Fictitious Play and No-Cycling Conditions*

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Abstract. We investigate the paths of pure strategy profiles induced by the *fictitious play* process. We present rules that such paths must follow. Using these rules we prove that every non-degenerate[1] 2x3 game has the *continuous fictitious play property*, that is, every *continuous fictitious play* process, independent of initial actions and beliefs, approaches equilibrium in such games.

1. Introduction

Consider n players that play repeatedly a game in strategic form. Each player has subjective beliefs about other players' future behavior. In each period each player chooses a best response according to his belief and updates his belief according to the past observations. We call such a process a belief-based learning process.

The basic belief based learning process is the *fictitious play* (*FP*) process proposed by Brown in 1951. In a *FP* process each player believes that each one of his opponents is using a stationary mixed strategy which is the empirical distribution of this opponent's past actions. Most of the research about the *FP* process has focused on the questions, whether players' beliefs approach equilibrium. A game in which every *FP* process, independent of initial actions and beliefs, approaches equilibrium, is called a game with the *FP property*.

Every learning process is a pair (X, B), where X is a path of pure strategy profiles, and B is a belief sequence. If we eliminate all successive repetitions in the path X, we get a new path of pure profiles which we call the reduced path. Studying reduced paths of the *fictitious play* process teaches us a lot about properties of the process. We present four rules which are called the four principles of motion that such paths must follow :

1. **The improvement principle** : Given a two person game, if the players are moving from the pure strategy profile x to the pure strategy profile y, then for each player i for which $y^i \neq x^i$, y^i is better for player i than x^i versus x^{-i}.
2. **The stability principle**[2] : If at some stage of the process the players choose a pure strategy equilibrium strategy x, then they will choose this x from this stage on.

[1] A game G is non-degenerate, if the payoff function of each player is one-to-one function for every joint strategy of his opponents.
[2] This is a simplification of the analogous principle established by Fudenberg and Kreps (1991).

3. **The separation principle** : If the players move from x to y and $y^i \neq x^i$, then x^i and y^i are not separated in the sense that $B(x^i) \cap B(y^i) \neq \emptyset$, where $B(x^i)$ is the set of all mixed strategy profiles of player i's opponents against which x^i is a best response for player i.
4. **The reduction principle** : Given a two person game, if the players do not use some strategies from a certain stage on, and if the sub game obtained by eliminating these unused strategies is a $2x2$ game, then the process approaches equilibrium.

We use these rules to prove either existence or non-existence of the *FP* property for some games. In particular, using these rules we introduce a simple proof to the well known example of Shapley (1964) which is an example to a class of games without the *FP* property. On the other hand we use these rules in order to prove that every non-degenerate $2x3$ game has the *FP* property in the continuous (time) case.

Indeed we know just about few classes of games with the *FP* property . Zero-sum games, i.e., bimatrix games of the form $(A,-A)$ (Robinson (1951)). Games with identical payoff functions, i.e., games in which all the players have the same payoff matrix (Monderer and Shapley(1996)). Games which are dominance solvable (Milgrom and Roberts (1991)). Games with strategic complementarities and diminishing returns (Krishna (1992)). 2x2 Games (Miyasawa (1964)). The results of Miyasawa (1964) have been proved under certain indifference breaking rules. Monderer and Sela (1996) have shown that if we do not assume any tie breaking rules about the particular best response at each stage (when the best response is not unique), there is a 2x2 game that does not have the *FP* property. In order to avoid such extreme cases we deal with non-degenerate games. Every 2x2 non-degenerate game has the *FP* property (see Monderer and Shapley (1996))[3]. The current conjecture is that this result can be extended for $2xn$ games, $n > 2$. As was mentioned, we affirm part of this conjecture. Using the four principles of motion, we prove that every 2x3 non-degenerate game has the *CFP* property (the *FP* property for the continuous case). In this case we find it convenient to work with the continuous time formulation of *fictitious Play* (*CFP*) rather than the discrete time formulation (*FP*). In some cases the

[3] Monderer and Shapley (1993) give a new proof for the *FP* property of such games.

CFP is more natural and more practical than the *FP*. For instance, the convergence of the *FP* in zero-sum games, which require a complex proof for the discrete process (Robinson (1951)), can be proved easily for the continuous case (Hofbauer (1994)). Although there are no results relating the discrete and the continuous processes, it seems that whenever the *CFP* exist, the *FP*'s path behaves similarly to the continuous one.

2. Notation

Let $N = \{0,1,2,\ldots,n\}$ be the set of players. For each $i \in N$, S^i is the finite strategy set of player i. For every $M \subseteq N$ we denote $S^M = \mathop{X}_{i \in M} S^i$. In particular we denote $S = \mathop{X}_{i \in N} S^i$ and $S^{-i} = \mathop{X}_{j \in N \setminus \{i\}} S^i$. Let $U^i : S \to R$ be player i's payoff function, where R denotes the set of real numbers. For each finite set A we denote by $\Delta(A)$ the set of probability measures over A. For $M \subseteq N$ we denote $\Delta^M = \mathop{X}_{i \in M} \Delta(S^i)$ and $\Delta_J^M = \Delta(S^M)$. The set of player i's mixed strategies $\Delta(S^i)$ is denoted by Δ^i. We denote $\Delta^{-i} = \Delta^{N \setminus \{i\}}$ and $\Delta_J^{-i} = \Delta_J^{N \setminus \{i\}}$. We identify $x^i \in S^i$ and $x^{-i} \in S^{-i}$ with extreme points in Δ^i and Δ_J^{-i} respectively.

3. No-Cycling Conditions

We define four order relations over S:

1. $x \phi_1 y$ if there exists a player i such that $x^{-i} = y^{-i}$ and $U^i(x) > U^i(y)$.
2. $x \phi_2 y$ if there exists a non-empty set of players M such that $x^{-M} = y^{-M}$ and $U^i(x^i, y^{-i}) > U^i(y)$ for every $i \in M$.
3. $x \phi_3 y$ if there exists a player i such that $x^{-i} = y^{-i}$ and x^i is a best response to y^{-i}.
4. $x \phi_4 y$ if there exists a non-empty set of players M such that $x^{-M} = y^{-M}$ and x^i is a best response to y^{-i} for every $i \in M$.

The order relations are related according to the next diagram :

$$x \,\phi_3\, y \;\Rightarrow\; x \,\phi_4\, y$$
$$\Downarrow \qquad\qquad \Downarrow$$
$$x \,\phi_1\, y \;\Rightarrow\; \mathrm{x} \,\phi_2\, y$$

A path in S is a sequence, $X = (x(t)),\ t \geq 0$, of elements in S. The path is $k-$ *increasing* , $1 \leq k \leq 4$, if $x(t+1)\ \phi_k\ x(t)$ for every $t \geq 1$.

A path which is $k-$increasing, $k = 1,2$ is called *path generated by the better reply dynamic*. And a path which is $k-$increasing, $k = 3,4$ is called *path generated by the best reply dynamic.*

We say that a game G satisfies the k*-no-cycling* condition or G is $k-$acyclic if every $k-$ *increasing* sequence is finite[4]. The set of all k-no-cycling games is denoted by $NC(k)$. The relations between these sets of games are :

$$NC(3) \;\supset\; NC(4)$$
$$\mathrm{Y} \qquad\qquad \mathrm{Y}$$
$$NC(1) \;\supset\; NC(2)$$

The following example shows a game that satisfies the 3-no cycling condition and does not satisfy the 1-no-cycling condition. Similarly it can be shown that all inclusion relations in the above diagram are strict.

Example 3.1

Let

$$G = \begin{bmatrix} (10,0) & (-10,2) & (0,-10) \\ (0,6) & (0,9) & (15,5) \\ (-20,-1) & (-2,3) & (14,4) \end{bmatrix}$$

Denote by (*i*,*j*) the point in G such that, the row player chooses row *i* and the column player chooses column *j*. The cycle:

$(1,1) \to (1,2) \to (3,2) \to (3,3) \to (2,3) \to (2,1) \to (1,1)$ is generated 1-increasing.

A simple inspection reveals that there are no 3-cycles[5].

The games $G(U^1, U^2, \ldots, U^n)$ and $G(V^1, V^2, \ldots, V^n)$ (U^i and V^i denote player i 's utility functions) are *better reply equivalent* if for every player i, $U^i(\ , x^{-i})$ and

[4] Monderer and Shapley (1993) called a "1-increasing path" an "improvement path" and they called a 1-acyclic game a "game with the finite improvement property".

$V^i(\ ,x^{-i})$ induce the same preference relation on S^i for every $x^{-i} \in S^{-i}$. The games are *best reply equivalent* if for every player i, $U^i(\ ,x^{-i})$ and $V^i(\ ,x^{-i})$ attain their maximal values at the same subset of S^i for every $x^{-i} \in S^{-i}$.

The game G is *non-degenerate* if for every i, $U^i(\ ,x^{-i})$ is a one-to-one function for every $x^{-i} \in S^{-i}$.

Monderer and Shapley (1996) showed that a non-degenerate game belongs to *NC(1)* if and only if it is better reply equivalent in pure strategies to a game with identical payoff functions. The analogous result can be similarly proved for the ϕ_3 relation.

<u>Lemma 3.2</u>: A non-degenerate game satisfies the 3-no-cycling condition if and only if it is best reply equivalent in pure strategies to a game with identical payoff functions[6].

The following example appears in Monderer and Shapley (1996). It shows the necessity of the "non-degenerate game" requirement in Lemma 3.2 :

<u>Example 3.3</u>:

$$G(U^1,U^2) = \begin{bmatrix} (0,0) & (1,0) \\ (1,0) & (0,1) \end{bmatrix}$$

If $G(U^1,U^2)$ is best reply equivalent to $G(U,U)$, then the following contradiction follows :

$$U(1,1) < U(2,1) < U(2,2) < U(1,2) = U(1,1).$$

4. Fictitious Play Process

There are two versions of the *FP* process. In the first version each player believes that each one of his opponents is using a stationary mixed strategy which is the empirical distribution of this opponent's past actions. Such a player will be called a *IFP* player ("I" stands for Independent). In the second version, each player believes that his opponents are using a joint correlated mixed strategy, which is the empirical distribution of his opponents' past actions. Such a player will be called a *JFP* player

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[5] Actually, this game has strategic complementarities and diminishing returns and therefore by Krishna (1992) it belongs to NC(3).

[6] Actually the "best reply" version can be proved for every class of games as well. We just need that the best reply correspondences (in pure strategies) are single values.

(“J” stands for Joint). In two person games, the concepts of *IFP* player and *JFP* player coincide. Since most of the research on *FP* process concentrated in two-person games, the difference between *JFP* and *IFP* has been hardly noticed. In this paper we refer to the *IFP* as *FP*.

A *belief sequence* $B=\{b(t)\}$, for $t \geq 1$, consist of elements of $\underset{i \in n}{\mathrm{X}} \Delta^{-i}$, i.e. $b^i(t) \in \Delta^{-i}$, $i \in N$, is the belief of player i about the other players’ strategies at stage t. $b_k^i(t) \in \Delta^k$, $i \neq k,$ is the belief of player i about player k's strategy at stage t.

A *joint belief sequence* $B=\{b(t)\}$, for $t \geq 1$, consist of elements of $\underset{i \in n}{\mathrm{X}} \Delta_J^{-i}$, i.e. $b^i(t) \in \Delta_J^{-i}$, $i \in N$, is the belief of player i about the joint strategy of the other players at stage t.

Let $b^i(t) \in \Delta_J^{-i}$ and $M \subseteq N \setminus \{i\}$. Denote by $b_{\{M\}}^i(t)$ the marginal distribution on M.

That is , $b_{\{M\}}^i(t)(x^M) = \sum_{x \in S^{N \setminus \{M,i\}}} b^i(x, x^M)(t)$.

A *learning process* is a pair (X,B), where X is a path in S, and B is a belief sequence or joint belief sequence, such that for every $t \geq 1$ and every player i, the strategy $x^i(t)$ is a best response to $b^i(t)$.

A learning process (X,B) is a *fictitious play (FP)* process, if for every player i, and for every $k \neq i$, $b_k^i(t) = \frac{1}{t}\sum_{s=0}^{t-1} x^k(s)$, $t \geq 1$, (here $x^k(s)$ is a point in Δ^k).

Note that in a *FP* process $b_k^i(t) = b_k^s(t)$ for all $i \neq k$ and $s \neq k$. We denote by $b_k(t)$ the identical belief of all the players about player $k's$ strategy at stage t .

A *FP* process (X,B) *approaches equilibrium,* if for every $\varepsilon > 0$ there exist $t_0 > 0$, such that for every $t \geq t_0$, there exist a mixed equilibrium profile $p \in \Delta^N$, such that, $\left|(b_1(t), b_2(t), \ldots, b_n(t)) - p\right| < \varepsilon$.

We say that a game has the *FP property* , if every *FP* process, independent of initial actions and beliefs, approaches equilibrium.

A learning process (X,B) is a *joint fictitious play* (*JFP*) process, if for every player i, $b^i(t) = \frac{1}{t}\sum_{s=0}^{t-1} x^{-i}(s)$, $t \geq 1$, (here $x^{-i}(s)$ is a point in Δ_J^{-i}).

Note that in a *JFP* process, for every two players $i \neq k$, and for all $M \subseteq N \setminus \{i,k\}$, $b^i_{\{M\}}(t) = b^k_{\{M\}}(t)$. We denote by $b_k(t)$ the identical belief of all the players about player k's strategy at stage t.

5. The Four Principles of Motion

Every learning process is a pair (X, B), where X is a path of pure strategy profiles and B is a belief sequence. If we eliminate all successive repetitions in the path of pure profiles we get a new path of pure profiles which we call the *reduced path*. For instance, x, y, x, z, y is the reduced path of $x, y, y, x, x, x, z, z, y$.

By investigation the reduced paths structure induced by the *FP* process, we obtain four principles of motion that such paths must follow in non-degenerate games.

1. <u>The Improvement Principle</u>: Consider a two person game**.** The reduced path of pure strategies generated by a *FP* process is 2-increasing[7].

<u>Proof :</u> It suffices to show that if $x(t) \neq x(t-1)$ then for every player

$i \in M(t) = \{k \in N\colon x^k(t) \neq x^k(t-1)\}$, $U^i(x^i(t), x^{-i}(t-1)) > U^i(x(t-1))$.

Indeed, note that $U^i(x^i(t-1), b^i(t-2)) \geq U^i(x^i(t), b^i(t-2))$, and $U^i(x^i(t), b^i(t-1)) \geq U^i(x^i(t-1), b^i(t-1))$. As $b^i(t-1)$ is a convex combination of $b^i(t-2)$ and $x^{-i}(t-1)$, and the game is non-degenerate, the result follows. ■

<u>Corollary 5.1</u>: Every 2-acyclic game has the FP property.

The class of 2-acyclic games is quite restricted. The following example (due to Foster and Young (1995)) give us a reasonable doubt about the conjecture that corollary 5.1 is valid also for 1-acyclic games :

[7] The reduced path generated by a *JFP* process is 2-increasing for every n, $n \geq 1$, player game.

$$G = \begin{bmatrix} (24,24) & (6,6) & (0,18) & (0,18) & (18,0) & (18,0) & (5,0) & (0,0) \\ (6,6) & (24,24) & (0,18) & (0,18) & (18,0) & (18,0) & (4,0) & (0,0) \\ (18,0) & (18,0) & (24,24) & (6,6) & (0,18) & (0,18) & (3,0) & (0,0) \\ (18,0) & (18,0) & (6,6) & (24,24) & (0,18) & (0,18) & (2,0) & (0,0) \\ (0,18) & (0,18) & (18,0) & (18,0) & (24,24) & (6,6) & (1,0) & (0,0) \\ (0,18) & (0,18) & (18,0) & (18,0) & (6,6) & (24,24) & (0,0) & (0,0) \\ (0,4) & (0,5) & (0,2) & (0,3) & (0,0) & (0,1) & (25,24) & (-25,-25) \\ (0,0) & (0,0) & (0,0) & (0,0) & (0,0) & (0,0) & (-25,-25) & (24,25) \end{bmatrix}$$

Foster and Young (1995) introduced this game as an example of a coordination two player game without the *FP* property. This game is not 1-acyclic game and therefore this game is not a counter example to the conjecture that every 1-acyclic game has the *FP* property. Still, the reduced path generated by a *FP* in this case is :

$(1,2) \to (2,1) \to (3,4) \to (4,3) \to (5,6) \to (6,5) \to (1,2).....$

This path is 2-increasing. Therefore the question whether a game is either 1-acyclic or not seems irrelevant . Further, by the Improvement principle we show that the reduced path generated by a *FP* is a better reply dynamic. We can see by this example that the reduced path generated by the *FP* process is not necessarily a best reply dynamic for some stage on.

Shapley's Example

Now, we proceed to show an application of the improvement principle. Consider the following game satisfying the ordinal properties of Shapley (1964):

$$G_1 = \begin{bmatrix} (0,0) & (a,b) & (b,a) \\ (b,a) & (0,0) & (a,b) \\ (a,b) & (b,a) & (0,0) \end{bmatrix} \quad \text{where } a > b > 0 \text{ , and } a < 2b .$$

Proposition (Shapley, 1964) : G_1 does not have the *FP* property.

We prove this result by the improvement principle :

Proof : This game has a unique equilibrium $(p,q) \in \Delta^1 x \Delta^2$, where $p = q = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.

Therefore, If a *FP* process approaches equilibrium it must actually converges to

(p,q). Consider a process starting from one of the non-zero entries. We show that this process does not converge to (p,q).

Consider the following game G_2 obtained by adding a 4^{th} row to G_1 :

$$G_2 = \begin{bmatrix} (0,0) & (a,b) & (b,a) \\ (b,a) & (0,0) & (a,b) \\ (a,b) & (b,a) & (0,0 \\ (c,0) & (c,0) & (c,0) \end{bmatrix}$$

where $\frac{a+b}{3} < c < b$.

By the improvement principle (because $b < c$), player 1 never chooses the 4^{th} row. Therefore the process generates the same play of pure strategies in G_1 and G_2. Suppose in negation that the process converges to (p,q) in G_1, then it must converge to $(\overline{p},q)$ in G_2, where $\overline{p} = (p_1, p_2, p_3, 0)$. Note that the 4^{th} row is the unique best reply of player 1 to q. Therefore it is the unique best reply to mixed strategies of player 2 that are sufficiently close to q. Hence, if The *FP* converges to $(\overline{p},q)$, player 1 eventually switches to the 4^{th} row, contradicting the improvement principle. ■

The proof of the non-approach result for the generalized *fictitious play* in the examples given by Deschamps, Elison and Fudenberg is very similar (simpler actually) so we omit it.

Let $$G = \begin{bmatrix} (0,0) & (a,b) & (b,a) & (0,c) \\ (b,a) & (0,0) & (a,b) & (0,c) \\ (a,b) & (b,a) & (0,0) & (0,c) \\ (c,0) & (c,0) & (c,0) & (d,d) \end{bmatrix}$$

Where $d > a > b > c > \frac{a+b}{3} > 0$.

Note that by the improvement principle, in every *FP* process starting at one of the (a,b) or (b,a) entries, the players will not use their 4^{th} strategies. On the other hand, the unique equilibrium in this game is attained at the (d,d) entry.

The second principle is an immediate conclusion of the improvement principle :

2. <u>The Stability Principle</u>: If a pure strategy equilibrium is played at some stage in

a *FP* process, then it is played from this stage on for ever.

Denote by $B(x)$, $x \in S^i$, the set of all joint mixed strategy profiles in Δ^{-i} against which x is best reply. We now establish the third principle :

3. <u>The Separation Principle</u>: Let x and y be pure strategies of player i .Suppose that $B(x)$ and $B(y)$ do not intersect. Then in every *FP* process, for sufficiently late stage, player i will never change its choice from x to y and vice versa.

<u>Proof :</u> The distance between two successive beliefs of player i , $b^i(t)$ and $b^i(t+1)$ is at most $\frac{2}{t+1}$. Therefore, if player i chooses x at stage t, he will not choose y in the next dt stages, where d is the distance between the sets $B(x)$ and $B(y)$. ■

The fourth principle is derived from Miyasawa's proof (1961) about the existence of the *FP* property in $2x2$ games

4. <u>The Reduction Games</u>: Consider a two person game. If the reduced path generated by a *FP* process always lies for $t \geq T$ in the strategy profile set of a *MxN* sub game, where $M + N \leq 4$. Then the process approaches equilibrium[8].

6. Continuous Fictitious Play (CFP)

We find it convenient to work with the continuous time formulation of *fictitious play* rather than the discrete time formulation, in order to prove the existence of the *FP* property for all non-degenerate $2x3$ games. Although there are no results relating the discrete and the continuous processes, it seems that whenever the *continuous fictitious play* exist, the *fictitious play*'s path behaves similarly to the continuous one. Since our proof based on the four principles of motion relating to both processes, we assume that this result holds for the discrete case as well.

While the time space is continuous, a *path* x in S is a right continuous function $x:[0,\infty) \to S$ such that the set of discontinuity points of x does not have an accumulation point in $[0,\infty)$.

[8] We conjecture that this principle holds for every sub-game with the *FP* property. That is, for every *MxN* sub-game such that $M \geq 1$ and $N \geq 1$.

A *belief path* is a pair (b,t_1), where $t_1>0$, and $b:[t_1,\infty)\to\Delta$ is the belief function.

A *continuous learning process* is a pair $(x,(b,t_1))$, where x is a path in S, and (b,t_1) is a belief path, such that $x^i(t)$ is a best response to $b^i(t)$ for every $t\ge t_1$.

A continuous learning process $(x,(b,t_1))$ is a *continuous fictitious play (CFP)* process, if for every i, and for every $j\ne i$, $b^i_j(t)=\int_{s=0}^{t}x^j(s)ds$, for every $t\ge t_1$.

A *CFP* $(x,(b,t_1))$ process approaches equilibrium if for every $\varepsilon>0$ there exist $t_0>0$, such that for every $t\ge t_0$, there exist a mixed equilibrium profile $p\in\Delta^N$, such that, $\left|((b_1(t),b_2(t),\ldots,b_n(t))-p\right|<\varepsilon$.

We say that a game has the *CFP property* , if every *CFP* process, independent of initial actions and beliefs, approaches equilibrium.

Let $(x,(b,t_1))$ be a *CFP* process. Since x is right continuous and takes values in a finite set S, it must be a step function. Further, since x has only finite number of discontinuity points at any bounded interval, there exists an increasing sequence $\{T_k\}$, $0\le k\le\bar{k}$ of times ($2\le\bar{k}\le\infty$), where $0\le T_0<t_1=T_1$, that will be called the *reduced time sequence*. Likewise, there exists a sequence $\{z_k\}$, $0\le k\le\bar{k}$, in S that will be called the *reduced path,* such that :

(6.1) $z(k)\ne z(k-1)$, $2\le k\le\bar{k}$.

(6.2) $z(k)=x(t)$, $T_k\le t<T_{k+1}$, $0\le k\le\bar{k}$,

where $T_{\bar{k}+1}=\infty$ if $\bar{k}$ is finite.

It can be verified that the four principles of motion for the *FP* process hold for the *CFP* as well. Further, The improvement, stability, and separation principles can be shown by the same arguments as in the discrete case. For the reduction principle see the discussion in the following section.

7. 2x2 Games

The result of Miyasawa (1961) that every $2x2$ game has the *FP* property proved under a tie-breaking rule about the particular best reply at each period (when there are some best replies). Without tie-breaking rules, Monderer and Shapley (1996) showed that every $2x2$ game that satisfies the *diagonal property* [9] is best response equivalent in mixed strategies to either a game with identical payoff functions (A, A), or to a zero-sum game $(A,-A)$. Therefore every such game has the *FP* property. Hofbauer (1994) showed that every two person zero-sum game has the *CFP* property. Likewise, it can be verified that every game with identical payoff functions, and particularly every two person $2x2$ game has the *CFP* property[10]. Hence, every $2x2$ game with the diagonal property has the *CFP* property. If a game does not have the diagonal property, it implies that for one of the players has either dominated strategy, or identical strategies. The case of dominated strategies is easy to analyze. The harder case is where there are identical strategies. As was shown by Monderer and Sela (1996) there is such a $2x2$ game without the *FP* property. It can be shown that this game does not have also the *CFP* property. Hence, we discuss only about non-degenerate games. Every $2x2$ non-degenerate game is either a game with dominated strategies, or a game with the diagonal property, and therefore weobtain the following result :

Corollary 7: Every non-degenerate $2x2$ game has the *CFP* property.

8. 2x3 Games

We conjecture that corollary 7.1 holds for every $2xk$, $k > 2$ non-degenerate game. By the principles of motion we affirm this conjecture for the case $k = 3$:

Theorem 8.1: Every non-degenerate 2x3 game has the *CFP* property.

Proof : Let G_1 be a $2x3$ game as follows :

$$G_1 = (A, B) = \begin{bmatrix} (a_{11}, b_{11}) & (a_{12}, b_{12}) & (a_{13}, b_{13}) \\ (a_{21}, b_{21}) & (a_{22}, b_{22}) & (a_{23}, b_{23}) \end{bmatrix}$$

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[9] We say that a game $G = (A, B) = (a(i, j), b(i, j))_{i,j=1}^{2}$ has the diagonal property if $c \neq 0$ and $d \neq 0$ where,
$d = b(1,0) - b(2,1) - b(1,2) + b(2,2)$ and $c = a(1,1) - a(2,1) - a(1,2) + a(2,2)$.

[10] In the following we actually need only the existence of the reduction principle in the continuous case for zero-sum $2x2$ sub-game. This result is derived from Hofbauer (1994).

Without loss of generality we assume that $b_{11} > b_{12} > b_{13}$. We can assume that there are no strictly dominated strategies (non-degenerate game does not have weakly dominated strategies). Thus, $b_{23} > b_{22} > b_{21}$. We can apply utility transformation that do not change the best response structure of the game. In particular, we can multiply a payoff matrix by a positive constant. We can also add a constant to a column in A, and we can add a constant to a row in B. Applying these transformations to G_1 yields:

$$G_2 = \begin{bmatrix} (a,w) & (b,0) & (c,z) \\ (0,u) & (0,0) & (0,v) \end{bmatrix} \quad a \neq 0,\ b \neq 0,\ c \neq 0,\ z > 0,\ w < 0,\ v < 0,\ u > 0 .$$

Depending on the signs of a, b and c, there are eight different forms of G_2. But there are only four different better response structures (each class of better response structure includes two symmetric forms). The four classes are as follows :

Class 1 : $a > 0,\ b > 0,\ c > 0$ (the symmetric form is : $a < 0,\ b < 0,\ c < 0$).

Class 2 : $a < 0,\ b < 0,\ c > 0$ $(a < 0,\ b > 0,\ c > 0)$.

Class 3 : $a > 0,\ b < 0,\ c > 0$ $(a < 0,\ b > 0,\ c < 0)$.

Class 4 : $a > 0,\ b > 0,\ c < 0$ $(a > 0,\ b < 0,\ c < 0)$.

Classes 1-3

We use now the four principles in order to prove the *CFP* property in the three first cases.

Let $G_2 = \begin{matrix} (a,w) & (b,0) & (c,z) \\ (0,u) & (0,0) & (0,v) \end{matrix}$ $(a, b, c, u, v, w, z) > 0$.

Class 1: $a > 0$, $b > 0$, $c > 0$ [11].

In this class of games row 2 is strictly dominated, and therefore we have actually a $1x3$ game. It is obvious that every $1xn$, $n \geq 1$, has the *CFP* property. ■

Class 2: $a < 0$, $b > 0$, $c > 0$ [12].

By the improvement principle, the following diagram describes all possible moves along a reduced path, that can be generated by a CFP process.

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[11] The same argument holds for the case : $a < 0,\ b < 0,\ c < 0$.

[12] The same argument holds for the case : $a < 0,\ b > 0,\ c > 0$.

$$\begin{array}{ccc} (1,1) & \rightarrow (1,2) & \rightarrow (1,3) \\ \downarrow & \uparrow & \uparrow \\ (2,1) & \leftarrow (2,2) & \leftarrow (2,3) \end{array}$$

Note that simultaneously moves $(1,1) \leftrightarrow (2,2)$ and $(1,2) \leftrightarrow (2,3)$ are impossible in the *CFP* process, since the meaning of such moves according to the structure of the game, is that at least one of the points in $A = \{(1,1),(2,2),(1,2),(2,3)\}$ is played only once among successive plays of other point in A. And this is a contradiction to the increasing property of the reduced time sequence generated by the *CFP* process.

The separation principle implies that player 2 will not move between column 1 and column 3. So, we can omit all arrows between these columns. This leaves us with :

$$\begin{array}{ccc} (1,1) & \rightarrow (1,2) & \rightarrow (1,3) \\ \downarrow & \uparrow & \uparrow \\ (2,1) & \leftarrow (2,2) & \leftarrow (2,3) \end{array}$$

Note that all the paths lead to one of the pure equilibrium point, and by the stability principle, if the process lands on pure equilibrium point, it will stay there forever. So, we can conclude that every *CFP* process approaches equilibrium in case 2. ■

<u>Class 3</u>: $a > 0$, $b < 0$, $c > 0$[13].

By the improvement and the separation principle, the following diagram describes all possible moves along a reduced path generated by a *CFP* process.

$$\begin{array}{ccc} (1,1) & \rightarrow (1,2) & \rightarrow (1,3) \\ \uparrow & \downarrow & \uparrow \\ (2,1) & \leftarrow (2,2) & \leftarrow (2,3) \end{array}$$

By the stability principle, without loss of generality, we can consider only arrows that do not point towards the equilibrium point $(1,3)$. This give us the following diagram :

$$\begin{array}{ccc} (1,1) & \rightarrow (1,2) & (1,3) \\ \uparrow & \downarrow & \\ (2,1) & \leftarrow (2,2) & \leftarrow (2,3) \end{array}$$

So the only potential non-converging process must cycle between :

$(1,1) \rightarrow (1,2) \rightarrow (2,2) \rightarrow (2,1) \rightarrow (1,1)\ldots$

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[13] The same argument holds for the case : $a > 0,\ b < 0,\ c > 0$.

Apply the reduction principle to conclude that the process must approach equilibrium in this case.[14] ■

Class 4

The following game G is a typical game of class 4:

$$G = \begin{matrix} (a,-w) & (b,0) & (0,z) \\ (0,u) & (0,0) & (1,-1) \end{matrix} \qquad (a,b,u,w,z) > 0 .$$

Without loss of generality, assume that $w > uz$, since otherwise column 2 is almost always[15] strictly dominated, and by the continuation of the *CFP*, the column player will not use this column for ever, since otherwise we have a contradiction to the increasing of the reduced time sequence generated by the *CFP* process.

Let $(x,(b,t_1))$ be a CFP in G , and let $\{T_k\}$ and $\{z_k\}$, $0 \le k \le k^*$, be the time sequence and the reduced path associated with the process respectively. By the stability principle , we can assume without loss of generality , that these sequences are infinite. By the separation principle, for sufficient late stage, the column player will not switch from column 1 to column 3 and vice versa. By the improvement principle, the player may move from $z(k)$ to $z(k+1)$ only according to the following arrows :

$$\begin{matrix} (1,1) & \rightarrow (1,2) & \rightarrow (1,3) \\ \uparrow & \uparrow & \downarrow \\ (2,1) & \leftarrow (2,2) & \leftarrow (2,3) \end{matrix}$$

Since the simultaneously move $(2,2) \rightarrow (1,1)$ is impossible by the *CFP* process , the reduced path induces only two cycle forms:

$$\begin{matrix} (1,1) & \rightarrow (1,2) & \rightarrow (1,3) \\ \uparrow & & \downarrow \\ (2,1) & \leftarrow (2,2) & \leftarrow (2,3) \end{matrix} \qquad\qquad \begin{matrix} (1,1) & (1,2) & \rightarrow (1,3) \\ & \uparrow & \downarrow \\ (2,1) & (2,2) & \leftarrow (2,3) \end{matrix}$$

six point cycle — four point cycle

Since the process lands in the point (1,2) infinite times, round n, $n = 1,2,\ldots.$ is defined as the n^{th} time in which the process lands in the point (1,2). The initial point of each round is naturally the point (1,2).

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[14] Note that the $2x2$ sub-game induced by the process is best response equivalent to a $2x2$ zero-sum game.

[15] We say that a strategy x of player i is almost always strictly dominated, if the set $B(x)$ has measure 0.

We proceed to prove that a *CFP* process can not generate only *six point cycles*, and therefore it must generate also *four point cycles*. Later we will prove that if the process generates once the *four point cycle*, then it generates only this cycle for ever. Apply the reduction principle to conclude that the process approaches equilibrium in this case.

Denote by x_{ij}^{n} $i = 1,2$, $j = 1,2,3$, the amount of time spent playing in the point (i, j) up to (including) round n , $n = 1,2,3,\ldots$

Denote by y_{ij}^{n} $i = 1,2$, $j = 1,2,3$, the amount of time spent playing in the point (i, j) in round n , $n = 1,2,3,\ldots$

A *CFP* generating only *six point cycles*, can be described by the difference equations ((8.3)-(8.8)). Each equations describes the variables x_{ij}^{n} at times in which the *CFP* switches (given in parentheses) from point to point in the *six point cycle* (equations (8.1) and (8.2) refer to previous round). $n > 0$ indicates the number of the round, while x_{ij}^{m} $i = 1,2$, $j = 1,2,3$, $m \le 0$, assigns the initial amount of the point (i, j) before the first time generating this cycle.

The equations are as follows :

(8.1) $ax_{11}^{n-2} + ax_{21}^{n-1} + bx_{12}^{n-1} + bx_{22}^{n-1} = x_{13}^{n-1} + x_{23}^{n-1}$ $((2,1) \to (1,1))$.

(8.2) $wx_{11}^{n-1} + wx_{12}^{n-1} + wx_{13}^{n-1} = ux_{21}^{n-1} + ux_{22}^{n-1} + ux_{23}^{n-1}$ $((1,1) \to (1,2))$.

(8.3) $zx_{11}^{n-1} + zx_{12}^{n} + zx_{13}^{n-1} = x_{21}^{n-1} + x_{22}^{n-1} + x_{23}^{n-1}$ $((1,2) \to (1,3))$.

(8.4) $ax_{11}^{n-1} + ax_{21}^{n-1} + bx_{12}^{n} + bx_{22}^{n-1} = x_{13}^{n} + x_{23}^{n-1}$ $((1,3) \to (2,3))$.

(8.5) $zx_{11}^{n-1} + zx_{12}^{n} + zx_{13}^{n} = x_{21}^{n-1} + x_{22}^{n-1} + x_{23}^{n}$ $((2,3) \to (2,2))$.

(8.6) $wx_{11}^{n-1} + wx_{12}^{n} + wx_{13}^{n} = ux_{21}^{n-1} + ux_{22}^{n} + ux_{23}^{n}$ $((2,2) \to (2,1))$.

(8.7) $ax_{11}^{n-1} + ax_{21}^{n} + bx_{12}^{n} + bx_{22}^{n} = x_{13}^{n} + x_{23}^{n}$ $((2,1) \to (1,1))$.

(8.8) $wx_{11}^{n} + wx_{12}^{n} + wx_{13}^{n} = ux_{21}^{n} + ux_{22}^{n} + ux_{23}^{n}$ $((1,1) \to (1,2))$.

We will show that these equations yield a contradiction. By subtraction pairs of equations (given in parentheses) we obtain :

(8.9) $zy_{13}^{n} = y_{23}^{n}$ $((8.5) - (8.3))$.

(8.10) $wy_{12}^{n} + wy_{13}^{n} = wy_{22}^{n} + uy_{23}^{n}$ $((8.6) - (8.2))$.

(8.11) $wy_{11}^{n} = uy_{21}^{n}$ $((8.8) - (8.6))$.

(8.12) $ay_{21}^{n} + by_{22}^{n} = y_{23}^{n}((8.7)-(8.4))$.

(8.13) $ay_{11}^{n-1} + by_{12}^{n} = y_{13}^{n}((8.4)-(8.1))$.

Using equations (8.9)-(8.13) give us :

(8.14) $y_{11}^{n} = y_{11}^{n-1}(\frac{zu}{w} - b(1-\frac{zu}{w})) - y_{12}^{n}(\frac{b}{aw}(w-uz)(1+b))$.

This yields : $y_{11}^{n} < ry_{11}^{n-1}$ where $0 < r = zu / w < 1$. Thus, $\lim_{n\to\infty} y_{11}^{n} = 0$.

On the other hand, since $(w-uz)/((1+z)(w+u))$ is the length of the set $B($ column $i)$, we obtain :

(8.15) $y_{12}^{n} > (w-uz)/((1+z)(w+u)) > 0 \quad \forall \mathrm{n} > 1$.

But equation (8.15) contradicts, by equation (8.14), the convergence of y_{11}^{n} to zero when n approaches infinity.

We showed that there is no a stage, such that a *CFP* process generates only *six point cycles* from this stage on. Thus, every *CFP* process generates *four point cycles* infinitely often. Now we will show that if a *CFP* generates the *four point cycle* at some stage, then it will generate only *four point cycles* from this stage on.

A *CFP* process generating the four point cycle for some $n > 0$, can be described by the following equations ((8.16)-(8.19)) :

(8.16) $zx_{11}^{n-1} + zx_{12}^{n} + zx_{13}^{n-1} = x_{21}^{n-1} + x_{22}^{n-1} + x_{23}^{n-1}$ $((1,2) \to (1,3))$.

(8.17) $zx_{11}^{n-1} + zx_{12}^{n} + zx_{13}^{n-1} = x_{21}^{n-1} + x_{22}^{n-1} + x_{23}^{n-1}$ $((1,2) \to (1,3))$.

(8.18) $zx_{11}^{n-1} + zx_{12}^{n} + zx_{13}^{n} = x_{21}^{n-1} + x_{22}^{n-1} + x_{23}^{n}$ $((2,3) \to (2,2))$.

(8.19) $ax_{11}^{n-1} + ax_{21}^{n-1} + bx_{12}^{n} + bx_{22}^{n} = x_{13}^{n} + x_{23}^{n}$ $((2,2) \to (1,2))$.

The subtraction (8.19)-(8.17) yields :

(8.20) $by_{22}^{n} = y_{23}^{n}$.

By the improvement principle, the unique escape route from the *four point cycle*, could be by moving from the point (2,2) to the point (2,1). We will show that if the process lands on (2,2) after one round through the *four point cycle*, then the next move will be necessarily to (1,2) and not to (2,1). That is, the process will induce only four *point cycles* for ever.

Suppose in negation that the process induces the *four point cycle* in round $n-1$, $n > 1$, and immediately after that the process moves from (2,2) to (2,1), that is, leaves the four point cycle. Then, the next move according to the improvement principle is from (2,1) to (1,1). The route $(2,2) \to (2,1) \to (1,1)$ implies :

(8.21) $ay_{21}^{n} + b\bar{y}_{22}^{n} = y_{23}^{n}$.

Where $\bar{y}_{22}^{n}$ is the time which is needed to move from (2,2) to (2,1), and y_{22}^{n} is the time which is needed to move from (2,2) to (1,2). Because of the geometrical structure of the sets $B(\text{column } i)$ $i = 1,2,3$, we obtain that,

(8.22) $\bar{y}_{22}^{n} > y_{22}^{n}$.

Equations (8.20) , (8.21), and (8.22) yield that $y_{21}^{n} = 0$. But this is a contradiction to our assumption about the escape of the *four point cycle*, that is, the move from (2,2) to (2,1). ■

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