \cap \mathrm{Proj}_{X_*}(\mathrm{NE}(\Gamma))$ can be explicitely written as union and intersection of sets of the form $\{z \in \mathbb{R}^N,\ P(z) < 0\}$ or $\{z \in \mathbb{R}^N,\ P(z) \leq 0\}$, and by the previous paragraph we conclude that there is an algorithm that decide if F contains a point with rational coordinates. Hence there is an algorithm that decides if Γ has an equilibrium in which all players play their first action with a probability in $\mathbb{Q}^*$.

Similarly, $\mathrm{NEP}(\Gamma)$ is a projection of the semi algebraic set $\{(z, g^1(z), \cdots, g^N(z))\} \cap (\mathrm{NE}(\Gamma) \times \mathbb{R}^N)$ hence there is an algorithm that decides if Γ has an equilibrium in which all players get a payoff in $\mathbb{Q}^*$.

Remark 29. It is straightforward to adapt the proof if one replace in b) $\mathbb{Q}^*$ by $\mathbb{Q} \setminus \{r\}$ for any fixed rational $r \in [0,1[$; or in c) $\mathbb{Q}^*$ by $\mathbb{Q} \setminus \{r\}$ for any fixed rational r. The situation seems different for $\mathbb{Q}$ however: one proves along the same line that if Hilbert tenth problem is decidable on $\mathbb{Q}$ then one can solve b) and c) for $\mathbb{Q}$, but proving the reverse implication seems more difficult.

□

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