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The replicator dynamics does not lead to correlated equilibria

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Abstract

It is shown that, under the replicator dynamics, all strategies played in correlated equilibrium may be eliminated, so that only strategies with zero marginal probability in all correlated equilibria survive. This occurs in particular in a family of 4×4 games built by adding a strategy to a Rock-Paper-Scissors game.

JEL classification: C73

Keywords: Replicator dynamics; Correlated equilibrium; Rock-Paper-Scissors

1. Introduction

For a fairly wide class of evolutionary dynamics, if all strategies are initially played with positive probability and if the solution of the dynamics converges to a point, then this point is a Nash equilibrium (Weibull, 1995). Weak dynamic stability (Lyapunov stability) also implies Nash equilibrium behavior, and solutions of prominent evolutionary dynamics have been shown to converge to the set of Nash equilibria in important classes of games (e.g. potential games, zero-sum games, supermodular games). However, in general, solutions of evolutionary dynamics need not converge to the set of Nash equilibria (Hofbauer and Sigmund, 1998, Section 8.6).

Whether there is nonetheless a general connection between the outcome of evolutionary dynamics and equilibrium concepts is not clear. First, even though the solution of a dynamics does not converge to the set of Nash equilibria, its time-average might. For instance, provided that no

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pure strategy comes close to being eliminated,[1] the time-average of the single or two-population replicator dynamics converges to the set of Nash equilibria (Hofbauer and Sigmund, 1998).

Second, a number of recent articles, surveyed by Hart (2005), show that there exist adaptive processes converging, in a time-average sense, towards the set of correlated equilibria. Though these processes are different from standard evolutionary dynamics, this suggests that the outcome of evolutionary dynamics might be more strongly connected to correlated equilibrium than to Nash equilibrium.

Third, convergence is not all that matters. Of great importance to our understanding of an evolutionary process are the strategies that survive and those that are eliminated under this process. Even if evolutionary dynamics do not always converge to the set of correlated equilibria, there might still be a link between strategies that survive and strategies that belong to the support of correlated equilibria.

This note shows, however, that this is not the case, at least for the replicator dynamics. Specifically, we present a family of 4×4 symmetric games for which, under the replicator dynamics and from a large set of initial conditions, all strategies used in correlated equilibrium are eliminated (hence only strategies that are *not* used in equilibrium remain). In particular, no kind of time-average of the replicator dynamics can converge to the set of correlated equilibria. The same results actually hold for an open set of games and for wide classes of dynamics (Viossat, 2005).

The remainder of this note is organized as follows. First, we introduce the notations and basic definitions, and recall some known results on Rock-Paper-Scissors (RPS) games. In addition, we prove that these games have a unique correlated equilibrium. We then introduce a family of 4×4 symmetric games built by adding a strategy to a RPS game. We describe in details the orbits of the replicator dynamics in these games and show that, from an open set of initial conditions, all strategies used in correlated equilibrium are eliminated. We conclude by discussing a variety of related results.

2. Notations and basic definitions

We consider finite, two-player symmetric games played within a single population. Such a game is given by a set $I = \{1, \ldots, N\}$ of pure strategies and a payoff matrix $\mathbf{U} = (u_{ij})_{1 \leqslant i,j \leqslant N}$. Here u_{ij} is the payoff of a player playing strategy i against a player playing strategy j. We use bold characters for vectors and matrices and normal characters for numbers.

The proportion of the population playing strategy i at time t is denoted by $x_i(t)$. Thus, the vector $\mathbf{x}(t) = (x_1(t), \ldots, x_N(t))^T$ denotes the population profile (or mean strategy) at time t. It belongs to the $N-1$ dimensional simplex over I

$$S_N := \left\{ \mathbf{x} \in \mathbb{R}_+^I \colon \sum_{i \in I} x_i = 1 \right\}$$

(henceforth, "the simplex") whose vertices $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_N$ correspond to the pure strategies of the game. We study the evolution of the population profile $\mathbf{x}$ under the single-population replicator dynamics (Taylor and Jonker, 1978):

$$\dot{x}_i(t) = x_i(t)\left[\left(\mathbf{U}\mathbf{x}(t)\right)_i - \mathbf{x}(t) \cdot \mathbf{U}\mathbf{x}(t)\right]. \tag{1}$$

[1] That is, provided that there exists $\epsilon > 0$ such that, at each time $t \geqslant 0$, the frequency of each pure strategy is greater than ϵ.

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Remark. For notational simplicity, we often write x_i and $\mathbf{x}$ instead of $x_i(t)$ and $\mathbf{x}(t)$.

We now define correlated equilibrium distributions. Consider a (not necessarily symmetric) bimatrix game with strategy set I (respectively J) for player 1 (respectively 2). Let $g_k(i,j)$ denote the payoff of player k when player 1 plays i and player 2 plays j. A correlated equilibrium distribution (Aumann, 1974) is a probability distribution μ on the set $I \times J$ of pure strategy profiles which satisfies the following inequalities:

$$\sum_{j \in J} \mu(i,j)\big[g_1(i,j) - g_1(i',j)\big] \geqslant 0 \quad \forall i \in I,\ \forall i' \in I \tag{2}$$

and

$$\sum_{i \in I} \mu(i,j)\big[g_2(i,j) - g_2(i,j')\big] \geqslant 0 \quad \forall j \in J,\ \forall j' \in J. \tag{3}$$

With some abuse of terminology, we may write "correlated equilibrium" for "correlated equilibrium distribution." Though the above definition applies to general bimatrix games, from now on, we only consider *symmetric* bimatrix games.

Definition. The pure strategy i is *used in correlated equilibrium* if there exists a correlated equilibrium μ and a pure strategy j such that $\mu(i,j) > 0$.[2]

Definition. The pure strategy i is *eliminated* (for some initial condition $\mathbf{x}(0)$) if $x_i(t)$ goes to zero as $t \to +\infty$.

3. A reminder on Rock-Paper-Scissors games

A RPS (Rock-Paper-Scissors) game is a 3×3 symmetric game in which the second strategy (Paper) beats the first (Rock), the third (Scissors) beats the second, and the first beats the third. Up to normalization (i.e. putting zeros on the diagonal) the payoff matrix is of the form:

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 0 & -a_2 & b_3 \\ b_1 & 0 & -a_3 \\ -a_1 & b_2 & 0 \end{pmatrix} \end{array} \quad \text{with } a_i > 0,\ b_i > 0 \text{ for all } i = 1,2,3. \tag{4}$$

Any RPS game has a unique Nash equilibrium: $(\mathbf{p}, \mathbf{p})$ where

$$\mathbf{p} = \frac{1}{\Sigma}(a_2a_3 + a_3b_2 + b_2b_3,\ a_1a_3 + a_1b_3 + b_3b_1,\ a_1a_2 + a_2b_1 + b_1b_2) \tag{5}$$

with $\Sigma > 0$ such that $\mathbf{p} \in S_4$ (see Zeeman, 1980; Gaunersdorfer and Hofbauer, 1995, or Hofbauer and Sigmund, 1998). Actually, as essentially noted by Martin Cripps (1991, Example 2, p. 433)[3]:

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[2] Note that if μ is a correlated equilibrium of a two-player symmetric game, then so is μ^T (defined by $\mu^T(i,j) = \mu(j,i)$), hence so is $(\mu + \mu^T)/2$. Thus, if a strategy is used in a correlated equilibrium, it is also used in a symmetric correlated equilibrium.

[3] Cripps (1991) mentions that, in a subclass of the class of RPS games (4), all games have a unique correlated equilibrium.

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Proposition 1. *Any RPS game has a unique correlated equilibrium*: $\mathbf{p} \otimes \mathbf{p}$.

(For $\mathbf{x} \in S_N$, $\mathbf{x} \otimes \mathbf{x}$ denotes the probability distribution on S_N induced by $\mathbf{x}$.)

Proof. Let μ be a correlated equilibrium of (4). For $i = 1$ and, respectively, $i' = 2$ and $i' = 3$, the incentive constraint (2) reads:

$$\mu(1,1)(-b_1) + \mu(1,2)(-a_2) + \mu(1,3)(a_3 + b_3) \geqslant 0, \tag{6}$$

$$\mu(1,1)a_1 + \mu(1,2)(-a_2 - b_2) + \mu(1,3)b_3 \geqslant 0. \tag{7}$$

Add (6) multiplied by a_1 to (7) multiplied by b_1. This gives

$$-\mu(1,2)(a_1a_2 + a_2b_1 + b_1b_2) + \mu(1,3)(a_1a_3 + a_1b_3 + b_3b_1) \geqslant 0.$$

That is, recalling (5):

$$p_2\mu(1,3) \geqslant p_3\mu(1,2).$$

Every choice of a player and a strategy i yields a similar inequality. So we get six inequalities which together read:

$$p_2\mu(1,3) \geqslant p_3\mu(1,2) \geqslant p_1\mu(3,2) \geqslant p_2\mu(3,1) \geqslant p_3\mu(2,1) \geqslant p_1\mu(2,3) \geqslant p_2\mu(1,3).$$

Therefore all the above inequalities hold as equalities. Letting λ be such that the common value of the above expressions is $\lambda p_1 p_2 p_3$, we have: $\mu(i,j) = \lambda p_i p_j$ for every $j \neq i$. Together with (6) and (7), this implies that we also have $\mu(1,1) = \lambda p_1^2$ (and similarly $\mu(i,i) = \lambda p_i^2$ for all i). Therefore $\lambda = 1$ and $\mu = \mathbf{p} \otimes \mathbf{p}$. □

The behavior of the replicator dynamics in RPS games has been totally analyzed by Zeeman (1980). In particular, letting $\partial S_3 := \{\mathbf{x} \in S_3\colon\ x_1x_2x_3 = 0\}$ denote the boundary of the simplex:

Proposition 2. (Zeeman, 1980) *If $a_1a_2a_3 > b_1b_2b_3$, then for every initial condition $\mathbf{x}(0) \neq \mathbf{p}$, the solution $\mathbf{x}(t)$ converges to the boundary of the simplex ∂S_3 as $t \to +\infty$.*

In the case of *cyclic symmetry* (i.e. $a_1 = a_2 = a_3$ and $b_1 = b_2 = b_3$) then the unique Nash equilibrium is $\mathbf{p} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. Furthermore, up to division of all payoffs by the common value of the a_i, the payoff matrix may be taken of the form:

$$\hat{\mathbf{U}} = \begin{pmatrix} 0 & -1 & \epsilon \\ \epsilon & 0 & -1 \\ -1 & \epsilon & 0 \end{pmatrix} \quad \text{with } \epsilon > 0. \tag{8}$$

The condition $a_1a_2a_3 > b_1b_2b_3$ then reduces to $\epsilon < 1$ and in this case Proposition 2 may be proved as follows: for $\epsilon < 1$, the Nash equilibrium $\mathbf{p}$ is globally inferior in the sense that:

$$\forall \mathbf{x} \in S_3,\ \mathbf{x} \neq \mathbf{p} \ \Rightarrow\ \mathbf{p} \cdot \hat{\mathbf{U}}\mathbf{x} < \mathbf{x} \cdot \hat{\mathbf{U}}\mathbf{x}.$$

More precisely,

$$\mathbf{p} \cdot \hat{\mathbf{U}}\mathbf{x} - \mathbf{x} \cdot \hat{\mathbf{U}}\mathbf{x} = -(\mathbf{p} - \mathbf{x}) \cdot \hat{\mathbf{U}}(\mathbf{p} - \mathbf{x}) = -\left(\frac{1-\epsilon}{2}\right) \sum_{1 \leqslant i \leqslant 3} (p_i - x_i)^2 \tag{9}$$

where the first equality follows from the fact that $(\mathbf{p}, \mathbf{p})$ is a completely mixed equilibrium, hence $(\hat{\mathbf{U}}\mathbf{p})_i - \mathbf{p} \cdot \hat{\mathbf{U}}\mathbf{p} = 0$ for all i. Now, let $\hat{V}(\mathbf{x}) := (x_1x_2x_3)^{1/3}$. Note that the function $\hat{V}$ takes its

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minimal value 0 on the boundary of the simplex ∂S_3 and its maximal value $1/3$ at $\mathbf{p}$. Letting $\hat{v}(t) := \hat{V}(\mathbf{x}(t))$ we get:

$$\dot{\hat{v}}(t) = (\mathbf{p}\cdot\hat{\mathbf{U}}\mathbf{x} - \mathbf{x}\cdot\hat{\mathbf{U}}\mathbf{x})\hat{v}(t) = -\hat{v}(t)\left(\frac{1-\epsilon}{2}\right)\sum_{1\leqslant i\leqslant 3}(p_i - x_i)^2. \tag{10}$$

The above expression is negative whenever $\hat{v}(t) \neq 0$ and $\mathbf{x} \neq \mathbf{p}$. It follows that for every initial condition $\mathbf{x}(0) \neq \mathbf{p}$, $\hat{v}(t)$ decreases to zero hence $\mathbf{x}(t)$ converges to the boundary.

4. A family of 4 × 4 games

Fix ϵ in $]0, 1[$, $\alpha \geqslant 0$, and consider the following 4×4 symmetric game which is built by adding a strategy to a RPS game:

$$\mathbf{U}_\alpha = \left(\begin{array}{ccc|c} 0 & -1 & \epsilon & -\alpha \\ \epsilon & 0 & -1 & -\alpha \\ -1 & \epsilon & 0 & -\alpha \\ \hline \frac{-1+\epsilon}{3} + \alpha & \frac{-1+\epsilon}{3} + \alpha & \frac{-1+\epsilon}{3} + \alpha & 0 \end{array}\right). \tag{11}$$

For $0 < \alpha < (1-\epsilon)/3$, the interesting case, this game is very similar to the example used by Dekel and Scotchmer (1992) to show that a discrete-time version of the replicator dynamics need not eliminate all strictly dominated strategies.[4] We now describe the main features of the above game.

Let $\mathbf{n}_{123} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0)$ denote the rest-point of the replicator dynamics corresponding to the Nash equilibrium of the underlying RPS game.

The case $\alpha = 0$. The strategies $\mathbf{n}_{123}$ and $\mathbf{e}_4$ always earn the same payoff:

$$\mathbf{n}_{123}\cdot\mathbf{U}_0\mathbf{x} = \mathbf{e}_4\cdot\mathbf{U}_0\mathbf{x} \quad \forall\mathbf{x}\in S_4. \tag{12}$$

Furthermore, all strategies earn the same payoff against strategy 4:

$$\mathbf{x}\cdot\mathbf{U}_0\mathbf{e}_4 = \mathbf{x}'\cdot\mathbf{U}_0\mathbf{e}_4 \quad \forall\mathbf{x}\in S_4,\ \forall\mathbf{x}'\in S_4. \tag{13}$$

The set of symmetric Nash equilibria is the segment $E_0 = [\mathbf{n}_{123}, \mathbf{e}_4]$, i.e. the set of convex combinations of $\mathbf{n}_{123}$ and $\mathbf{e}_4$. This shall be clear from the proof of Proposition 3 below. A key property is that whenever the population profile $\mathbf{x}$ does not belong to the segment of equilibria E_0, every strategy in E_0 earns less than the mean payoff. Formally,

$$\forall\mathbf{x}\notin E_0, \forall\mathbf{p}\in E_0,\ \mathbf{p}\cdot\mathbf{U}_0\mathbf{x} < \mathbf{x}\cdot\mathbf{U}_0\mathbf{x}.$$

More precisely, for $\mathbf{x} \neq \mathbf{e}_4$, define $\hat{x}_i$ as the share of the population that plays i relative to the share of the population that plays 1, 2 or 3. Formally,

$$\hat{x}_i = x_i/(x_1 + x_2 + x_3). \tag{14}$$

[4] More precisely, the game obtained from (11) by multiplying all payoffs by −1 belongs to the family of games *à la* Dekel and Scotchmer considered by Hofbauer and Weibull (1996). In particular, Fig. 1 of Hofbauer and Weibull (1996, p. 570) describes the dynamics on the boundary of the simplex in game (11), up to reversal of all arrows and permutation of strategies 2 and 3.

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Lemma 4.1. *For every* $\mathbf{p}$ *in* E_0 *and every* $\mathbf{x} \neq \mathbf{e}_4$,

$$\mathbf{p} \cdot \mathbf{U}_0\mathbf{x} - \mathbf{x} \cdot \mathbf{U}_0\mathbf{x} = -\frac{(1-\epsilon)}{2}(1-x_4)^2 \sum_{1 \leqslant i \leqslant 3} (\hat{x}_i - 1/3)^2. \tag{15}$$

Proof. Let $K = \mathbf{p} \cdot \mathbf{U}_0\mathbf{x} - \mathbf{x} \cdot \mathbf{U}_0\mathbf{x} = (\mathbf{p} - \mathbf{x}) \cdot \mathbf{U}_0\mathbf{x}$. It follows from (12) that $\mathbf{p} \cdot \mathbf{U}_0\mathbf{x} = (p_4\mathbf{e}_4 + (1-p_4)\mathbf{n}_{123}) \cdot \mathbf{U}_0\mathbf{x} = \mathbf{n}_{123} \cdot \mathbf{U}_0\mathbf{x}$ so that $K = (\mathbf{n}_{123} - \mathbf{x}) \cdot \mathbf{U}_0\mathbf{x}$. Now let $\mathbf{y} = (\hat{x}_1, \hat{x}_2, \hat{x}_3, 0)$. Using (13) we get:

$$K = (\mathbf{n}_{123} - \mathbf{x}) \cdot \mathbf{U}_0\big[(1-x_4)\mathbf{y} + x_4\mathbf{e}_4\big] = (1-x_4)(\mathbf{n}_{123} - \mathbf{x}) \cdot \mathbf{U}_0\mathbf{y}.$$

Noting that $\mathbf{n}_{123} - \mathbf{x} = (1-x_4)(\mathbf{n}_{123} - \mathbf{y}) + x_4(\mathbf{n}_{123} - \mathbf{e}_4)$ and using (12), we get: $K = (1-x_4)^2(\mathbf{n}_{123} - \mathbf{y}) \cdot \mathbf{U}_0\mathbf{y}$. Now apply (9). This gives (15) and concludes the proof. □

The case $\alpha > 0$. The mixed strategy $\mathbf{n}_{123}$ is no longer an equilibrium. Actually:

Proposition 3. *If* $\alpha > 0$*, then the game with payoffs* (11) *has a unique correlated equilibrium*: $\mathbf{e}_4 \otimes \mathbf{e}_4$.

Proof. Assume, by contradiction, that there exists a correlated equilibrium μ different from $\mathbf{e}_4 \otimes \mathbf{e}_4$. Since $\mathbf{e}_4$ is a strict Nash equilibrium, there exists $1 \leqslant i, j \leqslant 3$ such that $\mu(i,j) > 0$. Define the correlated distribution of the underlying RPS game $\hat{G}$ by:

$$\hat{\mu}(i,j) = \frac{\mu(i,j)}{K} \quad 1 \leqslant i, j \leqslant 3$$

with $K = \sum_{1 \leqslant i,j \leqslant 3} \mu(i,j) > 0$. For $1 \leqslant i, i' \leqslant 3$, we have $u_{i4} = u_{i'4} (= -\alpha)$, so that:

$$\sum_{j=1}^{3} \hat{\mu}(i,j)[u_{ij} - u_{i'j}] = \sum_{j=1}^{3} \frac{\mu(i,j)}{K}[u_{ij} - u_{i'j}] = \frac{1}{K} \sum_{j=1}^{4} \mu(i,j)[u_{ij} - u_{i'j}] \geqslant 0.$$

(The latter inequality holds because μ is a correlated equilibrium.)

Together with symmetric inequalities, this implies that $\hat{\mu}$ is a correlated equilibrium of $\hat{G}$. By Proposition 1, this implies that for every $1 \leqslant i, j \leqslant 3$, we have $\hat{\mu}(i,j) = 1/9$ hence $\mu(i,j) = K/9$. From this and the fact that strategy 4 is a best-response to itself, it follows that for any $1 \leqslant i, j \leqslant 3$,

$$\sum_{1 \leqslant j \leqslant 4} \mu(i,j)[u_{ij} - u_{4j}] \leqslant \sum_{1 \leqslant j \leqslant 3} \mu(i,j)[u_{ij} - u_{4j}] = -\frac{K\alpha}{3} < 0.$$

This contradicts the fact that μ is a correlated equilibrium. □

Nevertheless, for $\alpha < (1+2\epsilon)/3$, the above game has a best-response cycle: $\mathbf{e}_1 \to \mathbf{e}_2 \to \mathbf{e}_3 \to \mathbf{e}_1$. We will show that for $\alpha > 0$ small enough, the corresponding set

$$\Gamma := \{\mathbf{x} \in S_4, x_4 = 0 \text{ and } x_1x_2x_3 = 0\} \tag{16}$$

attracts all nearby orbits. We first show that the (replicator) dynamics in the interior of S_4 may be decomposed in two parts: an increase or decrease in x_4, and an outward spiraling movement around the segment $E_0 = [\mathbf{n}_{123}, \mathbf{e}_4]$.

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5. Decomposition of the dynamics

First, note that for every $\mathbf{x}$ in E_0, we have: $(\mathbf{U}_\alpha \mathbf{x})_1 = (\mathbf{U}_\alpha \mathbf{x})_2 = (\mathbf{U}_\alpha \mathbf{x})_3$. This implies that the segment E_0 is globally invariant.[5] Second, recall the definition (14) of $\hat{x}_i$. For $\mathbf{x} \neq \mathbf{e}_4$, let $\hat{\mathbf{x}} = (\hat{x}_1, \hat{x}_2, \hat{x}_3)$. Recall that $\hat{\mathbf{U}}$ denotes the payoff matrix (8) of the underlying RPS game.

Lemma 5.1. *Let* $\mathbf{x}(\cdot)$ *be a solution of the replicator dynamics* (1) *with* $\mathbf{x}(0) \neq \mathbf{e}_4$. *For every i in* $\{1, 2, 3\}$,

$$\dot{\hat{x}}_i = (1 - x_4)\hat{x}_i\big[\big(\hat{\mathbf{U}}\hat{\mathbf{x}}\big)_i - \hat{\mathbf{x}} \cdot \hat{\mathbf{U}}\hat{\mathbf{x}}\big]. \tag{17}$$

Proof. Let i in $\{1, 2, 3\}$. If $x_i = 0$, then (17) holds trivially. Otherwise, for every j in $\{1, 2, 3\}$ such that x_j is positive,

$$\frac{\dot{\hat{x}}_i}{\hat{x}_i} - \frac{\dot{\hat{x}}_j}{\hat{x}_j} = \frac{\mathrm{d}}{\mathrm{d}t} \ln\left(\frac{\hat{x}_i}{\hat{x}_j}\right) = \frac{\mathrm{d}}{\mathrm{d}t} \ln\left(\frac{x_i}{x_j}\right) = (\mathbf{U}_\alpha \mathbf{x})_i - (\mathbf{U}_\alpha \mathbf{x})_j = (1 - x_4)\big[\big(\hat{\mathbf{U}}\hat{\mathbf{x}}\big)_i - \big(\hat{\mathbf{U}}\hat{\mathbf{x}}\big)_j\big].$$

Multiplying the above equality by $\hat{x}_j$ and summing over all j such that $x_j > 0$ yields (17). □

The lemma means that, up to a change of velocity, $\hat{\mathbf{x}}$ follows the replicator dynamics for the game with payoff matrix $\hat{\mathbf{U}}$ (and thus spirals towards the boundary).[6] Now, recall that for $\mathbf{y} \in S_3$, $\hat{V}(\mathbf{y}) = (y_1 y_2 y_3)^{1/3}$. For $\mathbf{x} \in S_4 \backslash \{\mathbf{e}_4\}$, let $V(\mathbf{x}) := \hat{V}(\hat{\mathbf{x}})$. That is,

$$V(\mathbf{x}) = (\hat{x}_1 \hat{x}_2 \hat{x}_3)^{1/3} = \frac{(x_1 x_2 x_3)^{1/3}}{x_1 + x_2 + x_3}.$$

Corollary 5.2. *Let* $\mathbf{x}(\cdot)$ *be a solution of* (1) *with* $\mathbf{x}(0) \neq \mathbf{e}_4$. *The function* $v(t) := V(\mathbf{x}(t))$ *satisfies*:

$$\dot{v}(t) = -v(t) f\big(\mathbf{x}(t)\big) \quad \textit{with} \quad f(\mathbf{x}) = (1 - x_4)\left(\frac{1 - \epsilon}{2}\right) \sum_{1 \leqslant i \leqslant 3} (\hat{x}_i - 1/3)^2. \tag{18}$$

Proof. By definition of the function v, we have $v(t) = V(\mathbf{x}(t)) = \hat{V}(\hat{\mathbf{x}}(t))$. Let $\tau \in \mathbb{R}$ and let $\mathbf{y}(\cdot)$ be a solution of (1) in the RPS game (8), with initial condition $\mathbf{y}(0) = \hat{\mathbf{x}}(\tau)$. Let $\hat{v}(t) = \hat{V}(\mathbf{y}(t))$. It follows from Lemma 5.1 that

$$\dot{\hat{\mathbf{x}}}(\tau) = \big[1 - x_4(\tau)\big]\dot{\mathbf{y}}(0)$$

hence

$$\dot{v}(\tau) = \frac{\mathrm{d}\hat{V}(\hat{\mathbf{x}}(t))}{\mathrm{d}t}\bigg|_{t=\tau} = \big[1 - x_4(\tau)\big]\frac{\mathrm{d}\hat{V}(\mathbf{y}(t))}{\mathrm{d}t}\bigg|_{t=0} = \big[1 - x_4(\tau)\big]\dot{\hat{v}}(0),$$

[5] That is, denoting by $\phi(\mathbf{x}, t)$ the solution at time t of the replicator dynamics with initial condition $\mathbf{x}$ and letting $\phi(E_0, t) = \{\phi(\mathbf{x}, t), \mathbf{x} \in E_0\}$, we have: $\forall t \in \mathbb{R}, \phi(E_0, t) = E_0$.

[6] The fact that when the $N - 1$ first strategies earn the same payoff against the Nth (and last) strategy, the dynamics may be decomposed as in Lemma 5.1 was known to Josef Hofbauer (personal communication). This results from a combination of Theorem 7.5.1 and of Exercise 7.5.2 in (Hofbauer and Sigmund, 1998). I rediscovered it independently.

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but by (10),

$$\dot{\hat{v}}(0) = -\hat{v}(0)\left(\frac{1-\epsilon}{2}\right)\sum_{1\leqslant i\leqslant 3}\big(y_i(0)-1/3\big)^2.$$

Substituting $v(\tau)$ for $\hat{v}(0)$ and $\hat{x}_i(\tau)$ for $y_i(0)$ yields the result. □

Note that $v(t)$ is nonnegative and that the function f is positive everywhere but on the interval $[\mathbf{n}_{123}, \mathbf{e}_4[$, where V attains its maximal value $1/3$. Therefore, it follows from (18) that V decreases along all interior trajectories, except the ones starting (hence remaining) in the interval $]\mathbf{n}_{123}, \mathbf{e}_4[$. We now exploit this fact to build a Lyapunov function for the set Γ defined in (16).[7]

6. Main result

Let $W(\mathbf{x}) = \max(x_4, 3V(\mathbf{x}))$ for $\mathbf{x} \neq \mathbf{e}_4$ and $W(\mathbf{e}_4) = 1$, so that W is continuous. Note that W takes its maximal value 1 on the segment $E_0 = [\mathbf{n}_{123}, \mathbf{e}_4]$ and its minimal value 0 on Γ. The former follows from the fact that the function V, defined on $S_4\backslash\{\mathbf{e}_4\}$, takes its maximal value $1/3$ on $E_0\backslash\{\mathbf{e}_4\}$. For $\delta \geqslant 0$, let K_δ denote the compact set:

$$K_\delta := \big\{\mathbf{x} \in S_4,\ W(\mathbf{x}) \leqslant \delta\big\}$$

so that $K_0 = \Gamma$ and $K_1 = S_4$.

Proposition 4. *Let $0 < \delta < 1$. There exists $\gamma > 0$ such that for every game (11) with $0 < \alpha < \gamma$ and for every initial condition $\mathbf{x}(0)$ in K_δ,*

$$W\big(\mathbf{x}(t)\big) \leqslant W\big(\mathbf{x}(0)\big)\exp(-\gamma t) \quad \forall t \geqslant 0.$$

In particular, the set Γ attracts all solutions starting in K_δ.

Proof. Fix ϵ in $]0, 1[$ and recall that $\mathbf{U}_\alpha$ denotes the payoff matrix (11) with parameters ϵ, α. Since $E_0 = \{\mathbf{x}: W(\mathbf{x}) = 1\}$, $K_\delta = \{\mathbf{x}: W(\mathbf{x}) \leqslant \delta\}$, and $\delta < 1$, it follows that K_δ is disjoint from E_0. By (15) applied to $\mathbf{p} = \mathbf{e}_4$, this implies that for every $\mathbf{x}$ in K_δ, the quantity $(\mathbf{U}_0\mathbf{x})_4 - \mathbf{x}\cdot\mathbf{U}_0\mathbf{x}$ is negative. Similarly, it follows from the definition of the function f in (18) that for every $\mathbf{x}$ in K_δ, $f(\mathbf{x})$ is positive. Therefore, by compactness of K_δ, there exists a positive constant γ such that

$$\max_{\mathbf{x}\in K_\delta}\big((\mathbf{U}_0\mathbf{x})_4 - \mathbf{x}\cdot\mathbf{U}_0\mathbf{x}, -f(\mathbf{x})\big) \leqslant -3\gamma < 0. \tag{19}$$

We now fix α in $]0, \gamma[$ and consider the replicator dynamics in the game with payoff matrix $\mathbf{U}_\alpha$. For every $\mathbf{x}$ in S_4 and every i in I, $|[(\mathbf{U}_\alpha - \mathbf{U}_0)\mathbf{x}]_i| \leqslant \alpha$. Therefore, it follows from (19) that

$$\forall \mathbf{x} \in K_\delta,\ (\mathbf{U}_\alpha\mathbf{x})_4 - \mathbf{x}\cdot\mathbf{U}_\alpha\mathbf{x} \leqslant -3\gamma + 2\alpha \leqslant -\gamma.$$

Since $(\mathbf{U}_\alpha\mathbf{x})_4 - \mathbf{x}\cdot\mathbf{U}_\alpha\mathbf{x}$ is the growth rate of strategy 4, this implies that

$$\mathbf{x}(t) \in K_\delta \quad\Rightarrow\quad \dot{x}_4(t) \leqslant -\gamma x_4(t). \tag{20}$$

Now, recall the definition of $v(t)$ in Corollary 5.2. It follows from (18) that $\dot{v}(t) = -v(t)f(\mathbf{x}(t))$ and from (19) that if $\mathbf{x}(t) \in K_\delta$ then $-f(\mathbf{x}(t)) \leqslant -3\gamma$. Therefore,

$$\mathbf{x}(t) \in K_\delta \quad\Rightarrow\quad \dot{v}(t) \leqslant -3\gamma v(t) \leqslant -\gamma v(t). \tag{21}$$

[7] For an introduction to Lyapunov functions, see, e.g., Bhatia and Szegö, 1970.

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Let $w(t) := W(\mathbf{x}(t)) = \max(x_4(t), 3v(t))$. Equations (20) and (21) imply that if $\mathbf{x}(t)$ is in K_δ (i.e. $w(t) \leqslant \delta$) then w decreases weakly. This implies that K_δ is forward invariant. Therefore, for every initial condition $\mathbf{x}(0)$ in K_δ, Eqs. (20) and (21) apply for all $t \geqslant 0$. It follows that for all $t \geqslant 0$, $x_4(t) \leqslant x_4(0)\exp(-\gamma t)$ and $v(t) \leqslant v(0)\exp(-\gamma t)$. The result follows. □

It follows from Proposition 3 and Proposition 4 that if $\alpha > 0$ is small enough, then in the game (11) the unique strategy used in correlated equilibrium is strategy 4, but $x_4(t) \to 0$ from an open set of initial conditions.

7. Discussion

This note analyzed the behavior of the single population replicator dynamics in a family of 4×4 symmetric games, built by adding a strategy to a Rock-Paper-Scissors game. The added strategy is equivalent to the Nash equilibrium of the underlying RPS game, but for a fixed additional gain. It was shown that, provided that this fixed additional gain is small enough and for an open set of initial conditions, the unique strategy in the support of a correlated equilibrium is eliminated. We conclude with a few remarks.

(1) The results of this note also show that the *two-population* replicator dynamics may eliminate all strategies used in correlated equilibrium along interior solutions, though maybe not from an open set of initial conditions. See the remark in (Hofbauer and Weibull, 1996, p. 571).

(2) The basic idea of the proof is that if an attractor is disjoint from the set of equilibria, then it is likely that we may add a strategy in a way that strongly affects the set of equilibria but does not perturb much the dynamics in the neighborhood of the attractor.

(3) As mentioned in the introduction, elimination of all strategies used in correlated equilibrium actually occurs on an open set of games and for vast classes of dynamics (Viossat, 2005, Chapter 10, part B). This robustness is crucial for the relevance of our results. Indeed, in many situations, we are unlikely to have an exact knowledge of the payoffs or of the population dynamics of individual behaviors.

(4) Elimination of all strategies used in correlated equilibrium does not occur in 2×2 games nor in 3×3 symmetric games. Actually, in every 3×3 symmetric game, from any interior initial condition, and under any convex monotonic dynamics (Hofbauer and Weibull, 1996), all pure strategies that have probability zero in all correlated equilibria are eliminated (Viossat, 2005, Chapter 9, part B).

(5) Let $\mu(t) = \mathbf{x}(t) \otimes \mathbf{x}(t)$ denote the joint distribution of play at time t. Since $\mu(t)$ is a product distribution, it follows that it converges to the set of correlated equilibria if and only if $\mathbf{x}(t)$ converges to the set of Nash equilibria. The same is not true for time-averages: convergence of $\bar{\mu}(T) = \frac{1}{T}\int_0^T \mu(t)\,\mathrm{d}t$ to the set of correlated equilibria does not imply convergence of $\bar{\mathbf{x}}(T) = \frac{1}{T}\int_0^T \mathbf{x}(t)\,\mathrm{d}t$ to the set of Nash equilibria. Nevertheless, our results imply that there are games for which neither $\bar{\mu}$, nor any generalized time-average of the joint distribution of play, converges to the set of correlated equilibria.[8]

[8] We say that $\nu(\cdot)$ is a generalized time-average of the joint distribution of play if there exists an absolutely continuous, strictly increasing function $\tau : \mathbb{R} \to \mathbb{R}$, with $\tau(0) = 0$ and $\tau(t) \to +\infty$ as $t \to +\infty$ such that, for all $T > 0$, $\nu(T) = \frac{1}{\tau(T)}\int_0^T \dot{\tau}(t)\mu(t)\,\mathrm{d}t$. ($\tau$ is a rescaled time and the standard time-average corresponds to $\tau(t) = t$ for all t.)

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(6) Consider a bimatrix game with pure strategy sets I and J, and payoff function g_k for player $k = 1, 2$. A *weak correlated equilibrium* (Moulin and Vial, 1978) is a probability distribution μ on $I \times J$ such that:

$$\sum_{(i,j)\in I\times J} \mu(i,j)\big[g_1(i,j) - g_1(i',j)\big] \geqslant 0 \quad \forall i' \in I \tag{22}$$

and such that the corresponding inequalities for player 2 are satisfied.[9] The definition can be extended to n-player games. The set of weak correlated equilibria is also called the *Hannan set*, after Hannan (1957). Recent interest for this notion arose from the construction of simple adaptive processes converging to the Hannan set, and such that more sophisticated versions of these processes converge to the set of correlated equilibria (Hart and Mas-Colell, 2001; Young, 2004; Hart, 2005).[10]

In contrast with our results, Hofbauer (2005) shows that under the standard version of the n-population replicator dynamics, for any n-player game and any interior initial condition, the average joint distribution of play converges to the Hannan set. This indicates that, for evolutionary dynamics, there is a sharp difference between converging to the Hannan set and to the set of correlated equilibria; or equivalently, between having "no regret" and "no conditional regret" (for definitions of these concepts see, e.g., Hart and Mas-Colell, 2003 or the book of Young, 2004).

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[9] An interpretation is as follows: assume that before play, a mediator draws a pure strategy profile (i, j) with probability $\mu(i, j)$, never communicates i nor j to the players, but tell each player: you can either let me play for you and then I will play your component of the strategy profile I drew, or you can play whatever you like. The distribution μ is a weak correlated equilibrium if it is in the best interest of each player to let the mediator play for her, provided that the other player does so.

[10] Young (2004) uses the expression "coarse correlated equilibrium" instead of "weak correlated equilibrium."

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