Equilibrium Selection

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An old dream of Harsanyi and Selten is to select a unique (the 'best') equilibrium in a game.

J.C. Harsanyi, R. Selten: *A General Theory of Equilibrium Selection in Games*, MIT Press (1988).

Evolutionary/dynamic approaches to equilibrium selection
stochastic models:
Foster & Young (1990), Young (1993), Kandori, Mailath, Rob (1993)

Today: How replicator dynamics (or some other deterministic evolutionary game dynamics) can help to achieve this.

2 approaches based on **replicator dynamics**

$n \times m$ bimatrix game (A, B)

$$\dot{x}_i = x_i\Big((Ay)_i - x \cdot Ay\Big), \quad i = 1, \dots n$$

$$\dot{y}_j = y_j\Big((B^T x)_j - x \cdot By\Big) \quad j = 1, \dots m$$

meaning of payoffs?
risk–dominance

2×2 coordination games $\begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} \quad (a_i > 0)$

$$\begin{pmatrix} a_1, b_1 & 0, 0 \\ 0, 0 & a_2, b_2 \end{pmatrix} \quad (a_i, b_i > 0)$$

$\mathbf{1}$ is **risk dominant** over $\mathbf{2}$, if $a > b$, resp.

$$a_1 b_1 > a_2 b_2.$$

Every strict equilibrium is an attractor for every evolutionary/adaptive dynamics.

quantity vs. quality ??

Equilibrium selection via travelling waves

J. Hofbauer, V. Hutson and G.T. Vickers, *Travelling waves for games in economics and biology,* Nonlinear Analysis **30** (1997) 1235–1244.

P. Fife: *Mathematical aspects of reacting and diffusing systems*, Springer Lecture Notes in Biomathematics **28** (1979).

spatial version of REP goes back to
R. Cressman, G. T. Vickers: *Spatial and density effects in evolutionary game theory*, J. Theor. Biol. **184** (1997), 359-369.

The spatio–temporal model: Reaction diffusion equation

Players interact locally and migrate randomly (like particles in a Brownian motion).

n player populations distributed in space (= line) $p_i = p_i(x,t)$: functions of space $x \in \mathbb{R}$ and time t.

$$\frac{\partial p_i}{\partial t} = f_i(p) + d\frac{\partial^2 p_i}{\partial x^2}.$$

diffusion term: random migration of players at a uniform, strategy– and player–independent rate $d > 0$.

reaction term: $f(p)$ models the local interaction of players and the resulting adaptations of their strategy according to their local experience.

Symmetric 2×2 games

$$\begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} \qquad (a_i > 0)$$

$$\frac{\partial p}{\partial t} = p(1-p)(a_1 + a_2)p - a_2) + d\frac{\partial^2 p}{\partial x^2} \qquad (RD)$$

Solutions $0 \leq p(x,t) \leq 1$

Stationary solutions $p(x,t) = P(x):\quad dP'' + f(P) = 0$

Travelling wave solutions $p(x,t) = P(x-ct)$: $(c \ldots$ wave speed)

$$dP'' + cP' + f(P) = 0$$

Travelling wave solutions $p(x,t) = P(x - ct)$: ($c \ldots$ wave speed)

$$dP'' + cP' + f(P) = 0$$

bistable wave: $P(-\infty) = 0, \qquad P(\infty) = 1$

$c < 0$: $p(x,t) \to 1$ .... $\mathbf{1}$ drives out $\mathbf{0}$

$c = 0$: standing wave

Theorem: $a_1 > a_2 \Longleftrightarrow c < 0 \Longleftrightarrow \mathbf{1}$ drives out $\mathbf{2}$

Theorem: (see Fife, 1979)

For a bistable RD equation the following conditions are equivalent.

(1) There exists a unique 'bistable' travelling wave, and its speed $c < 0$.

(2) For any initial condition satisfying $p(x,0) = 0$ for $x < a$ and $p(x,0) = 1$ for $x > b$, one has $p(x,t) \to 1$ as $t \to \infty$, for each $x \in \mathbb{R}$.

(3) $\mathbf{1}$ is asymptotically stable in the c–o–topology:
$\exists L > 0, \varepsilon > 0$: for each initial function satisfying $p(x,0) > 1 - \varepsilon$ for $x \in [-L, L]$: $p(x,t) \to 1$ as $t \to \infty$, uniformly on compact subsets of $\mathbb{R}$.

(4) There exists a 'standing pulse' solution, i.e., a stationary solution $P(x)$ satisfying $\lim_{x\to\pm\infty} P(x) = 0$.

(5) $\int_0^1 f(p)dp > 0$.

(6) $V(1) > V(0)$, where V is a potential function for the reaction term: $V'(p) = f(p)$.

2×2 **bimatrix games**

$$\begin{pmatrix} a_1, b_1 & 0, 0 \\ 0, 0 & a_2, b_2 \end{pmatrix}$$

$a_i, b_i > 0$ coordination game (unanimity game)

Two strict equilibria $\mathbf{1}$, $\mathbf{2}$, and a mixed equilibrium (p, q)

1 is **risk dominant** over 2, if

$$a_1 b_1 > a_2 b_2,$$

($\mathbf{1}$ has the higher Nash product), or

$$p + q < 1.$$

replicator dynamics

$$\dot{u} = u(1-u)\left((a_1+a_2)v - a_1\right)$$

$$\dot{v} = v(1-v)\left((b_1+b_2)u - b_1\right)$$

$$\dot{u} = (a_1+a_2)u(1-u)(v-q)$$

$$\dot{v} = (b_1+b_2)v(1-v)(u-p)$$

$$p = \frac{b_1}{b_1+b_2}, \quad q = \frac{a_1}{a_1+a_2}$$

$r = \frac{b_1+b_2}{a_1+a_2}$ ratio of payoff scales of two players

$p = \frac{b_1}{b_1+b_2}$, $\quad q = \frac{a_1}{a_1+a_2}$ mixed equilibrium

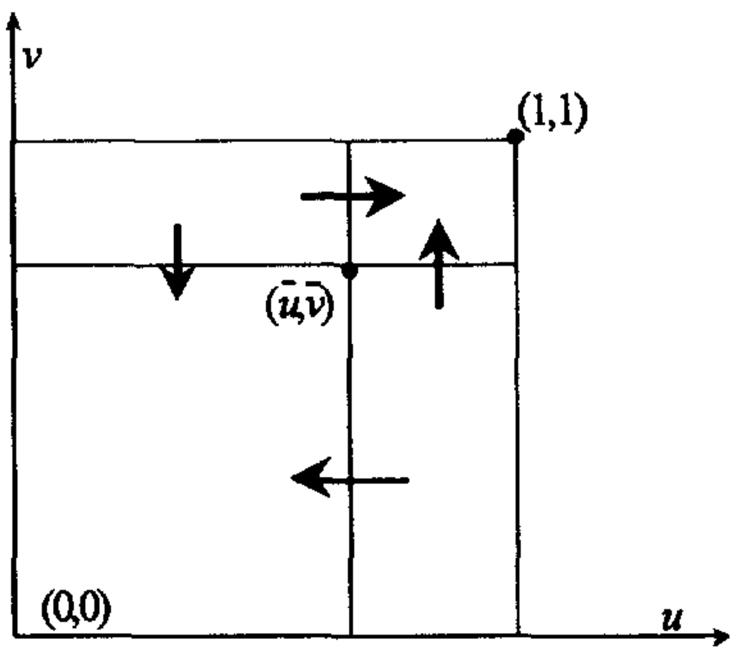
$$\dot{u} = u(1-u)(v-q)$$

$$\dot{v} = rv(1-v)(u-p)$$

$\mathbf{1}$ risk-dominates $\mathbf{2}$ iff $p+q<1$

$$u_t = u(1-u)(v-q) + du_{xx}$$

$$v_t = rv(1-v)(u-p) + dv_{xx}$$



Theorem. For (RD) with REP dynamics as reaction term, there exists a *unique* monotone travelling wave for (RD) that connects the two strict equilibria.

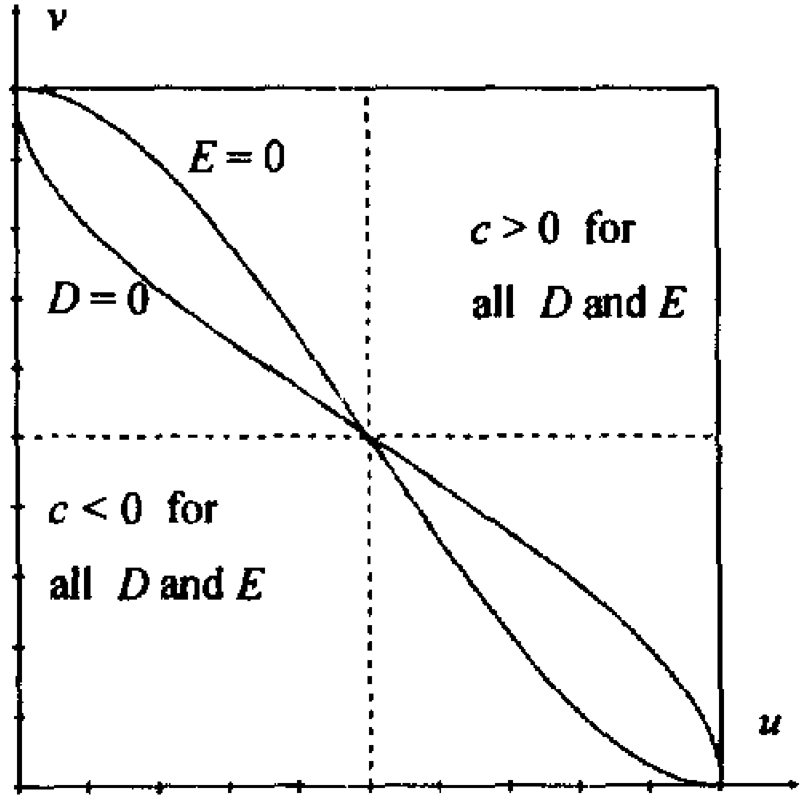
wave speed $c = 0$ if
1) $r = 1$, $p + q = 1$, or
2) $p = q = \frac{1}{2}$

wave speed $c < 0$ ($\mathbf{1}$ drives out $\mathbf{2}$) if
1) $r = 1$, $p + q < 1$, or
2) $p, q < \frac{1}{2}$

$r \to 0$: singular perturbation theory (Hutson, Mischaikow 1996)

$c = 0$ at $p = 2q^3 - 3q^2 + 1$

$r \to \infty$:
$c = 0$ at $q = 2p^3 - 3p^2 + 1$



Inside the petal: outcome depends on weights (i.e., on r)
outside the petal: outcome depends on p, q only!

Selection + mutation

(JH + Boyu ZHANG)

$$\dot{x}_i = x_i\left((Ax)_i - x\cdot Ax\right) + \varepsilon(1 - nx_i), \quad i = 1, \dots, n$$

with $\varepsilon > 0$ uniform mutation rate.

rest points (selection–mutation balance) $x(\varepsilon)$
$\varepsilon \to 0$: $x(\varepsilon) \to$ Nash equilibria of game A
$\varepsilon \to \infty$: $x(\varepsilon) \to (\frac{1}{n}, \dots, \frac{1}{n})$ centroid

selection + mutation homotopy method

$$\dot{x}_i = 0 = x_i\,((Ax)_i - x\cdot Ax) + \varepsilon(1 - nx_i), \quad i = 1, \ldots, n$$

For almost every game, there is a unique, smooth path of selection–mutation equilibria that connects the centroid ($\varepsilon = \infty$) to a single NE ($\varepsilon = 0$).

The set of selection–mutation equilibria forms a 1d submanifold of $\Delta \times [0, \infty)$.

Harsanyi's logarithmic games

$\pi(y, x, t) = (1-t)y \cdot Ax + t \sum_i \log y_i$

equilibria: $(1-t)\left((Ax)_i - (Ax)_j)\right) + t(\frac{1}{x_i} - \frac{1}{x_j}) = 0 \qquad \forall i, j$

$\varepsilon = \frac{t}{1-t}$

J.C. Harsanyi: *Oddness of the number of equilibrium points: a new proof*, Int. J. Game Theory **2** (1973), 235–250.

closely related to **quantal response equilibria**

$$x_i = \frac{e^{(Ax)_i/\varepsilon}}{\sum_j e^{(Ax)_j/\varepsilon}}, \quad \text{or} \quad x = L(\frac{Ax}{\varepsilon})$$

with $L : \mathbb{R}^n \to \Delta$, $\quad L_k(u) = \frac{e^{u_k}}{\sum_j e^{u_j}}$ logit equilibria

$\varepsilon \to \infty$: $x(\varepsilon) \to (\frac{1}{n}, \ldots, \frac{1}{n})$ centroid
$\varepsilon \to 0$: $x(\varepsilon) \to$ NE
LLE 'limiting logit equilibrium'
McKelvey, R. D. and T. D. Palfrey: Quantal response equilibria for normal form games. *Games Econ. Behav.* **10** (1995), 6–38.

T.L. Turocy: A dynamic homotopy interpretation of the logistic quantal response equilibrium correspondence.
Games Econ. Behav. 51, 243-263 —— implemented in Gambit

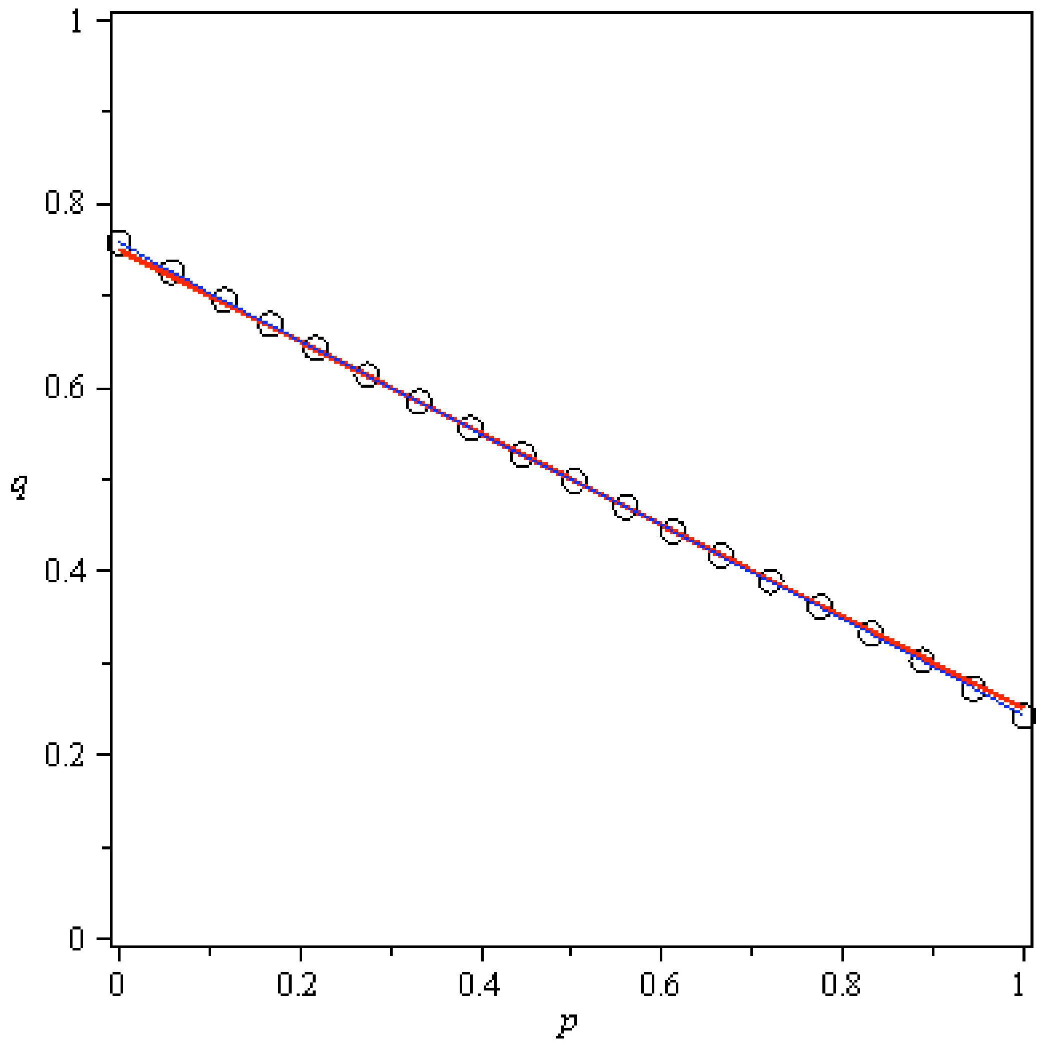
2×2 bimatrix game

$$\dot{u} = u(1-u)(v-q) + \varepsilon(1-2u)$$

$$\dot{v} = rv(1-v)(u-p) + \varepsilon(1-2v)$$

$\mathbf{1}$, i.e. $(u,v) = (1,1)$ is selected by selection-mutation homotopy if

1) $r = 1$, $p + q < 1$, or
2) $p, q < \frac{1}{2}$, or
3) $r \approx 0$, $q < \frac{1}{2}$, or
4) $r \approx \infty$, $p < \frac{1}{2}$



$r = \frac{1}{4}$

$$0 = u(1-u)(v-q) + \varepsilon(1-2u)$$

$$0 = rv(1-v)(u-p) + \varepsilon(1-2v)$$

eliminate ε:

$$r\frac{1-2u}{u(1-u)}(u-p) - \frac{1-2v}{v(1-v)}(v-q) = 0$$

$$S(u,v) = rf(u)(u-p) - f(v)(v-q) = 0$$

look for critical points (works for general QRE)

$S_u = rf'(u)(u-p) + rf(u) = 0$

$S_v = -f'(v)(v-q) - f(v) = 0$

$$H(q) = rH(p)$$

$$H(q) = rH(p)$$

$$H(p) = \frac{1}{2} - \sqrt{p(1-p)}$$

The mutation–selection homotopy leads to $\mathbf{1}$ (A_1B_1) iff

$$\sqrt{a_1} + \sqrt{b_1} > \sqrt{a_2} + \sqrt{b_2}$$

1 is **risk dominant** over $\mathbf{2}$, iff

$$a_1b_1 > a_2b_2$$