Sex and Evolutionary Stability

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Hawk/Dove/Retaliator Game
Resource Game

Extended Form

Strategic Form

0 < k < b
\[ \dot{s} = s(\pi_S - \pi_O) = ks(1 - s)(1 - p) \]
\[ \dot{p} = p(\pi_P - \pi_I) = p(1 - p)(k - s(v + k)) \]
\[ \dot{s} = s(\pi_S - \pi_O) + \mu_F (1 - s) \pi_F - \mu_S s \pi_S \]
\[ \dot{p} = p(\pi_P - \pi_I) + \mu_C (1 - p) \pi_C - \mu_P p \pi_P \]

\text{drift terms}
\[
\dot{s} = s(\pi_S - \pi_O) + \mu_F (1 - s) \pi_F - \mu_S s \pi_S \\
\dot{p} = p(\pi_P - \pi_I) + \mu_C (1 - p) \pi_C - \mu_P p \pi_P
\]

\[
\frac{s(1 - s)(1 - p)k}{p(1 - p)[k - s(v + k)]} = \frac{\mu_S s \pi_S - \mu_F (1 - s) \pi_F}{\mu_P p \pi_P - \mu_C (1 - p) \pi_C}.
\]

\[
\mu_P = (1 - \beta) \delta_I, \quad \mu_C = \beta \delta_I, \quad \alpha = \beta = \frac{1}{2}, \quad \mu_S = (1 - \alpha) \delta_C, \quad \mu_F = \alpha \delta_O.
\]
\[ \dot{s} = s(\pi_S - \pi_O) + \mu_F (1 - s) \pi_F - \mu_S s \pi_S \]

\[ \dot{p} = p(\pi_P - \pi_I) + \mu_C (1 - p) \pi_C - \mu_P p \pi_P \]

\[ s^* = \frac{k}{k + v} \]

\[ \frac{ss^*(1-s)}{2(s-\frac{1}{2})(s^*-s)} = \frac{b + v \delta_O}{b} \frac{\delta_O}{\delta_I} > 1 ? \]

Drift terms

Background fitness
Figure 4: Stability with different assumptions on mutation rates. Figure 4(a) reproduces the configuration of rest points from Figure 3, for the case of asexual dynamics without drift. Figure 4(b) shows that only the asymptotic attractor $M$ survives when we allow only mutations from aggressive to passive behavior. Figures 4(c) illustrates the case when the probability that an aggressive allele mutates to a passive allele is the same as the probability of a mutation in the other direction. These rates may differ between the owner and intruder loci, with $\mu_S = \mu_F = \frac{1}{2} \delta_O$ and $\mu_P = \mu_C = \frac{1}{2} \delta_I$. Note that rest points near $N$ occur in pairs: an asymptotic attractor marked with a star and a saddle point marked with a circle.
## Sexual Dynamics

<table>
<thead>
<tr>
<th>genotype</th>
<th>population proportion</th>
<th>gametes per individual</th>
<th>successful (a) gametes</th>
<th>successful (A) gametes</th>
</tr>
</thead>
<tbody>
<tr>
<td>(aa)</td>
<td>(x)</td>
<td>(\pi_{aa})</td>
<td>((1 - \mu_{a})X)</td>
<td>(\mu_{a}X)</td>
</tr>
<tr>
<td>(aA) or (Aa)</td>
<td>(y)</td>
<td>(\pi_{aA})</td>
<td>(\frac{1}{2}(1 - \mu_{a})Y + \frac{1}{2}\mu_{A}Y)</td>
<td>(\frac{1}{2}\mu_{a}Y + \frac{1}{2}(1 - \mu_{A})Y)</td>
</tr>
<tr>
<td>(AA)</td>
<td>(z)</td>
<td>(\pi_{AA})</td>
<td>(\mu_{A}Z)</td>
<td>((1 - \mu_{A})Z)</td>
</tr>
</tbody>
</table>

### Probability that an offspring is of this genotype

<table>
<thead>
<tr>
<th>genotype</th>
<th>probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>(aa)</td>
<td>((X + \frac{1}{2}Y + d)^2/\pi^2)</td>
</tr>
<tr>
<td>(aA) or (Aa)</td>
<td>(2(X + \frac{1}{2}Y + d)(Z + \frac{1}{2}Y - d)/\pi^2)</td>
</tr>
<tr>
<td>(AA)</td>
<td>((Z + \frac{1}{2}Y - d)^2/\pi^2)</td>
</tr>
</tbody>
</table>

\[ \mu_{A}Z - \mu_{a}X + \frac{1}{2}Y(\mu_{A} - \mu_{a}) \]

\[ \overline{\pi} = X + Y + Z \]
Sexual Dynamics

\[ x(t + \tau) = \frac{x(t) + \tau \frac{1}{2} \pi (X + \frac{1}{2} Y + d)^2 / \pi^2}{1 + \tau \frac{1}{2} \pi} \]

\[ y(t + \tau) = \frac{y(t) + \tau \frac{1}{2} \pi 2(X + \frac{1}{2} Y + d)(Z + \frac{1}{2} Y - d) / \pi^2}{1 + \tau \frac{1}{2} \pi} \]

\[ z(t + \tau) = \frac{z(t) + \tau \frac{1}{2} \pi (Z + \frac{1}{2} Y - d)^2 / \pi^2}{1 + \tau \frac{1}{2} \pi}. \]
Sexual Dynamics

\[ x + y + z = 1, \]

\[ \dot{x} = \frac{(2X + Y + 2d)^2}{\pi} - 4\pi x \]

\[ \dot{z} = \frac{(2Z + Y - 2d)^2}{\pi} - 4\pi z \]
Sexual Dynamics

\[ x + y + z = 1, \]

\[ \dot{x} = \frac{(2X + Y + 2d)^2}{\pi} - 4\pi x \]
\[ \dot{z} = \frac{(2Z + Y - 2d)^2}{\pi} - 4\pi z \]

**At a rest point** where \( \dot{x} = \dot{z} = 0, \)

\[ (2X + Y + 2d)^2 = 4x\pi^2, \]
\[ (2Z + Y - 2d)^2 = 4z\pi^2. \]

When it is right to take the positive square roots, we obtain

\[ 2(X + Y + Z) = 2\pi(\sqrt{x} + \sqrt{z}), \]

from which we derive the useful equations:

\[ \sqrt{x} + \sqrt{z} = 1 \]
\[ y = 2\sqrt{xz} \]
Sexual Dynamics: Hawk-Dove Game

To accommodate sexual reproduction, assume that the allele \( a \) is recessive and that \( A \) is dominant. The strategy *dove* is played by the genotype \( aa \). The strategy *hawk* is played by both genotypes \( aA \) and \( AA \). We then have that \( \pi_{aa} = \pi_D \) and \( \pi_{aA} = \pi_{AA} = \pi_H \). Writing (6) and (7) into equation (4), we obtain that

\[
x(1 - \sqrt{x})(\pi_D - \pi_H) + d = 0,
\]

where \( x \) is the proportion of the population playing *dove*. Apart from the drift term \( d \), this equation differs from that obtained using the standard asexual replicator dynamics only in replacing the factor \( 1 - x \) by \( 1 - \sqrt{x} \).

In the absence of drift (\( d = 0 \)), either \( x = 0 \) or \( x = 1 \) or else

\[
\pi_H = \pi_D,
\]

which is the equation that determines the mixed Nash equilibrium in which *dove* is played with probability \( x = \frac{1}{2} \) and *hawk* is played with probability \( y + z = 1 - x = \frac{1}{2} \). This matches the outcome under asexual dynamics and provides an illustration of Grafen’s [7] general observation that the rest points are the same in both the sexual and the asexual dynamics. The frequency of the \( aa \) genotype at the mixed equilibrium is therefore \( x = 0.5 \). It follows from equations (6) and (7) that the frequencies of the \( aA \) and \( AA \) genotypes are respectively \( y = 0.415 \) and \( z = 0.086 \).
Sexual Dynamics: Hawk-Dove Game

To accommodate sexual reproduction, assume that the allele $a$ is recessive and that $A$ is dominant. The strategy dove is played by the genotype $aa$. The strategy hawk is played by both genotypes $aA$ and $AA$. We then have that $\pi_{aa} = \pi_D$ and $\pi_{aA} = \pi_{AA} = \pi_H$. Writing (6) and (7) into equation (4), we obtain that

$$x(1 - \sqrt{x})(\pi_D - \pi_H) + d = 0,$$

where $x$ is the proportion of the population playing dove. Apart from the drift term $d$, this equation differs from that obtained using the standard asexual replicator dynamics only in replacing the factor $1 - x$ by $1 - \sqrt{x}$.

In the absence of drift ($d = 0$), either $x = 0$ or $x = 1$ or else

$$\pi_H = \pi_D,$$

which is the equation that determines the mixed Nash equilibrium in which dove is played with probability $x = \frac{1}{2}$ and hawk is played with probability $y + z = 1 - x = \frac{1}{2}$. This matches the outcome under asexual dynamics and provides an illustration of Grafen’s [7] general observation that the rest points are the same in both the sexual and the asexual dynamics. The frequency of zebra at the mixed equilibrium is therefore $x = 0.5$. It follows from (6) and (7) that the frequencies of the $aA$ and $AA$ genotypes are respectively $y = 0.415$ and $z = 0.086$. 

phenotypical gambit
Impact of Sexual Transmission on Drift

\[
s(1 - \sqrt{s})(1 - p)k + d_O = 0, \\
p(1 - \sqrt{p})(k - s(v + k)) + d_I = 0,
\]

where the drift terms are given by

\[
d_O = (b + pv - k(1 - p))(1 - \sqrt{s})(\mu_A - \mu_a\sqrt{s}) - \mu_a(b + pv)s \\
d_I = (b + s(v + k) - k)(1 - \sqrt{p})(\mu_B - \mu_b\sqrt{p}) - \mu_bbp.
\]
Impact of Sexual Transmission on Drift

\[
\frac{ss^*(1 - s)}{2(s - \frac{1}{2})(s^* - s)} = \frac{b + u}{b} \frac{\delta_o}{\delta_I} \frac{(\sqrt{s} - \frac{1}{2})(1 - s)}{2(s - \frac{1}{2})(1 - \sqrt{s})}
\]

> 1 ?

(c) Mutation both ways
Impact of Sexual Transmission on Drift

\[
\frac{ss^*(1 - s)}{2(s - \frac{1}{2})(s^* - s)} = \frac{b + v}{b} \frac{\delta_o}{\delta_I} \frac{(\sqrt{s} - \frac{1}{2})(1 - s)}{2(s - \frac{1}{2})(1 - \sqrt{s})} > 1 ?
\]

(c) Mutation both ways