Discrete Stochastic Models vs Continuous Deterministic Models

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Introduction

Models of learning and evolution are typically either :

- **Discrete** (in time / space) and **stochastic** (microscopic models)

or

- **Continuous and deterministic** (macroscopic models)

Microscopic models can be used to justify macroscopic models and macroscopic models can be used to analyze microscopic ones

Introduction

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> *Microscopic models can be used to justify macroscopic models and macroscopic models can be used to analyze microscopic ones*

Purpose of the talk : explain this later sentence with a **math** perspective.

Outline

First model (Inspired from Benaim and Weibull, Econometrica, 2003)

Finite population of size N,
Each individual has a strategy $i \in \{1, \ldots, m\}$
$\Delta =$ unit simplex over $\{1, \ldots, m\}$.
State of the system at time t, $X^N(t) \in \Delta$,

$$\mathrm{P}(X^N(t+1) = x + \frac{1}{N}(e_k - e_i) | X^N(t) = x) = p_{ik}(x).$$

Mean Field:

$$\mathrm{E}(X^N(t+1) - X^N(t) | X^N(t) = x) = \frac{1}{N} F(x)$$

with

$$F_k(x) = \sum_i p_{ik}(x) - p_{ki}(x).$$

Example: Imitation dynamics

A Randomly chosen individual imitates another randomly chosen individual having a better payoff

$$p_{ik}(x) = x_i x_k \alpha (U(k,x) - U(i,x))^+$$

$$F_k(x) = \alpha x_k (U(k,x) - U(x,x)).$$

$\Rightarrow$ the mean field is the Replicator vector field.

Second Model (from Benaim, Schreiber, Tarres, Ann. App. Prob., 2004)

Population of size N_t at time $t \in \mathbb{N}$, $X(t) \in \Delta$
For $\omega = (\omega_1, \ldots, \omega_m) \in \mathbb{Z}^m$

$$\mathsf{P}(N_{t+1}X(t+1) - N_t X(t) = \omega | X(t) = x, N_t = n) = p_\omega(x)$$

For simplicity **here**, assume that $|\omega| = \sum_i \omega_i \in \{1, \ldots, K\}$

Mean Field :

$$\mathsf{E}(X(t+1) - X(t) | X(t) = x, N_t = n) = \frac{1}{n} F(x)$$

with

$$F(x) = \sum_\omega p_\omega(x)(x - x|\omega|).$$

Example : The Replicator Process

At each time t, two individuals are randomly chosen in the population.
If their strategies are i and j, the i (resp. j)-strategist gives birth to R_t^{ij} (resp. $\tilde{R}_t^{ji}$) i (resp j)-strategists.
$R_t, \tilde{R}_t, t \geq 1$ are i.i.d random variables

$$F_k(x) = x_k(U(k,x) - U(x,x))$$

with

$$U(i,j) = \mathsf{E}(R_{ij} + \tilde{R}_{ij}).$$

The mean field is the "Replicator" vector field.

Both models can be written as

$$X(t+1) - X(t) = \gamma_t(F(X(t) + U_{t+1})$$

with U_t a "noise" such that $E(U_{t+1}|\mathcal{F}_t) = 0$.

Model 1 : $\gamma_t = \gamma = \frac{1}{N}$: **Constant step size**
Model 2 : $\frac{1}{t} \leq \gamma_t \leq \frac{K}{t}$: **Decreasing step size**

Both models can be seen as a Noisy Cauchy Euler approximation to

$$\frac{dx}{dt} = F(x).$$

Decreasing step size

$$X(t+1) - X(t) = \gamma_t(F(X(t) + U_{t+1})$$

$$\sum_t \gamma_t = \infty, \gamma_t = o(1/\log(t)).$$

Decreasing step size

$$X(t+1) - X(t) = \gamma_t(F(X(t) + U_{t+1})$$

$$\sum_t \gamma_t = \infty, \gamma_t = o(1/\log(t)).$$

Results from Benaim, Benaim and Hirsch, in the late 90's lead to

Theorem

The limit set of $(X(t))$ is almost surely compact, connected, invariant and attractor free for the mean field ode

Decreasing step size

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Theorem

Under reasonable assumptions, for every attractor A $\Pr(\lim_{t\to\infty} d(X(t), A) = 0) > 0$.

Decreasing step size

Theorem

The limit set of $(X(t))$ is almost surely compact, connected, invariant and attractor free for the mean field ode

Theorem

Under reasonable assumptions, for every attractor A $\Pr(\lim_{t\to\infty} d(X(t),A)=0)>0.$

Results from Pemantle, Tarres, Benaim lead to

Theorem

Under reasonable assumptions, for every linearly unstable equilibrium or periodic orbit Γ $\Pr(\lim_{t\to\infty} d(X(t),\Gamma)=0)=1.$

Illustration

The Replicator process, with m = 3,

$$U = \begin{pmatrix} 0 & -a_2 & b_1 \\ b_1 & 0 & -a_3 \\ -a_1 & b_2 & 0 \end{pmatrix}$$

and $det(U) < 0$

By results of Zeeman, 1980 (see Hofbauer and Sigmund (Bull. AMS, 2003)) the phase portrait for the replicator ode is
on the black board

... Hence, the limit set of the replicator process is almost surely an heteroclinic cycle.

Other applications :

The method of *Stochastic Fictitious Play* leads to a mean ODE given by the *smooth best reply* dynamics

$\Rightarrow$ Almost sure convergence of SFP for

- *Two players, Two strategies games*

Benaim and Hirsch (GEB, 1999)

- *Two player symmetric games with an interior EES*
- *Two player zero sum games*
- *Potential games*

Hofbauer and Sandholm (Econometrica, 2002)

- *Supermodular games*

Benaim and Faure, 2010

Generalizations to set valued dynamics

$$X(t+1) - X(t) - \gamma_t U_{t+1} \in F(X(t))$$

$$\text{Mean Field : } \frac{dx}{dt} \in F(x)$$

Benaim, Hofbauer, Sorin (SIAM 2005, MOR 2006)
Faure and Roth (MOR, 2010)

Constant step size

$$X(t+1) - X(t) = \gamma(F(X(t) + U_{t+1})$$

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Set $\hat{X}(t\gamma) = X(t)$.

Benaim and Weibull (2003) leads to

Theorem

$\Pr(\sup_{0\leq s\leq T} \|\hat{X}(s) - x(s)\| \geq \epsilon | X(0) = x) \leq C(T)\exp(-\epsilon^2/\gamma)$

with $x(s)$ solution to

$$\dot{x} = F(x),\ x(0) = x.$$

Constant step size

Theorem

$$\Pr(\sup_{0\le s\le T} \|\hat{X}(s) - x(s)\| \ge \epsilon | X(0) = x) \le C(T)\exp(-\epsilon^2/\gamma)$$

Corollary

Let μ_γ be an invariant probability for $X(t)$, $\mu = \lim_{i\to\infty} \mu_{\gamma_i}$ is invariant for F. In particular $\mu\{x :\ x \in \overline{\omega(x)}\} = 1$.

Constant step size

Theorem

$\Pr(\sup_{0\leq s\leq T} \|\hat{X}(s) - x(s)\| \geq \epsilon | X(0) = x) \leq C(T)\exp(-\epsilon^2/\gamma)$

Corollary

Let μ_γ be an invariant probability for $X(t)$, $\mu = \lim_{i\to\infty} \mu_{\gamma_i}$ is invariant for F. In particular $\mu\overline{\{x : x \in \omega(x)\}} = 1$.

Corollary

Let U be a neighborhood of $\overline{\{x : x \in \omega(x)\}}$

$$\lim_{\gamma\to 0} \liminf_{t\to\infty} \frac{1}{t} \sharp\{i \leq t : X(i) \in U\} = 1.$$

Localisation of invariant measures

Let

$$\mu = \lim_i \mu_{\gamma_i}$$

If A is unstable, it is false (in general) that $\mu(A) = 0$!

It is true if
- F is a gradient vector field (Fort and Pages, SIAM, 1999)
- F is Morse Smale, Axiom A, or with simple dynamics and no cycle (Benaim, Erg.Th.Dyn Systems, 1999)