

Deterministic monotone dynamics and dominated strategies

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Abstract

We survey and unify results on elimination of dominated strategies by monotonic dynamics and prove some new results that may be seen as dual to those of Hofbauer and Weibull (J. Econ. Theory, 1996, 558-573) on convex monotonic dynamics.

Key-words: evolutionary dynamics, dominated strategies

1 Introduction

A key issue in evolutionary game theory is whether evolutionary consideration lend support to rationality based concepts. The most basic of these concepts is perhaps that strictly dominated strategies should not be played. Accordingly, whether evolutionary dynamics wipe out dominated strategies has been studied by a number of authors (Akin, 1980; Nachbar, 1990; Dekel and Scotchmer, 1992; Samuelson and Zhang, 1992; Cabrales and Sobel, 1992; Hofbauer and Weibull, 1996; Cabrales, 2000; Berger and Hofbauer, 2005; Mertikopoulos and Moustakas, 2010; Hofbauer and Sandholm, 2011, and others). Many results concern deterministic monotonic dynamics and are proved in a very similar way. We unify these results, and prove some new results on concave monotonic dynamics (defined in the next section). These may be seen as a dual of Hofbauer and Weibull's (1996) results on convex monotonic dynamics.

The material is organized as follows. In the next section, we precise the framework and define several classes of dynamics. The main results are given in section 3 for continuous-time dynamics and in section 4 . for discrete-time dynamics. Section 5 gives examples of survival of strictly dominated strategies for single-population dynamics. Section 6 reviews more general dynamics

and section 7 discusses the concept of mixed strategy elimination. Appendix A deals with elimination of iteratively strictly dominated strategies. Finally, Appendices B and C provide details on some proofs.

2 Framework and classes of dynamics

The setting we introduce is rather abstract in order to encompass a variety of situations, including 1-player games (decision problems), single population dynamics, and multipopulation dynamics where the different populations need not evolve according to the same dynamics.

We consider a large population of players from which individuals are repeatedly and randomly drawn to play a game against some unspecified opponent. For instance, if the underlying game is a n -person game, then we focus on player 1, which we call the focal player, and consider players 2 to n as a single entity, which we call the opponent (whether players 2 to n can correlate their actions or not will not be relevant).

We assume that the set of pure strategies of the focal player is finite and denote it by $I := \{1, \dots, N\}$. The set S_{opp} of strategies of the opponent is compact. We denote by S_N or $\Delta(I)$ the $N - 1$ dimensional simplex over I :

$$S_N = \Delta(I) = \left\{ \mathbf{x} \in \mathbb{R}_+^N, \sum_{i \in I} x_i = 1 \right\}$$

and by $\text{int } S_N := \{\mathbf{x} \in S_N : \forall i \in I, x_i > 0\}$ its interior. Let $x_i(t)$ denote the frequency of strategy i at time t in the focal population, and $\mathbf{x}(t) := (x_i(t))_{i \in I}$ the vector of these frequencies. Let $\mathbf{y}(t)$ be the opponent's strategy at time t .¹

The payoff of the pure strategy i against strategy \mathbf{y} is denoted $U_i(\mathbf{y})$, where $U_i : S_{opp} \rightarrow \mathbb{R}$ is continuous. The payoff of the mixed strategy \mathbf{p} is $U_{\mathbf{p}}(\mathbf{y}) := \sum_{i \in I} p_i U_i(\mathbf{y})$. Vectors are in bold characters.

Dynamics We assume that the focal population adapts to the opponent's strategy through some biological or sociological process (imitation, selection,...). This is modeled by assuming that $\mathbf{x}(t)$ follows a differential equation. We focus on dynamics of the form:

$$\dot{x}_i(t) = x_i(t) \left[g_i(\mathbf{x}(t), \mathbf{y}(t)) - \sum_{k \in I} x_k(t) g_k(\mathbf{x}(t), \mathbf{y}(t)) \right] \quad (1)$$

¹If the game models an interaction with an individual of another population, then $\mathbf{y}(t)$ is the mean strategy in this other population; if it models a symmetric interaction with an individual of the same population, then $\mathbf{y}(t) = \mathbf{x}(t)$.

To ensure uniqueness of the solution through each initial condition, we assume that the functions g_i and \mathbf{y} are locally Lipschitz. The simplex S_N and its faces are then invariant under (1).

This class of dynamics includes the payoff functional dynamics

$$\dot{x}_i = x_i [f(U_i(\mathbf{y})) - \bar{f}] \quad (2)$$

where $f : \mathbb{R} \rightarrow \mathbb{R}$ is Lipschitz continuous and $\bar{f} = \sum_{i \in I} x_i f(U_i(\mathbf{y}))$, with time indices suppressed. An interpretation of (2) is that $U_i(\mathbf{y})$ is a material gain (food, territory, money,...) and $f(U_i(\mathbf{y}))$ its translation in terms of fitness. The replicator dynamics (Taylor and Jonker, 1978) corresponds to f linear, that is when the payoffs are directly measured in terms of fitness.

Domination and elimination As usual, we say that the mixed strategy \mathbf{p} *strictly dominates* the mixed strategy \mathbf{q} if $U_{\mathbf{p}}(\mathbf{y}) > U_{\mathbf{q}}(\mathbf{y})$ for all \mathbf{y} in S_{opp} . Throughout, we focus on strict domination, and write “dominated” for “strictly dominated”. The pure strategy i is *eliminated* under a solution $\mathbf{x}(\cdot)$ of (1) if $x_i(t) \rightarrow 0$ as $t \rightarrow +\infty$; it *survives* otherwise. Following Samuelson and Zhang (1992) and Cabrales (2000), we say that the mixed strategy $\mathbf{q} \in S_N$ is *eliminated* if $\min_{\{i \in I: q_i > 0\}} x_i(t) \rightarrow 0$ (or equivalently $\prod_{i \in I} x_i^{q_i}(t) \rightarrow 0$). We discuss the usefulness of this concept in section 7.

Classes of dynamics We want to find conditions on dynamics of type (1) that ensure that dominated strategies are eliminated, or conversely, that dominated strategies survive in some games. To formulate these conditions, let us call $g_i(\mathbf{x}(t), \mathbf{y}(t))$ the (unnormalized) *growth rate* of strategy i at time t . By analogy, we call *growth rate of the mixed strategy* \mathbf{p} the quantity

$$g_{\mathbf{p}}(\mathbf{x}, \mathbf{y}) := \sum_{i \in I} p_i g_i(\mathbf{x}, \mathbf{y}) \quad (3)$$

Definition. A dynamics (1) is *aggregate monotonic* (Samuelson and Zhang, 1992) if the growth rates of mixed strategies are ordered by their payoffs: for any mixed strategies \mathbf{p} and \mathbf{q} in S_N ,

$$\forall (\mathbf{x}, \mathbf{y}) \in S_N \times S_{opp}, U_{\mathbf{p}}(\mathbf{y}) > U_{\mathbf{q}}(\mathbf{y}) \Rightarrow g_{\mathbf{p}}(\mathbf{x}, \mathbf{y}) > g_{\mathbf{q}}(\mathbf{x}, \mathbf{y}) \quad (4)$$

It is *monotonic* if (4) holds whenever \mathbf{p} and \mathbf{q} are pure strategies, and *convex monotonic* (Hofbauer and Weibull, 1996) if (4) holds whenever \mathbf{q} is pure. It is *concave monotonic* if (4) holds whenever \mathbf{p} is pure. Examples of convex and concave monotonic dynamics are discussed by Hofbauer and Weibull (1996).²

²Monotonicity (Samuelson and Zhang, 1992) is called *relative monotonicity* by Nachbar

To understand these names, note that a payoff functional dynamics (2) with f increasing is convex (resp. concave) monotonic if and only if f is convex (resp. concave). It is aggregate monotonic if and only if f is linear, in which case (2) is the replicator dynamics.³

3 Intuition and main results

Intuition. Convex monotonic dynamics favour mixed strategies over pure ones in that a payoff advantage of a mixed strategy over a pure one always translates in a higher growth rate; thus they should eliminate dominated pure strategies, but not necessarily dominated mixed strategies. Similarly, concave monotonic dynamics favour pure strategies, thus should eliminate mixed strategies dominated by pure ones.

Aggregate monotonic dynamics are a limit case: a better payoff always translates in a higher growth rate, hence all dominated strategies are eliminated. The same holds for monotonic dynamics when restricting attention to pure strategies. Formally:

Proposition 1 *Let \mathbf{p} and \mathbf{q} be two mixed strategies. If \mathbf{p} strictly dominates \mathbf{q} then, under any aggregate monotonic dynamics and for any interior initial condition, \mathbf{q} is eliminated.*

The same result holds for monotonic dynamics when \mathbf{p} and \mathbf{q} are both pure, for convex monotonic dynamics when \mathbf{q} is pure, and for concave monotonic dynamics when \mathbf{p} is pure.⁴

For generalizations to elimination of iteratively dominated strategies, see Appendix A. Before providing a formal proof, we note that these results are sharp in the class of payoff functional dynamics (2):

(1990), *order-compatibility* by Friedman (1991), and *payoff monotonicity* by Hofbauer and Weibull (1996). The definition of concave monotonic dynamics is new but related dynamics are studied by Björnerstedt (1995).

³Samuelson and Zhang (1992) have shown that in bimatrix games, for any aggregate monotonic dynamics, there exists a positive speed function λ such that $\dot{\mathbf{x}}(t) = \lambda(\mathbf{x}(t), \mathbf{y}(t)) \dot{\mathbf{x}}_{REP}(t)$, where $\dot{\mathbf{x}}_{REP} = x_i(U_i(\mathbf{y}) - U_{\mathbf{x}}(\mathbf{y}))$. For single-population dynamics ($\mathbf{y}(t) = \mathbf{x}(t) \forall t$), this implies that any aggregate monotonic dynamics has the same orbits than the replicator dynamics; for multi-population dynamics, this need not be so, because the speed function is population specific.

⁴For multipopulation dynamics, these results are due to Akin (1980) for the replicator dynamics, Nachbar (1990) for monotonic dynamics, Samuelson and Zhang (1992) for aggregate monotonic dynamics and Hofbauer and Weibull (1996) for convex monotonic dynamics. The result on concave monotonic dynamics seems to be new.

Proposition 2 Consider a payoff functional dynamics (2). If f is not convex, then (2) need not eliminate strictly dominated pure strategies: there exists a game with a strictly dominated pure strategy that survives for some interior initial conditions and some opponent's behaviors.

Similarly, if f is not concave, then (2) need not eliminate mixed strategies dominated by a pure strategy; if f is not linear (resp. not increasing), then (2) need not eliminate mixed strategies dominated by another mixed strategy; (resp. pure by pure).

The result on convex monotonic dynamics is due to Hofbauer and Weibull (1996). The others seem to be new.

Proof of proposition 1. First consider aggregate monotonic dynamics: let \mathbf{p} and \mathbf{q} be mixed strategies such that \mathbf{p} strictly dominates \mathbf{q} . Since the dynamics is aggregate monotonic, it follows that $g_{\mathbf{p}}(\mathbf{x}, \mathbf{y}) > g_{\mathbf{q}}(\mathbf{x}, \mathbf{y})$ for all $(\mathbf{x}, \mathbf{y}) \in S_N \times S_{opp}$. By compactness of $S_N \times S_{opp}$, there exists $\varepsilon > 0$ such that

$$g_{\mathbf{p}}(\mathbf{x}, \mathbf{y}) - g_{\mathbf{q}}(\mathbf{x}, \mathbf{y}) \geq \varepsilon \quad \forall (\mathbf{x}, \mathbf{y}) \in S_N \times S_{opp} \quad (5)$$

Fix an interior solution $\mathbf{x}(\cdot)$ and let $w(t) := \sum_{i \in I} (p_i - q_i) \ln x_i(t)$. By (5), at any point in time,

$$\dot{w} = \sum_{i \in I} (p_i - q_i) \frac{\dot{x}_i}{x_i} = \sum_{i \in I} (p_i - q_i) g_i(\mathbf{x}, \mathbf{y}) = g_{\mathbf{p}}(\mathbf{x}, \mathbf{y}) - g_{\mathbf{q}}(\mathbf{x}, \mathbf{y}) \geq \varepsilon \quad (6)$$

Therefore, $\lim_{t \rightarrow +\infty} w(t) = +\infty$. Since $\sum_{i \in I} p_i \ln x_i(t) \leq 0$ (as $x_i \leq 1$), this implies that $\sum_{i \in I} q_i \ln x_i(t) \rightarrow -\infty$. Therefore $\prod_{i \in I} x_i(t)^{q_i} \rightarrow 0$ and \mathbf{q} is eliminated.

The proof for the other classes of dynamics is the same, up to replacement of \mathbf{p} , \mathbf{q} , or both, by pure strategies. ■

Proof of proposition 2. The case f nonincreasing is obvious. The case f nonlinear follows from the cases f nonconvex and f nonconcave, which we now deal with. Assume first that f is not convex. Since f is continuous (as assumed throughout), there exist real numbers a , b , and $\varepsilon > 0$ such that

$$\frac{f(a) + f(b)}{2} < f\left(\frac{a+b}{2} - \varepsilon\right) \quad (7)$$

Consider the 3×2 game with payoff matrix:

$$\begin{array}{c} T \\ M \\ B \end{array} \begin{array}{cc} L & R \\ \left(\begin{array}{cc} a & b \\ \frac{a+b}{2} - \varepsilon & \frac{a+b}{2} - \varepsilon \\ b & a \end{array} \right) \end{array} \quad (8)$$

The pure strategy M is strictly dominated by $\mathbf{q} = (1/2, 0, 1/2)$. But if, approximately, $\mathbf{y}(t) = L$ half of the time and $\mathbf{y}(t) = R$ the other half, then the average (unnormalized) growth rate of strategies T and B is close to $[f(a) + f(b)]/2$, while the average growth rate of strategy M is $f([a + b]/2 - \varepsilon) > [f(a) + f(b)]/2$. It follows that $x_M \rightarrow 1$, hence strategy M survives.

If f is not concave, then take a , b , and $\varepsilon > 0$ such that $\frac{f(a)+f(b)}{2} > f\left(\frac{a+b}{2} + \varepsilon\right)$. Consider the same game as (8), but with $-\varepsilon$ changed into $+\varepsilon$. Strategy \mathbf{q} is strictly dominated by M . But against the same opponent's behaviour, the average growth rate of strategy \mathbf{q} is now higher than the average growth rate of M . It follows that $x_M \rightarrow 0$. Choosing $y(t)$ periodic and respecting the symmetry between strategies T and B ensures that x_T/x_B remains bounded. Together with $x_M \rightarrow 0$, this implies that $\liminf_{t \rightarrow +\infty} x_T(t)x_B(t) > 0$, hence \mathbf{q} is not eliminated. For details see Appendix B. ■

As detailed in Appendix C, a variant of this proof shows that proposition 2 extends to the wider class of payoff functional dynamics considered by Hofbauer and Weibull (1996).

4 Discrete-time dynamics

Dekel and Scotchmer (1992) show that the discrete-time replicator dynamics need not eliminate pure strategies strictly dominated by mixed strategies. Thus, proposition 1 does not extend to discrete-time dynamics. To see why, consider the discrete-time dynamics:

$$x_i(n+1) = x_i(n) \frac{C + g_i(\mathbf{x}, \mathbf{y})}{C + \sum_k x_k g_k(\mathbf{x}, \mathbf{y})} \quad (9)$$

where $\mathbf{x} = \mathbf{x}(n)$, $\mathbf{y} = \mathbf{y}(n)$, and the constant $C > -\min_{i, \mathbf{x}, \mathbf{y}} g_i(\mathbf{x}, \mathbf{y})$ may be interpreted as background fitness (Maynard-Smith, 1982). Eq. (9) is equivalent to

$$x_i(n+1) - x_i(n) = x_i(n) \frac{g_i(\mathbf{x}, \mathbf{y}) - \sum_k x_k g_k(\mathbf{x}, \mathbf{y})}{C + \sum_k x_k g_k(\mathbf{x}, \mathbf{y})} \quad (10)$$

which is the Euler discretization of (1) with step size $1/[C + \sum_k x_k g_k(\mathbf{x}, \mathbf{y})]$. The discrete-time replicator dynamics (Maynard-Smith, 1982) corresponds to $g_i(\mathbf{x}, \mathbf{y}) = U_i(\mathbf{y})$. Dekel and Scotchmer (1992) take $C = 0$.

Proposition 1 and 2 are based on the evolution of the quantity $w := \sum_{i \in I} (q_i - p_i) \ln x_i$, where \mathbf{p} and \mathbf{q} are two mixed strategies. In continuous-time, we had (6): $\dot{w} = \sum_{i \in I} (q_i - p_i) g_i(\mathbf{x}, \mathbf{y})$. But it follows from (9) that in

discrete-time,

$$w(n+1) - w(n) = \sum_{i \in I} (q_i - p_i) \tilde{g}_i(\mathbf{x}, \mathbf{y}) \text{ with } \tilde{g}_i = \ln(C + g_i). \quad (11)$$

It follows that propositions 1 and 2 hold for (9) if we replace g_i by \tilde{g}_i when defining our classes of dynamics. This was first understood, and shown for aggregate monotonic dynamics, by Cabrales and Sobel (1992).

For payoff functional dynamics

$$x_i(n+1) = x_i(n) \frac{C + f(U_i(\mathbf{y}))}{C + \sum_k x_k f(U(k, \mathbf{y}))} \quad (12)$$

we get:

Proposition 3 *The dynamics (12) eliminates pure (mixed) strategies strictly dominated by mixed (pure) strategies if and only if $\ln(C + f)$ is increasing and convex (concave). It eliminates mixed strategies strictly dominated by mixed strategies if and only if $\ln(C + f)$ is increasing and linear.*

Since the logarithm is concave, dynamics that are aggregate monotonic in continuous-time become concave monotonic in discrete-time, and thus do not eliminate dominated pure strategies. Dekel and Scotchmer's (1922) example is an instance of this general result.

For large C (a fine discretization), $\tilde{g}_i \simeq \ln C + g_i/C$ and discretizing concavifies less the dynamics. If we first fix the game and the functions g_i and then take C large enough, or if we let C depend on n , with $C_n \rightarrow +\infty$ (and $\sum_n 1/C_n = +\infty$ for the dynamics not to stop), then proposition 1 holds also in the discrete case, without modifying the definitions of the classes of dynamics (see Appendix C and Cabrales and Sobel, 1992). For more on discrete time dynamics, see Cabrales and Sobel (1992) and Björnerstedt et al. (1996).⁵

5 Survival of strictly dominated strategies in single-population dynamics

Proposition 2 is easy to prove because the opponent's behaviour may be chosen ad-hoc. This section shows that similar results may be obtained with

⁵Björnerstedt et al. (1996) consider an overlapping-generations dynamics which has as limit cases the discrete-time and the continuous-time replicator dynamics, for respectively no overlap and full overlap. The degree of overlap plays a role similar to that of the background fitness C in (9). They show that, for a fixed game, if the degree of overlap is large enough, strictly dominated strategies are eliminated.

more realistic opponent's behaviours. We first recall a result of Hofbauer and Weibull (1996).

5.1 Hofbauer and Weibull's result

Consider a single population dynamics with a payoff functional form (2), with $f \in C^1$. Up to a change of time, and suppressing time indices, it may be written as $\dot{x}_i = x_i[f(U_i) - \bar{f}]$ with $U_i = U_{\mathbf{x}}(\mathbf{x})$ and $\bar{f} = \sum_i x_i f(U_i)$. Hofbauer and Weibull (1996) show that if f is not convex, then there are games where pure strategies strictly dominated by mixed strategies survive, for many interior initial conditions. The game they consider is a 4×4 two-player symmetric game. It is built by adding a fourth strategy to a generalized Rock-Paper-Scissors (RPS) game. The payoff matrix has the following structure:

$$\begin{pmatrix} a & c & b & \gamma \\ b & a & c & \gamma \\ c & b & a & \gamma \\ a + \beta & a + \beta & a + \beta & 0 \end{pmatrix}$$

where: (i) $c < a < b$, (ii) $(a + b + c)/3 > a + \beta$ and $\gamma > 0$, and (iii) $\beta > 0$.

By (i), the 3×3 base game is a RPS game, and by (ii), strategy 4 is strictly dominated by $\mathbf{p} = (1/3, 1/3, 1/3, 0)$. Since β and γ are positive, there is a rest point \mathbf{z}_i in the relative interior of the edge connecting strategy $i \in \{1, 2, 3\}$ to strategy 4. It attracts any solution starting in the relative interior of this edge, and actually, any solution starting in the relative interior of the face $x_{i+1} = 0$, where $i + 1$ is counted modulo 3. It follows that the \mathbf{z}_i form a heteroclinic cycle, that is, a loop of saddle rest points and orbits connecting them. Conditions (ii) and (iii) imply that $(a + b + c)/3 > a$ (or equivalently, $a < (b+c)/2$). This implies that, under the replicator dynamics, the base game is an inward cycling RPS game (Hofbauer and Sigmund, 1998, p93; Gaunersdorfer and Hofbauer, 1995, p291). It may be shown that under the replicator dynamics (i.e. f strictly increasing and linear), any solution starting in the interior of the state space converges to the Nash equilibrium $\mathbf{p} = (1/3, 1/3, 1/3, 0)$. In particular, strategy 4 is eliminated. However, if the function f is strictly increasing but not convex, then one may find such payoffs that furthermore satisfy $f(a) > [f(b) + f(c)]/2$. This means that in the RPS base game, even though the replicator dynamics is inward cycling, the current dynamics is outward cycling close to the RPS cycle; thus, this cycle is asymptotically stable on the face $x_4 = 0$. As detailed by Hofbauer and Weibull, this implies that for β small enough, the heteroclinic cycle between the \mathbf{z}_i is asymptotically stable. But along this cycle, $x_4 > 0$. Thus, though strictly dominated, strategy 4 survives for an open set of initial conditions.

5.2 A dual result.

Our aim is to give a dual example, showing that if f is strictly increasing and sufficiently regular but not concave, then mixed strategies that are strictly dominated by a pure strategy need not be eliminated.

To simplify the exposition, we assume that f is differentiable, with a strictly positive derivative, and that f is strictly convex. The strict convexity assumption may be replaced by any assumption that guarantees that f is strictly convex over some range, as we can always locate all the payoffs in this range. For instance, f C^2 and not concave suffices.

Consider the game with payoff matrix

$$\mathbf{A} = \begin{pmatrix} a & c & b & m - \gamma \\ b & a & c & m - \gamma \\ c & b & a & m - \gamma \\ m + \beta & m + \beta & m + \beta & m \end{pmatrix} \quad (13)$$

where $m = (a + b + c)/3$. Assume that : (i) $c < a < b$, (ii) $\beta > 0$ and $\gamma > 0$, and (iii) $a > (b+c)/2$. Condition (i) means that the 3×3 base game is a RPS game; condition (ii) implies that strategy 4 strictly dominates the strategy $\mathbf{p} = (1/3, 1/3, 1/3, 0)$; condition (iii) implies that in the RPS base game, the replicator dynamics cycles outwards.

Because f is strictly convex, we may find such payoffs that furthermore satisfy: (iv) $f(a) < [f(b) + f(c)]/2$. Consider first the dynamics on the face $x_4 = 0$, that is, in the RPS base game. Condition (iv) implies that on this face, even though the replicator dynamics is outward cycling, our dynamics is inward cycling close to the RPS heteroclinic cycle, i.e. the relative boundary of this face is repulsive (Hofbauer and Sigmund, 1998; Gaunersdorfer and Hofbauer, 1995). Using a Taylor expansion of f near m , it may be shown that, close to the rest point $\mathbf{p} = (1/3, 1/3, 1/3, 0)$, the dynamics behaves as if f was linear, that is, as the replicator dynamics. Due to condition (3) this implies that close to \mathbf{p} (except exactly at \mathbf{p}), the dynamics is outward cycling: $x_1 x_2 x_3$ strictly decreases along trajectories (Figure 1).⁶

By compactness, this implies that there exists $\rho \in]0, 1/27[$ and a positive constant C such that, if $x_4 = 0$ and $x_1 x_2 x_3 = \rho$, then $d(x_1 x_2 x_3)/dt < -C$.

⁶Technically, fix a solution $\mathbf{x}(\cdot)$ in the relative interior of the face $x_4 = 0$. Let $v(t) = (x_1 x_2 x_3)^{1/3}$ and $w(t) = \ln v(t)$. Let $\mathbf{h}(t) = \mathbf{x}(t) - \mathbf{p}$. Then $\dot{w}(t) = -[a - (b+c)/2] \|\mathbf{h}(t)\|^2 + o(\|\mathbf{h}(t)\|^2)$, hence near \mathbf{p} , by condition (iii), $x_1 x_2 x_3$ decreases along trajectories. To obtain this expression of \dot{w} , note that $(\mathbf{A}\mathbf{x})_i - m = (\mathbf{A}\mathbf{h})_i$, thus $f((\mathbf{A}\mathbf{x})_i) = m + f'(m)(\mathbf{A}\mathbf{h})_i + o([\mathbf{A}\mathbf{h}]_i)$. Using this and $\sum_i h_i = 0$, we get $\dot{w} = -f'(m)\mathbf{h} \cdot \mathbf{A}\mathbf{h} + o(\|\mathbf{h}\|^2)$. A standard computation shows that $\mathbf{h} \cdot \mathbf{A}\mathbf{h} = [a - (b+c)/2] \|\mathbf{h}(t)\|^2$ hence the result. This may be seen as a variant of exercise 8.1.1 in Hofbauer and Sigmund (1998).

By continuity of f , this implies the existence of a small positive ε such that, if $x_4 \leq \varepsilon$ and $x_1x_2x_3 = \rho$, then $d(x_1x_2x_3)/dt < -C/2$. Furthermore, for $\beta = 0$, except if $x_1 = x_2 = x_3$, the payoff of strategy 4 is less than the mean payoff: $(\mathbf{Ax})_4 < \mathbf{x} \cdot \mathbf{Ax}$ (Viossat, 2007), hence $\dot{x}_4 < 0$ by convexity of f . By a compactness and continuity argument, this implies that for β small enough, if \mathbf{x} belongs to the compact set $K = \{\mathbf{x} \in S_4 : x_1x_2x_3 \leq \rho \text{ and } x_4 \leq \varepsilon\}$, then $\dot{x}_4 < -C'$ for some positive constant C' .

Now consider any solution with initial condition in K . The solution cannot leave K , which implies that it converges to the face $x_4 = 0$. Moreover, since on this face the relative boundary is repulsive, this implies that $\liminf x_1x_2x_3 > 0$. Thus, though $\mathbf{p} = (1/3, 1/3, 1/3, 0)$ is strictly dominated by strategy 4, it is not eliminated.⁷

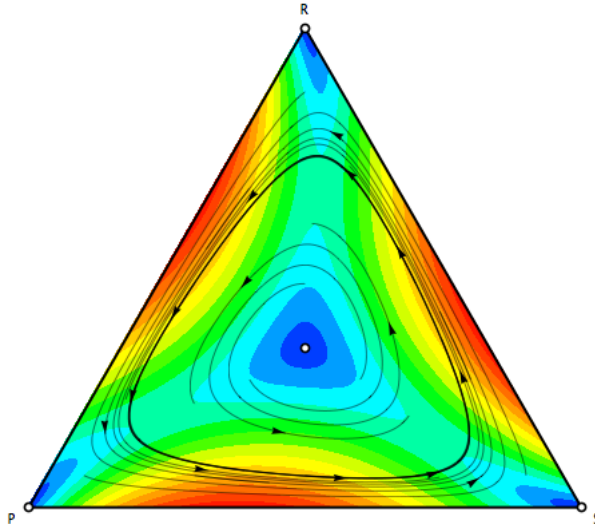


Figure 1: Dynamics on the face $x_4 = 0$. Both the center and the boundary are repulsive. In the neighborhood of this face, for suitable payoffs, x_4 decreases outside a small neighborhood of $(1/3, 1/3, 1/3, 0)$. Thus the limit cycle (or annulus) in the interior of this face is asymptotically stable in the whole game. Graphic produced with Dynamo (Sandholm et al., 2011).

5.3 Discrete-time analog

For discrete-time dynamics (12), the survival examples of sections 5.1 and 5.2 go through if the assumptions on the convexity of f are replaced by

⁷This proof is similar to the proof that the Brown-von-Neumann-Nash dynamics may eliminate all strategies used in correlated equilibria (Viossat, 2008)

the corresponding assumptions on $\ln f$. Indeed, for reasons similar to those of section 4, the stability condition of the RPS cycle of the base game for (12) is : $\ln f(a) > [\ln f(b) + \ln f(c)]/2$ (Gaunersdorfer, 1992, section 5 ; Hofbauer and Schlag, 2000, p533 ; see also Weissing, 1991). For Hofbauer and Weibull's result, the proof goes through with no other changes. For our dual result, two other ingredients are needed: first, $(\mathbf{Ax})_4 < \mathbf{x} \cdot \mathbf{Ax}$ should imply $f[(\mathbf{Ax})_4] < \sum_i x_i f[(\mathbf{Ax})_i]$; but if $\ln f$ is convex, so is f , hence this holds. Second, in the RPS base game, $x_1 x_2 x_3$ should decrease near the equilibrium. In continuous time, when $a > [b + c]/2$, this holds for any C^1 monotonic dynamics; but the discrete-time dynamics trajectories are tangent to their continuous-time counterpart and the set of mixed strategies \mathbf{x} such that $x_1 x_2 x_3 \geq \lambda$ is convex; thus, under the discrete-time dynamics, $x_1 x_2 x_3$ decreases a fortiori (see also Hofbauer and Schlag, 2000, p534). The rest of the proof is unchanged.

6 Other dynamics

Continuous set of pure strategies Elimination of dominated strategies by monotonic dynamics is generalized to games with a continuous set of pure strategies by Heifetz et al (2007a, 2007b) and by Cressman and Hofbauer (2005) and Cressman et al (2006) [see also, Cressman, 2005] for the replicator dynamics.

Stochastic dynamics Cabrales (2000) studies a stochastic version of the replicator dynamics and gives conditions under which dominated strategies are eliminated. Mertikopoulos and Moustakas (2010) show that for another version of the stochastic replicator dynamics, dominated strategies are always eliminated.

Best-reply dynamics. Under the best-reply dynamics (Gilboa and Matsui, 1991; Matsui 1992), the frequency of pure strategies that are not best replies to the current strategy of the opponent decreases exponentially. It follows that strictly dominated pure strategies are eliminated. This extends to iteratively strictly dominated pure strategies. However, strictly dominated mixed strategies need not be eliminated. This follows from (Viossat, 2008) or from a variant of the proof of proposition (2). These results are not surprising, as the best reply dynamics may be seen as a limit case of convex monotonic dynamics (Hofbauer and Weibull, 1996)

Survival of pure strategies strictly dominated by other pure strategies. Building on Berger and Hofbauer (2006), Hofbauer and Sandholm (forthcoming) [see also Sandholm (2011)] show that under a large class of dynamics, a pure strategy strictly dominated by another pure strategy may survive. Their

key feature is that, except at Nash equilibria, the growth rate of an unused best-reply is always positive. Combined with regularity requirements, this gives a small advantage to rare strategies which allows to maintain strictly dominated strategies at nonvanishing frequencies.⁸

7 Discussion

One of the contribution of this article is to study elimination of dominated mixed strategies by concave monotonic dynamics. Elimination of dominated mixed strategies has a natural interpretation if we assume that agents may play mixed strategies, but even if we think that agents play only pure strategies, we find the concept useful on at least three grounds. First, to compare the outcome of evolutionary dynamics with rationalizable strategies (Bernheim, 1984; Pearce, 1984). Second, to clarify the logic of Hofbauer and Weibull's (1996) results by showing that elimination of pure strategies by convex monotonic dynamics and of mixed strategies by concave monotonic dynamics are two faces of the same coin. Third, to analyse the behaviour of some dynamics.

Consider for instance a single-population dynamics modeling the evolution of behavior in the two-player symmetric game

$$\begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 2 & 2 & 1 \end{pmatrix}$$

The mixed strategy $\mathbf{q} = (1/2, 1/2, 0)$ is strictly dominated by the third pure strategy. Thus, if the dynamics is concave monotonic, then for any interior initial condition $x_1(t)x_2(t) \rightarrow 0$, hence the solution converges to the union of the edges $x_1 = 0$ and $x_2 = 0$. Together with the dynamics on these edges (which is qualitatively the same for all monotonic dynamics), this implies that any interior solution converges to one of the three pure Nash equilibria, or to one of the rest points in the relative interior of these edges, that is $(1/2, 0, 1/2)$ and $(0, 1/2, 1/2)$. This need not be the only way to prove this result, but probably the most elementary.

⁸An example of a game and a dynamics such that a pure strategy dominated by *all* other pure strategies survives for most initial conditions is given by Björnerstedt et al. (1996).

A Iterative elimination

Proposition (1) extends to elimination of iteratively strictly dominated strategies. For simplicity, we focus on two-player finite games. The extension to n -player games is immediate.

Let I and J denote the finite sets of pure strategies of player 1 and 2, respectively, and for any finite set K , let $\Delta(K)$ denote the simplex of probability distribution over K . Thus, the set of mixed strategies of player 1 is $\Delta(I)$.

Let $I^0 = \tilde{I}^0 = I$ and $S^0 = \tilde{S}^0 = \Delta(I)$. Similarly, let $J^0 = \tilde{J}^0 = J$ and $T^0 = \tilde{T}^0 = \Delta(J)$. Inductively, let I^{k+1} (resp. \tilde{I}^{k+1} , S^{k+1} , \tilde{S}^{k+1}) denote the set of strategies in I^k (resp. \tilde{I}^k , S^k , \tilde{S}^k) that are not strictly dominated by any strategy in I^k (resp. $\Delta(\tilde{I}^k)$, $I \cap S^k$, \tilde{S}^k) when player 2 chooses strategies in J^k (resp. \tilde{J}^k , T^k , \tilde{T}^k). Similar definitions apply to player 2.

Definition. A pure strategy $i \in I$ (resp. a mixed strategy $\mathbf{x} \in \Delta(I)$) is *iteratively strictly dominated by pure strategies* if there exists k in \mathbb{N} such that $i \notin I^k$ (resp. $\mathbf{x} \notin S^k$). A pure strategy $i \in I$ (resp. a mixed strategy $\mathbf{x} \in \Delta(I)$) is *iteratively strictly dominated* if there exists k in \mathbb{N} such that $i \notin \tilde{I}_k$ (resp. $\mathbf{x} \notin \tilde{S}^k$).⁹

Proposition 4 *For every interior initial condition: if both $\mathbf{x}(t)$ and $\mathbf{y}(t)$ follow a monotonic (resp. concave monotonic) dynamics, then every pure (resp. mixed) strategy iteratively strictly dominated by pure strategies is eliminated; if both $\mathbf{x}(t)$ and $\mathbf{y}(t)$ follow a convex (resp. aggregate) monotonic dynamics, then every iteratively strictly dominated pure (resp. mixed) strategy is eliminated.*

Proof. We first prove the result on convex monotonic dynamics. It follows from proposition 1 that the pure strategies in $\tilde{I}^0 \setminus \tilde{I}^1$ and in $\tilde{J}^0 \setminus \tilde{J}^1$ are eliminated. By induction assume that for k in \mathbb{N} , the strategies in $\tilde{I}^0 \setminus \tilde{I}^k$ and in $\tilde{J}^0 \setminus \tilde{J}^k$ are eliminated, hence in particular

$$\forall \eta > 0, \exists T \in \mathbb{R}, \forall t \geq T, \max_{j \in J \setminus \tilde{J}^k} y_j(t) \leq \eta \quad (14)$$

⁹ Since a pure strategy is also a mixed strategy, there seems to be two definitions of iteratively strictly dominated pure strategies, but they are equivalent. This is because iterative elimination of dominated mixed strategies boils down to iterative elimination of dominated pure strategies followed by one round of elimination of dominated mixed strategies, as is easily shown. Similarly, iterative elimination of mixed strategies dominated by pure strategies boils down to iterative elimination of pure strategies dominated by other pure strategies, followed by one round of elimination of mixed strategies dominated by pure ones. Thus, both definitions of pure strategies iteratively strictly dominated by pure strategies are also equivalent.

Let $i \in \tilde{I}^k \setminus \tilde{I}^{k+1}$. Since strategy i is strictly dominated in the game restricted to $\tilde{I}^k \times \tilde{J}^k$, there exists $\varepsilon > 0$, $\eta > 0$ and $\mathbf{p} \in S_N$ such that

$$\max_{j \in J \setminus \tilde{J}^k} y_j \leq \eta \Rightarrow U_i(\mathbf{y}(t)) < U_{\mathbf{p}}(\mathbf{y}(t)) + \varepsilon$$

Therefore, it follows from (14) and the proof of proposition 1 that strategy i is eliminated. The result follows.

The proof for monotonic dynamics is identical up to replacement of \tilde{I}^k , \tilde{J}^k by I^k , J^k , and of “ $p \in S_N$ ” by “ $i \in I$ ”. Concave monotonic dynamics eliminate pure strategies iteratively strictly dominated by pure strategies, as any monotonic dynamics, and then mixed strategies that become dominated by pure ones after this elimination, by an argument similar to the iteration step for convex monotonic dynamics. This and footnote 9 proves the result. The proof for aggregate monotonic dynamics relies similarly on the fact that they are convex monotonic and footnote 9. ■

These results are due to Nachbar (1990) for monotonic dynamics, Samuelson and Zhang (1992) for aggregate monotonic dynamics and Hofbauer and Weibull (1996) for convex monotonic dynamics.

B Details on survival of dominated mixed strategies under non concave monotonic dynamics

B.1 For payoff functional dynamics (2)

If f is not concave, then there exist reals a and b such that

$$\frac{f(a) + f(b)}{2} > f\left(\frac{a+b}{2}\right)$$

By continuity of f , there exists $\varepsilon > 0$ such that:

$$\alpha := \frac{f(a) + f(b)}{2} - f\left(\frac{a+b}{2} + \varepsilon\right) > 0 \quad (15)$$

Assume that player 1 plays a 3×2 game with payoff matrix:

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{cc} L & R \\ \left(\begin{array}{cc} a & b \\ \frac{a+b}{2} + \varepsilon & \frac{a+b}{2} + \varepsilon \\ b & a \end{array} \right) \end{array}$$

Let $y_l(t)$ denote the probability at time t that the opponent chooses the left column. Fix some large positive constant T (more precisely, $T > (2C + 1)/\alpha + 1$ with $C = \max_{[a,b]} |f|$; this condition will appear later). Assume that the function $y_l(\cdot)$ is $2T$ -periodic with $y_l(t) = 1$ if $t \in [0, T - 1]$, $y_l(t) = 0$ if $t \in [T, 2T - 1]$, and linear variation on $[T - 1, T]$ and $[2T - 1, 2T]$.

Assume that $\mathbf{x}(0) \in \text{int}S_N$ and let

$$w(t) := \ln \left(\frac{x_2}{\sqrt{x_1 x_3}} \right) (t)$$

Letting $g_i(t) := f(U_i(\mathbf{y}(t)))$, we have:

$$\dot{w}(t) = g_2(t) - \frac{g_1(t) + g_3(t)}{2}$$

hence

$$w((k+1)T) - w(kT) = \int_{kT}^{(k+1)T} \left(g_2(t) - \frac{g_1(t) + g_3(t)}{2} \right) dt$$

Since for all t , $g_2(t) = f((a+b)/2 + \varepsilon)$ and since

$$\forall t \in [kT, (k+1)T - 1], g_1(t) + g_3(t) = f(a) + f(b)$$

it follows from (15) that:

$$w((k+1)T) - w(kT) \leq -(T-1)\alpha + 2C$$

with $C = \max_{[a,b]} |f|$. Since we assumed $T > (2C + 1)/\alpha + 1$, it follows that

$$w((k+1)T) \leq w(kT) - 1$$

Therefore $w(kT) \rightarrow -\infty$ as $k \rightarrow +\infty$. Since the variation of w between kT and $(k+1)T$ is bounded (less than $2CT$), it follows that $w(t) \rightarrow -\infty$ hence $x_2(t) \rightarrow 0$ as $t \rightarrow +\infty$. Furthermore, it is easy to see by the same kind of computation that $x_1(t)/x_3(t)$ is $2T$ -periodic. It follows that $x_1 x_3 \not\rightarrow 0$. Actually, as is easily seen,

$$\inf_{t \in \mathbb{R}_+} x_1(t)x_3(t) = \min_{t \in [0, 2T]} x_1(t)x_3(t) > 0$$

Therefore, though $\mathbf{q} = (1/2, 0, 1/2)$ is strictly dominated by strategy 2, the mixed strategy \mathbf{q} is not eliminated.¹⁰

¹⁰Note also that for T sufficiently large or $x_1(0)$ sufficiently close to $x_3(0)$, there exists t in $[0, 2T]$ such that $x_1(t) = x_3(t)$, hence $x_1(t+2kT) = x_3(t+2kT)$ for all $k \in \mathbb{N}$. Together with $x_2(t) \rightarrow 0$, this implies that $\limsup_{t \rightarrow +\infty} x_1(t)x_3(t) = 1/4$.

B.2 For the wider class of payoff functional dynamics considered by Hofbauer and Weibull (1996)

In the context of n -player normal form games (hence with the opponent corresponding to players 2 to n), Hofbauer and Weibull (1996) consider payoff functional dynamics of the form (time indices suppressed):

$$\dot{x}_i = \lambda(\mathbf{x}, \mathbf{y}) x_i [f(U_i(\mathbf{y})) - \bar{f}] \quad (16)$$

This is more general than (2) because the speed of the dynamics may now depend on the population's and opponent's mean strategies. Assuming that λ is continuous, the previous proof is easily adapted to deal with these more general dynamics.

Fix a large constant T . Let (t_n) be a strictly increasing sequences of times, with $t_0 = 0$. Assume that :

- (i) $y(t)$ is piecewise linear, with for all $n \in \mathbb{N}$: $y_L(t) = 1$ on $[t_{6n}, t_{6n+1}]$, $y_L(t) = 0$ on $[t_{6n+3}, t_{6n+4}]$, $y_L(t_{6n+2}) = y_L(t_{6n+5}) = 1/2$, and linear variation on $[t_{6n+1}, t_{6n+2}]$, $[t_{6n+2}, t_{6n+3}]$, $[t_{6n+4}, t_{6n+5}]$ and $[t_{6n+5}, t_{6n+6}]$. Furthermore :
- (ii) Choose t_{6n+1} and t_{6n+4} such that

$$\int_{t_{6n}}^{t_{6n+1}} \lambda(x(s), y(s)) ds = \int_{t_{6n+3}}^{t_{6n+4}} \lambda(x(s), y(s)) ds = T;$$

- (iii) Let $t_{6n+3} = t_{6n+1} + 1$ and $t_{6n+6} = t_{6n+4} + 1$;
- (iv) if $a = b$, choose t_{6n+2} and t_{6n+5} arbitrarily in $]t_{6n+1}, t_{6n+3}[$ and $]t_{6n+4}, t_{6n+6}[$, respectively. Otherwise, assume w.l.o.g. $a < b$, note that $g_1 - g_3$ is negative on $]t_{6n+1}, t_{6n+2}[$ and positive on $]t_{6n+2}, t_{6n+3}[$, and choose t_{6n+2} such that $\int_{t_{6n+1}}^{t_{6n+3}} [g_1(\mathbf{x}(s), \mathbf{y}(s)) - g_3(\mathbf{x}(s), \mathbf{y}(s))] ds = 0$. Similarly, choose t_{6n+5} such that

$$\int_{t_{6n+4}}^{t_{6n+6}} [g_1(\mathbf{x}(s), \mathbf{y}(s)) - g_3(\mathbf{x}(s), \mathbf{y}(s))] ds = 0.$$

The same kind of computation as in the case $\lambda \equiv 1$ show that

$$w(t_{6n+6}) - w(t_{6n}) < -2T\alpha + 4v_{max}C$$

where $v_{max} = \max_{(\mathbf{x}, \mathbf{y}) \in S_N \times S_{opp}} \lambda(\mathbf{x}, \mathbf{y})$, and $C = \max_{[a,b]} |f|$. Thus for $T > 2v_{max}C/\alpha$, $w(t) \rightarrow_{t \rightarrow +\infty} -\infty$. Thus $x_2(t) \rightarrow 0$ and $x_1(t) + x_3(t) \rightarrow 1$. Conditions (ii) and (iv) ensure that $[x_1/x_3](t_{6n+6}) = [x_1/x_3](t_{6n})$. Together with $x_1(t) + x_3(t) \rightarrow 1$, this implies that $\limsup x_1(t)x_3(t) \geq x_1(0)x_3(0) > 0$, hence \mathbf{q} is not eliminated.

C More on discrete dynamics

We now consider the limit $C \rightarrow +\infty$. If $C \gg \max_{(i,\mathbf{x},\mathbf{y})} |g_i(\mathbf{x},\mathbf{y})|$ then $\ln(1 + g_i/C)$ is approximately equal to g_i/C . From this remark we obtain the following result, which was shown by Cabrales and Sobel (1992), in the aggregate monotonic case.

Proposition 5 *Fix a game and functions g_i ; fix a mixed strategy \mathbf{q} strictly dominated by a mixed strategy \mathbf{p} . Assume either that the dynamics is aggregate monotonic (in the continuous-time sense) or that it is convex monotonic and that \mathbf{q} is pure. Then there exists \bar{C} in \mathbb{R} such that for all $C \geq \bar{C}$, the discrete dynamics (9) eliminates strategy \mathbf{q} .*

Proof. There exists $\varepsilon > 0$ such that

$$\forall(\mathbf{x}, \mathbf{y}), \sum_{i \in I} (p_i - q_i) U_i(\mathbf{y}) \geq \varepsilon$$

Therefore, under the above assumptions, there exists $\alpha > 0$ such that

$$\forall(\mathbf{x}, \mathbf{y}), \sum_{i \in I} (p_i - q_i) g_i(\mathbf{x}, \mathbf{y}) \geq \alpha$$

For C large enough,

$$\forall(i, \mathbf{x}, \mathbf{y}), \left| \ln \left(1 + \frac{g_i(\mathbf{x}, \mathbf{y})}{C} \right) - \frac{g_i(\mathbf{x}, \mathbf{y})}{C} \right| < \frac{\alpha}{4C}$$

so that

$$\forall(\mathbf{x}, \mathbf{y}), \sum_{i \in I} (p_i - q_i) \tilde{g}_i(\mathbf{x}, \mathbf{y}) \geq \alpha/2C > 0$$

where $\tilde{g}_i = \ln C + \ln \left(1 + \frac{g_i}{C} \right) = \ln(C + g_i)$. The result follows. ■

Note that the constant \bar{C} depends not only on the game and the functions g_i , but also, through ε , on the strategies \mathbf{p} and \mathbf{q} . Compare proposition 5 and proposition 3. Note that for aggregate monotonic dynamics, the order of the quantifiers (whether we first fix the game, the functions g_i and the strategies \mathbf{p} and \mathbf{q} , or we first fix the constant C) is crucial.

To conclude, note that if the constant C depends on the step n :

$$x_i(n+1) = x_i(n) \frac{C_n + g_i(\mathbf{x}, \mathbf{y})}{C_n + \sum_k x_k g_k(\mathbf{x}, \mathbf{y})} \quad (17)$$

then we have:

Proposition 6 Assume that $C_n \rightarrow +\infty$ as $n \rightarrow +\infty$ and that

$$\sum_{n \in \mathbb{N}} \frac{1}{C_n} = +\infty, \quad (18)$$

Then for any game, any functions g_i and any mixed strategies \mathbf{p} and \mathbf{q} such that \mathbf{p} strictly dominates \mathbf{q} , if the dynamics is aggregate monotonic or if it is convex monotonic and \mathbf{q} is pure, then the discrete dynamics (17) eliminates strategy \mathbf{q} .

Proof. This follows from proposition 5. The condition (18) is needed for the dynamics not to slow down too much and “stop”. ■

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