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Evolutionary dynamics may eliminate all strategies used in correlated equilibrium

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Abstract

We show on a 4×4 example that many dynamics may eliminate all strategies used in correlated equilibria, and this for an open set of games. This holds for the best-response dynamics, the Brown–von Neumann–Nash dynamics and any monotonic or weakly sign-preserving dynamics satisfying some standard regularity conditions. For the replicator dynamics and the best-response dynamics, elimination of all strategies used in correlated equilibrium is shown to be robust to the addition of mixed strategies as new pure strategies.

JEL classification: C73; C72

Keywords: Correlated equilibrium; Evolutionary dynamics; Elimination; As-if rationality

1. Introduction

A number of positive connections have been found between Nash equilibria and the outcome of evolutionary dynamics. For instance, for a wide class of dynamics, if a solution converges to a point from an interior initial condition, then this point is a Nash equilibrium (Weibull, 1995). However, solutions of evolutionary dynamics need not converge and may cycle away from the set of Nash equilibria (Zeeman, 1980; Hofbauer and Sigmund, 1998).

Since the set of correlated equilibria of a game is often much larger than its set of Nash equilibria, it might be hoped that correlated equilibria better capture the outcome of evolutionary dynamics than Nash equilibria. This hope is reinforced by the recent literature on adaptive processes converging, in a time-average sense, to the set of correlated equilibria (Hart, 2005).

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It was found, however, that there are games for which, for some initial conditions, the replicator dynamics eliminate all strategies belonging to the support of at least one correlated equilibrium (Viossat, 2007). Thus, only strategies that do not take part in any equilibrium remain, ruling out convergence of any kind of time-average to the set of correlated equilibria.

The purpose of this article is to show, on a 4×4 example, that elimination of all strategies used in correlated equilibrium does not only occur under the replicator dynamics and for very specific games, but for many dynamics and for an open set of games. We also study the robustness of this result when agents are explicitly allowed to use mixed strategies.

The article is organized as follows. After presenting the framework and notations, we introduce the games we consider and explain the technique used to show that all strategies used in correlated equilibrium are eliminated (Section 2). Sections 3–5 deal in turn with monotonic or weakly sign-preserving dynamics, the best-response dynamics and the Brown–von Neumann–Nash dynamics. Section 6 and the Appendix show that elimination of all strategies used in correlated equilibrium still occurs when agents are explicitly allowed to play mixed strategies. Section 7 concludes.

Framework and notation. We study single-population dynamics in two-player, finite symmetric games. The set of pure strategies is $I = \{1, 2, \ldots, N\}$ and S_N denotes the simplex of mixed strategies (henceforth, "the simplex"). Its vertices $\mathbf{e}_i$, $1 \leq i \leq N$, correspond to the pure strategies of the game. We denote by $x_i(t)$ the proportion of the population playing strategy i at time t and by $\mathbf{x}(t) = (x_1(t), \ldots, x_N(t)) \in S_N$ the population profile (or mean strategy). We study its evolution under dynamics of type $\dot{\mathbf{x}}(t) = f(\mathbf{x}(t), \mathbf{U})$, where $\mathbf{U} = (u_{ij})_{1 \leq i,j \leq N}$ is the payoff matrix of the game. We often skip the indication of time. For every $\mathbf{x}$ in S_N, the probability distribution on $I \times I$ induced by $\mathbf{x}$ is denoted by $\mathbf{x} \otimes \mathbf{x}$. If A is a subset of S_N, then $\mathrm{conv}(A)$ denotes its convex hull.

We assume known the definition of a correlated equilibrium distribution (Aumann, 1974) and, with a slight abuse of vocabulary, we write throughout correlated equilibrium for correlated equilibrium distribution. A pure strategy i is *used in correlated equilibrium* if there exists a correlated equilibrium μ under which strategy i has positive marginal probability (since the game is symmetric, whether we restrict attention to symmetric correlated equilibria or not is irrelevant; see footnote 2 in Viossat (2007)). Finally, the pure strategy i is *eliminated* (for a given solution $\mathbf{x}(\cdot)$ of a given dynamics) if $x_i(t) \to 0$ as $t \to +\infty$.

2. A family of games with a unique correlated equilibrium

The games considered in Viossat (2007) were 4×4 symmetric games with payoff matrix

$$\mathbf{U}_\alpha = \left(\begin{array}{ccc|c} 0 & -1 & \varepsilon & -\alpha \\ \varepsilon & 0 & -1 & -\alpha \\ -1 & \varepsilon & 0 & -\alpha \\ \hline \frac{-1+\varepsilon}{3} + \alpha & \frac{-1+\varepsilon}{3} + \alpha & \frac{-1+\varepsilon}{3} + \alpha & 0 \end{array}\right) \tag{1}$$

with ε in $]0, 1[$, and $0 < \alpha < (1-\varepsilon)/3$. The 3×3 game obtained by omitting the fourth strategy is a Rock–Paper–Scissors game (RPS). This game has a unique Nash equilibrium: $(1/3, 1/3, 1/3)$, which is also the unique correlated equilibrium. When $\alpha = 0$, the fourth strategy of the full game earns the same payoff as $\mathbf{n} = (1/3, 1/3, 1/3, 0)$, and there is a segment of symmetric Nash equilibria : for every $\mathbf{x} \in [\mathbf{n}, \mathbf{e}_4] = \{\lambda \mathbf{n} + (1-\lambda)\mathbf{e}_4, \lambda \in [0, 1]\}$, $(\mathbf{x}, \mathbf{x})$ is a Nash equilibrium. For $\alpha > 0$, $\mathbf{e}_4$ earns more than $\mathbf{n}$, so $(\mathbf{e}_4, \mathbf{e}_4)$ is a strict Nash equilibrium, and the unique correlated

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equilibrium is $\mathbf{e}_4 \otimes \mathbf{e}_4$. However, for α small enough, the best-response cycle $\mathbf{e}_1 \to \mathbf{e}_2 \to \mathbf{e}_3 \to \mathbf{e}_1$ remains and the corresponding set :

$$\Gamma = \{\mathbf{x} \in S_4 : x_4 = 0 \text{ and } x_1 x_2 x_3 = 0\}. \tag{2}$$

is asymptotically stable under the replicator dynamics

$$\dot{x}_i(t) = x_i(t)\left[(\mathbf{U}\mathbf{x}(t))_i - \mathbf{x}(t) \cdot \mathbf{U}\mathbf{x}(t)\right].$$

It follows that there exist games for which, for an open set of initial conditions, the replicator dynamics eliminate all strategies used in correlated equilibrium (Viossat, 2007).

This article shows that elimination of all strategies used in correlated equilibrium does not only occur for non-generic games and the replicator dynamics, but for an open set of games and many other dynamics. This is done by showing that, for many dynamics, there are values of α and ε such that, for every game in a neighborhood of (1):

(i) the unique correlated equilibrium is $\mathbf{e}_4 \otimes \mathbf{e}_4$;

(ii) for an open set of initial conditions, strategy 4 is eliminated.

Point (i) is the object of the following proposition:

Proposition 2.1. *For every ε in $]0,1[$ and every α in $]0,(1-\varepsilon)/3[$, every game in the neighborhood of (1) has a unique correlated equilibrium: $\mathbf{e}_4 \otimes \mathbf{e}_4$.*

Proof. Since the set of games with a unique correlated equilibrium is open (Viossat, 2008) and game (1) has a unique correlated equilibrium (Viossat, 2007), it follows that every game in a neighborhood of (1) has a unique correlated equilibrium. Since $\mathbf{e}_4 \otimes \mathbf{e}_4$ is clearly a correlated equilibrium of every game sufficiently close to (1), the result follows. ■

To prove (ii), a first method is to show that in (1), and every nearby game, the cyclic attractor of the underlying RPS game is still asymptotically stable. This is the method we use for monotonic dynamics and for weakly sign-preserving dynamics. When in the underlying RPS game the attractor is not precisely known, but the Nash equilibrium is repelling, another method may be used. It consists in showing that there is a tube surrounding the segment $[\mathbf{n}, \mathbf{e}_4]$ which repels solutions and such that outside of this tube, x_4 decreases along all trajectories. We use this method for the Brown–von Neumann–Nash dynamics. For the best-response dynamics, both methods work.

3. Monotonic or weakly sign-preserving dynamics

We first need some definitions. Consider a dynamics of the form

$$\dot{x}_i = x_i g_i(\mathbf{x}) \tag{3}$$

where the C^1 functions g_i have the property that $\sum_{i \in I} x_i g_i(\mathbf{x}) = 0$ for all $\mathbf{x}$ in S_4, so that the simplex S_4 and its boundary faces are invariant. Such a dynamics is *monotonic* if the growth rates of the different strategies are ranked according to their payoffs:[1]

$$g_i(\mathbf{x}) > g_j(\mathbf{x}) \Leftrightarrow (\mathbf{U}\mathbf{x})_i > (\mathbf{U}\mathbf{x})_j \quad \forall i \in I, \forall j \in I.$$

It is *weakly sign-preserving* (WSP) (Ritzberger and Weibull, 1995) if whenever a strategy earns below average, its growth rate is negative:

[1] This property goes under various names in the literature: *relative monotonicity* in Nachbar (1990), *order-compatibility* of pre-dynamics in Friedman (1991), *monotonicity* in Samuelson and Zhang (1992), which we follow, and *payoff monotonicity* in Hofbauer and Weibull (1996).

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$[(\mathbf{Ux})_i < \mathbf{x} \cdot \mathbf{Ux}] \Rightarrow g_i(\mathbf{x}) < 0.$

Dynamics[2] of type (3) implicitly depend on the payoff matrix $\mathbf{U}$. Thus, a more correct writing of (3) would be: $\dot{x}_i = x_i g_i(\mathbf{x}, \mathbf{U})$. Such a dynamics *depends continuously on the payoff matrix* if, for every i in I, g_i depends continuously on $\mathbf{U}$. A prime example of a dynamics of type (3) which is monotonic, WSP, and depends continuously on the payoff matrix is the replicator dynamics.

Finally, a closed subset C of S_4 is *asymptotically stable* if it is both:

(a) Lyapunov stable: for every neighborhood N_1 of C, there exists a neighborhood N_2 of C such that, for every initial condition $\mathbf{x}(0)$ in N_2, $\mathbf{x}(t) \in N_1$ for all $t \geq 0$.

(b) locally attracting: there exists a neighborhood N of C such that, for every initial condition $\mathbf{x}(0)$ in N, $\min_{c \in C} \|\mathbf{x}(t) - c\| \to_{t \to +\infty} 0$ (where $\| \cdot \|$ is any norm on $\mathbb{R}^I$).

Proposition 3.1. *Fix a monotonic or WSP dynamics* (3) *that depends continuously on the payoff matrix. For every* α *in* $]0, 1/3[$, *there exists* $\varepsilon > 0$ *such that for every game in the neighborhood of* (1), *the set* Γ *defined by* (2) *is asymptotically stable.*

Proof for monotonic dynamics. Consider a monotonic dynamics (3). Under this dynamics, for every game in the neighborhood of (1), the set Γ is a heteroclinic cycle. That is, a set consisting of saddle rest points and the saddle orbits connecting these rest points. Thus we may use the asymptotic stability's criteria for heteroclinic cycles developed by Hofbauer (1994) (a more accessible reference for this result is Theorem 17.5.1 in Hofbauer and Sigmund (1998)). Specifically, associate with the heteroclinic cycle Γ its so-called characteristic matrix. That is, the 3×4 matrix whose entry in row i and column j is $g_j(\mathbf{e}_i)$ (for $i \neq j$, this is the eigenvalue in the direction of $\mathbf{e}_j$ of the linearization of the vector field at $\mathbf{e}_i$):

	1	2	3	4
$\mathbf{e}_1$	0	$g_2(\mathbf{e}_1)$	$g_3(\mathbf{e}_1)$	$g_4(\mathbf{e}_1)$
$\mathbf{e}_2$	$g_1(\mathbf{e}_2)$	0	$g_3(\mathbf{e}_2)$	$g_4(\mathbf{e}_2)$
$\mathbf{e}_3$	$g_1(\mathbf{e}_3)$	$g_2(\mathbf{e}_3)$	0	$g_4(\mathbf{e}_3)$

($g_i(\mathbf{e}_i) = 0$ because $\mathbf{e}_i$ is a rest point of (3)).

Call $\mathbf{C}$ this matrix. If $\mathbf{p}$ is a real vector, let $\mathbf{p} < 0$ (resp. $\mathbf{p} > 0$) mean that all coordinates of $\mathbf{p}$ are negative (resp. positive). Hofbauer (1994) shows that if the following conditions are both satisfied, then Γ is asymptotically stable:

$$\text{There exists a vector } \mathbf{p} \text{ in } \mathbb{R}^4 \text{ such that } \mathbf{p} > 0 \text{ and } \mathbf{Cp} < 0. \tag{4}$$

$$\Gamma \text{ is asymptotically stable within the boundary of } S_4.^{3} \tag{5}$$

Therefore, to prove Proposition 3.1, it is enough to show that for every α in $]0, 1/3[$, there exists $\varepsilon > 0$ such that, for every game in the neighborhood of (1), conditions (4) and (5) are satisfied. We begin with a lemma. In the remainder of this section, $i \in \{1, 2, 3\}$ and $i - 1$ and $i + 1$ are counted modulo 3.

[2] Instead of dynamics of type (3), Ritzberger and Weibull (1995) consider dynamics of the more general type $\dot{x}_i = h_i(\mathbf{x})$, that need not leave the faces of the simplex positively invariant. Thus, we only consider a subclass of their WSP dynamics.

[3] That is, for each proper face (subsimplex) F of S_4, if $\Gamma \cap F$ is nonempty, then it is asymptotically stable for the dynamics restricted to F.

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Lemma 3.2. *For every α in* $]0, 1/3[$, *there exists* $\varepsilon > 0$ *such that in game* (1), *for every* i *in* $\{1, 2, 3\}$,

$$g_4(\mathbf{e}_i) < 0 \quad \textit{and} \quad 0 < g_{i+1}(\mathbf{e}_i) < -g_{i-1}(\mathbf{e}_i). \tag{6}$$

Proof. For $\varepsilon > 0$, at the vertex $\mathbf{e}_i$, the payoff of strategy 4 (resp. $i + 1$) is strictly smaller (greater) than the payoff of strategy i. Since the growth rate of strategy i at $\mathbf{e}_i$ is 0, this implies by monotonicity $g_4(\mathbf{e}_i) < 0$ (resp. $g_{i+1}(\mathbf{e}_i) > 0$). It remains to show that $g_{i+1}(\mathbf{e}_i) < -g_{i-1}(\mathbf{e}_i)$. For $\varepsilon = 0$, we have: $(\mathbf{U}\mathbf{e}_i)_i = (\mathbf{U}\mathbf{e}_i)_{i+1} > (\mathbf{U}\mathbf{e}_i)_{i-1}$ so that $0 = g_{i+1}(\mathbf{e}_i) > g_{i-1}(\mathbf{e}_i)$. Therefore $g_{i+1}(\mathbf{e}_i) < -g_{i-1}(\mathbf{e}_i)$ and since the dynamics depends continuously on the payoff matrix, this still holds for small positive ε. ■

We now prove Proposition 3.1. Fix α and ε as in Lemma 3.2. Note that since the dynamics we consider depends continuously on the payoff matrix, there exists a neighborhood of the game (1) in which the strict inequalities (6) still hold. Thus, to prove Proposition 3.1, it suffices to show that (6) implies (4) and (5).

(6) ⇒ (4): It follows from (6) that if $p_1 = p_2 = p_3 = 1$ and $p_4 > 0$, then $\mathbf{Cp} < 0$. Therefore, condition (4) is satisfied.

(6) ⇒ (5): We use again characteristic matrices. Let $\hat{\mathbf{C}}$ denote the 3×3 matrix obtained from $\mathbf{C}$ by omitting the fourth column. This corresponds to the characteristic matrix of Γ, when viewed as a heteroclinic cycle of the underlying 3×3 RPS game. In this RPS game, the set Γ is trivially asymptotically stable on the relative boundary of S_3 (Γ *is* the relative boundary!). Furthermore, for $\hat{\mathbf{p}} = (1/3, 1/3, 1/3) > 0$, the last inequality in (6) implies that $\hat{\mathbf{C}}\hat{\mathbf{p}} < 0$. Therefore, it follows from Theorem 1 of Hofbauer (1994) that, in the 4×4 initial game, Γ is asymptotically stable on the face spanned by $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$. Asymptotic stability on the face spanned by $\mathbf{e}_i, \mathbf{e}_{i+1}, \mathbf{e}_4$ follows easily from the following facts : on this face, $\mathbf{e}_{i+1}$ is a sink, $\mathbf{e}_i$ a saddle, every solution starting in $]\mathbf{e}_i, \mathbf{e}_{i+1}]$ converges to $\mathbf{e}_{i+1}$, and $\mathbf{x}(t)$ depends smoothly on $\mathbf{x}(0)$. This concludes the proof. ■

Proof of Proposition 3.1 for WSP dynamics. The proof is exactly the same, except for the proof of Lemma 3.2, which is as follows: Fix a WSP dynamics (3). For concreteness, set $i = 2$. At $\mathbf{e}_2$, strategy 4 earns less than average. Therefore $g_4(\mathbf{e}_2) < 0$. Now consider the case $\varepsilon = 0$: at every point $\mathbf{x}$ in the relative interior of the edge $[\mathbf{e}_1, \mathbf{e}_2]$, strategy 3 earns strictly less than average hence its growth rate is negative. By continuity at $\mathbf{e}_2$ this implies $g_3(\mathbf{e}_2) \leq 0$. Since at $\mathbf{e}_2$, strategy 1 earns strictly less than average, it follows that $g_1(\mathbf{e}_2) < 0$, hence $g_3(\mathbf{e}_2) < -g_1(\mathbf{e}_2)$. Since the dynamics depends continuously on the payoff matrix, this still holds for small positive ε.

To establish (6), it suffices to show that $g_3(\mathbf{e}_2)$ is positive for every sufficiently small positive ε. Let $\varepsilon > 0$. If $\lambda > 0$ is sufficiently small then, for all $\mu > 0$ small enough, the unique strategy which earns weakly above average at $\mathbf{x} = (\lambda\mu, 1 - \mu - \lambda\mu, \mu, 0)$ is strategy 3, hence $g_i(\mathbf{x}) < 0$ for $i \neq 3$. Since $\sum_{1 \leq i \leq 4} x_i g_i(\mathbf{x}) = 0$, it follows that $x_1 g_1(\mathbf{x}) + x_3 g_3(\mathbf{x}) > 0$, hence $\lambda\mu g_1(\mathbf{x}) + \mu g_3(\mathbf{x}) > 0$, hence $g_3(\mathbf{x}) > -\lambda g_1(\mathbf{x})$. Letting μ go to zero, we obtain $g_3(\mathbf{e}_2) \geq -\lambda g_1(\mathbf{e}_2) > 0$ ($g_1(\mathbf{e}_2) < 0$ was proved in the previous paragraph). ■

4. Best-response dynamics

4.1. Main result

The best-response dynamics (Gilboa and Matsui, 1991; Matsui, 1992) is given by the differential inclusion:

$$\dot{\mathbf{x}}(t) \in BR(\mathbf{x}(t)) - \mathbf{x}(t), \tag{7}$$

where $BR(\mathbf{x})$ is the set of best responses to $\mathbf{x}$:

$$BR(\mathbf{x}) = \{\mathbf{y} \in S_N : \mathbf{y} \cdot \mathbf{Ux} = \max_{\mathbf{z} \in S_N} \mathbf{z} \cdot \mathbf{Ux}\}.$$

A solution $\mathbf{x}(\cdot)$ of the best-response dynamics is an absolutely continuous function satisfying (7) for almost every t. For the games and the initial conditions that we will consider, there is a unique solution starting from each initial condition. [4]

Consider a 4×4 symmetric game with payoff matrix $\mathbf{U}$. Let

$$V(\mathbf{x}) := \max_{1 \le i \le 3} \left[(\mathbf{Ux})_i - \sum_{1 \le i \le 4} u_{ii} x_i \right] \quad \text{and} \quad W(\mathbf{x}) := \max(x_4, |V(\mathbf{x})|). \tag{8}$$

For every game sufficiently close to (1), the set

$$ST := \{\mathbf{x} \in S_4 : W(\mathbf{x}) = 0\} \tag{9}$$

is a triangle, which, following Gaunersdorfer and Hofbauer (1995), we call the Shapley triangle.

Proposition 4.1. *For every game sufficiently close to* (1)*, if strategy* 4 *is not a best response to* $\mathbf{x}(0)$*, then for all* $t \ge 0$*,* $\mathbf{x}(t)$ *is uniquely defined, and* $\mathbf{x}(t)$ *converges to the Shapley triangle* (9) *as* $t \to +\infty$*.*

Proof. We begin with a lemma, which is the continuous time version of the improvement principle of Monderer and Sela (1997):

Lemma 4.2 (*Improvement Principle*). *Let* $t_1 < t_2$*, let* $\mathbf{b}$ *be a best response to* $\mathbf{x}(t_1)$ *and let* $\mathbf{b}' \in S_4$*. Assume that* $\dot{\mathbf{x}} = \mathbf{b} - \mathbf{x}$ *(hence the solution points towards* $\mathbf{b}$*) for all* t *in* $]t_1, t_2[$*. If* $\mathbf{b}'$ *is a best response to* $\mathbf{x}(t_2)$ *then* $\mathbf{b}' \cdot \mathbf{Ub} \ge \mathbf{b} \cdot \mathbf{Ub}$*, with strict inequality if* $\mathbf{b}'$ *is not a best response to* $\mathbf{x}(t_1)$*.*

Proof of Lemma 4.2. Between t_1 and t_2, the solution points towards b. Therefore there exists λ in $]0, 1[$ such that

$$\mathbf{x}(t_2) = \lambda \mathbf{x}(t_1) + (1 - \lambda)\mathbf{b}. \tag{10}$$

If $\mathbf{b}'$ is a best response to $\mathbf{x}(t_2)$ then $(\mathbf{b}' - \mathbf{b}) \cdot \mathbf{Ux}(t_2) \ge 0$ so that, substituting the right-hand side of (10) for $\mathbf{x}(t_2)$, we get:

$$(1 - \lambda)(\mathbf{b}' - \mathbf{b}) \cdot \mathbf{Ub} \ge \lambda(\mathbf{b} - \mathbf{b}') \cdot \mathbf{Ux}(t_1). \tag{11}$$

Since $\mathbf{b}$ is a best response to $\mathbf{x}(t_1)$, the right-hand side of (11) is nonnegative, and positive if $\mathbf{b}'$ is not a best response to $\mathbf{x}(t_1)$. The result follows. ■

Proof of Proposition 4.1 for game (1). Fix a solution $\mathbf{x}(\cdot)$ of (7) such that strategy 4 is not a best response to $\mathbf{x}(0)$. Note that for any $\mathbf{x}$ in S_4, $BR(\mathbf{x}) \neq \text{conv}(\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\})$, because $\mathbf{e}_4$ strictly dominates $(1/3, 1/3, 1/3, 0)$. Thus, either there is a unique best response to $\mathbf{x}(0)$ or, counting i modulo 3, $BR(\mathbf{x}(0)) = \text{conv}(\{\mathbf{e}_i, \mathbf{e}_{i+1}\})$ for some i in $\{1, 2, 3\}$. Assume for concreteness that strategy 1 is the unique best response to $\mathbf{x}(0)$. The solution then initially points towards $\mathbf{e}_1$, until some other pure strategy becomes a best response. Due to the improvement principle (Lemma 4.2), this strategy can only be strategy 2. Thus, the solution must then point towards the edge $[\mathbf{e}_1, \mathbf{e}_2]$. Since strategy 2 strictly dominates strategy 1 in the game restricted to $\{1, 2\} \times \{1, 2\}$, strategy 2 immediately becomes the unique best response. Iterating this argument, we see that

[4] We focus on forward time and never study whether a solution is uniquely defined in backward time.

the solution will point towards $\mathbf{e}_2$, till 3 becomes a best response, then towards $\mathbf{e}_3$, till 1 becomes a best response again, and so on.

To show that this behaviour continues for ever, it suffices to show that the times at which the direction of the trajectory changes do not accumulate. This is the object of the following claim, which will be proved in the end:

Claim 4.3. *The time length between two successive times when the direction of* $\mathbf{x}(t)$ *changes is bounded away from zero.*

Now recall (8), and note that for game (1) the terms u_{ii} are zero, so that $V(\mathbf{x}) = \max_{1\le i\le 3}(\mathbf{Ux})_i$. Let $v(t) := V(\mathbf{x}(t))$, $w(t) := W(\mathbf{x}(t))$. When $\mathbf{x}(t)$ points towards $\mathbf{e}_i$ (with i in $\{1, 2, 3\}$), we have $\dot{x}_4 = -x_4$ and

$$\dot{v} = (\mathbf{U}\dot{\mathbf{x}})_i = (\mathbf{U}(\mathbf{e}_i - \mathbf{x}))_i = -v. \tag{12}$$

Therefore $\dot{w} = -w$. Since for almost all time t, $\mathbf{x}(t)$ points towards $\mathbf{e}_1$, $\mathbf{e}_2$ or $\mathbf{e}_3$, it follows that $w(t)$ decreases exponentially to 0. Therefore $\mathbf{x}(t)$ converges to the Shapley triangle. □

To complete the proof, we still need to prove Claim 4.3:

Proof of Claim 4.3. In what follows $i \in \{1, 2, 3\}$ and $i + 1$ is counted modulo 3. Fix an initial condition and let

$$g(t) := \max_{1\le i,j\le 3}\left[(\mathbf{Ux}(t))_i - (\mathbf{Ux}(t))_j\right]$$

denote the maximum difference between the payoffs of strategies in $\{1, 2, 3\}$. Let t_i^k denote the kth time at which strategy i becomes a best response and choose i such that $t_i^k < t_{i+1}^k$. Simple computations, detailed in (Viossat, 2006, p. 11–12), show that:

$$g(t_i^{k+1}) = \frac{1}{\varepsilon^3 + g(t_i^k)(1+\varepsilon+\varepsilon^2)}\,g(t_i^k). \tag{13}$$

Since $\varepsilon < 1$, it follows that for small $g(t_i^k)$, we have $g(t_i^{k+1}) > g(t_i^k)$; therefore $g(t_i^k)$ is bounded away from zero. Now, since $(\mathbf{Ux}(t))_i - (\mathbf{Ux}(t))_{i+1}$ decreases from $g(t_i^k)$ to 0 between t_i^k and t_{i+1}^k, and since the speed at which this quantity varies is bounded, it follows that $t_{i+1}^k - t_i^k$ is bounded away from zero too. That is, the time length between two successive times at which the direction of $\mathbf{x}(t)$ changes is bounded away from zero. ■

Proof of Proposition 4.1 for games close to (1). Counting i modulo 3, let $\alpha_i = u_{ii} - u_{i-1,i}$ and $\beta_i = u_{i+1,i} - u_{ii}$, $i = 1, 2, 3$. Let $i \in \{1, 2, 3\}$. For every game sufficiently close to (1), α_i and β_i are positive, $\alpha_1\alpha_2\alpha_3 > \beta_1\beta_2\beta_3$, $u_{4i} < u_{ii}$, and strategy 4 strictly dominates $(1/3, 1/3, 1/3, 0)$. Furthermore, for every game satisfying these conditions, the proof of Proposition 4.1 for game (1) goes through. The only differences are that Eq. (12) becomes

$$\dot{v} = (\mathbf{U}\dot{\mathbf{x}})_i - \sum_{1\le j\le 4} u_{jj}\dot{x}_j = (\mathbf{U}(\mathbf{e}_i - \mathbf{x}))_i - \left(u_{ii} - \sum_{1\le j\le 4} u_{jj}x_j\right) = -v$$

and Eq. (13) becomes

$$g(t_i^{k+1}) = \frac{\alpha_1\alpha_2\alpha_3}{\beta_1\beta_2\beta_3 + g(t_i^k)(\alpha_i\alpha_{i+1} + \alpha_i\beta_{i+2} + \beta_{i+1}\beta_{i+2})}\,g(t_i^k).$$

See Viossat (2006) for details. This completes the proof. ■

Note that for every $\eta > 0$, we may set the parameters of (1) so that the set $\{\mathbf{x} \in S_4 : \mathbf{e}_4 \in BR(\mathbf{x})\}$ has Lebesgue measure less than η. In this sense, the basin of attraction of the Shapley triangle can be made arbitrarily large. Similarly, for the replicator dynamics, the basin of attraction of the heteroclinic cycle (2) can be made arbitrarily large (Viossat, 2007). For additional results on the best-response dynamics and the replicator dynamics in 4×4 games based on a RPS game, see Viossat (2006).

5. Brown–von Neumann–Nash dynamics

The Brown–von Neumann–Nash dynamics (henceforth BNN) is given by:

$$\dot{x}_i = k_i(\mathbf{x}) - x_i \sum_{j \in I} k_j(\mathbf{x}) \tag{14}$$

where

$$k_i(\mathbf{x}) := \max(0, (\mathbf{U}\mathbf{x})_i - \mathbf{x} \cdot \mathbf{U}\mathbf{x}) \tag{15}$$

is the excess payoff of strategy i over the average payoff. We refer to Hofbauer (2000), Berger and Hofbauer (2006) and the references therein for a motivation of and results on BNN.

Let G_0 denote the game (1) with $\alpha = 0$. Recall that $\mathbf{U}_0$ denotes its payoff matrix and $\mathbf{n} = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0\right)$ the mixed strategy corresponding to the Nash equilibrium of the underlying RPS game. It may be shown that the set of symmetric Nash equilibria of G_0 is the segment $E_0 = [\mathbf{n}, \mathbf{e}_4]$.[5] This section is devoted to a proof of the following proposition:

Proposition 5.1. *If C is a closed subset of S_4 disjoint from E_0, then there exists a neighborhood of G_0 such that, for every game in this neighborhood and every initial condition in C, $x_4(t) \to 0$ as $t \to +\infty$.*

Any neighborhood of G_0 contains a neighborhood of a game of kind (1), hence an open set of games for which $\mathbf{e}_4 \otimes \mathbf{e}_4$ is the unique correlated equilibrium. Together with Proposition 5.1, this implies that there exists an open set of games for which, under BNN, the unique strategy played in correlated equilibrium is eliminated from an open set of initial conditions.

The essence of the proof of Proposition 5.1 is to show that, for games close to G_0, there is a "tube" surrounding E_0 such that: (i) the tube repels solutions coming from outside; (ii) outside of the tube, strategy 4 earns less than average, hence x_4 decreases. We first show that in G_0 the segment E_0 is locally repelling.

The function

$$V_0(\mathbf{x}) := \frac{1}{2} \sum_{i \in I} k_i^2 = \frac{1}{2} \sum_{i \in I} [\max(0, (\mathbf{U}_0\mathbf{x})_i - \mathbf{x} \cdot \mathbf{U}_0\mathbf{x})]^2$$

is continuous, nonnegative and equals 0 exactly on the symmetric Nash equilibria, i.e. on E_0, so that $V_0(\mathbf{x})$ may be seen as a distance from $\mathbf{x}$ to E_0. Fix an initial condition and let $v_0(t) := V_0(\mathbf{x}(t))$.

Lemma 5.2. *There exists an open neighborhood N_{eq} of E_0 such that, under BNN in the game G_0, $\dot{v}_0(t) > 0$ whenever $\mathbf{x}(t) \in N_{eq} \setminus E_0$.*

[5] The game G_0 has other, asymmetric equilibria, but they will play no role.

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Proof. It is easily checked that:

$$\mathbf{n} \cdot \mathbf{U}_0\mathbf{x} = \mathbf{e}_4 \cdot \mathbf{U}_0\mathbf{x} \quad \forall \mathbf{x} \in S_4 \tag{16}$$

(that is, $\mathbf{n}$ and $\mathbf{e}_4$ always earn the same payoff) and

$$(\mathbf{x} - \mathbf{x}') \cdot \mathbf{U}_0\mathbf{e}_4 = (\mathbf{x} - \mathbf{x}') \cdot \mathbf{U}_0\mathbf{n} = 0 \quad \forall \mathbf{x} \in S_4, \forall \mathbf{x}' \in S_4 \tag{17}$$

(that is, against $\mathbf{e}_4$ [resp. $\mathbf{n}$], all strategies earn the same payoff). Furthermore, as it follows from lemma 4.1 in Viossat (2007), for every $\mathbf{p}$ in E_0 and every $\mathbf{x} \notin E_0$,

$$(\mathbf{x} - \mathbf{p}) \cdot \mathbf{U}_0\mathbf{x} = (\mathbf{x} - \mathbf{p}) \cdot \mathbf{U}_0(\mathbf{x} - \mathbf{p}) = \frac{1-\varepsilon}{2} \sum_{1 \le i \le 3} \left(x_i - \frac{1 - x_4}{3} \right)^2 > 0. \tag{18}$$

Hofbauer (2000) shows that the function v_0 satisfies

$$\dot{v}_0 = \bar{k}^2 \left[(\mathbf{q} - \mathbf{x}) \cdot \mathbf{U}_0(\mathbf{q} - \mathbf{x}) - (\mathbf{q} - \mathbf{x}) \cdot \mathbf{U}_0\mathbf{x} \right] \tag{19}$$

with $\mathbf{x} = \mathbf{x}(t)$, $\bar{k} = \sum_i k_i$ and $q_i = k_i/\bar{k}$. It follows from Eq. (17) that if $\mathbf{p} \in E_0$, then against $\mathbf{p}$ all strategies earn the same payoff. Therefore the second term $(\mathbf{q} - \mathbf{x}) \cdot \mathbf{U}_0\mathbf{x}$ goes to 0 as $\mathbf{x}$ approaches E_0. Thus, to prove Lemma 5.2, it suffices to show that as $\mathbf{x}$ approaches E_0, the first term $(\mathbf{q} - \mathbf{x}) \cdot \mathbf{U}_0(\mathbf{q} - \mathbf{x})$ is positive and bounded away from 0. But for $\mathbf{x} \notin E_0$,

$$\min_{1 \le i \le 3} (\mathbf{U}_0\mathbf{x})_i \le \mathbf{n} \cdot \mathbf{U}_0\mathbf{x} = (\mathbf{U}_0\mathbf{x})_4 < \mathbf{x} \cdot \mathbf{U}_0\mathbf{x} \tag{20}$$

(the first inequality holds because $\mathbf{n}$ is a convex combination of $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$, the equality follows from (16) and the strict inequality from (18) applied to $\mathbf{p} = \mathbf{e}_4$). It follows from $(\mathbf{U}_0\mathbf{x})_4 < \mathbf{x} \cdot \mathbf{U}_0\mathbf{x}$ that $k_4 = 0$ hence $q_4 = 0$; similarly, it follows from $\min_{1 \le i \le 3}(\mathbf{U}_0\mathbf{x})_i < \mathbf{x} \cdot \mathbf{U}_0\mathbf{x}$ that $q_i = 0$ for some i in $\{1, 2, 3\}$. Together with (18) applied to $\mathbf{x} = \mathbf{q}$, this implies that for every $\mathbf{p}$ in E_0,

$$(\mathbf{q} - \mathbf{p}) \cdot \mathbf{U}_0(\mathbf{q} - \mathbf{p}) = \frac{1-\varepsilon}{2} \sum_{1 \le i \le 3} \left(q_i - \frac{1}{3} \right)^2 \ge \frac{1-\varepsilon}{18}.$$

This completes the proof. ■

Proof of Proposition 5.1. Consider first the BNN dynamics in the game G_0. Recall Lemma 5.2 and let

$$0 < \delta < \min_{\mathbf{x} \in S_4 \setminus N_{eq}} V_0(\mathbf{x}) \tag{21}$$

(the latter is positive because V_0 is positive on $S_4 \setminus E_0$, hence on $S_4 \setminus N_{eq}$, and because $S_4 \setminus N_{eq}$ is compact). Note that if $V_0(\mathbf{x}) \le \delta$ then $\mathbf{x} \in N_{eq}$. Therefore it follows from Lemma 5.2 and $\delta > 0$ that

$$v_0(t) = \delta \Rightarrow \dot{v}_0(t) > 0. \tag{22}$$

Let

$$C_\delta := \{\mathbf{x} \in S_4 : V_0(\mathbf{x}) \ge \delta\}.$$

Since $\delta > 0$, the sets C_δ and E_0 are disjoint. Therefore, by (18) applied to $\mathbf{p} = \mathbf{e}_4$,

$$\mathbf{x} \in C_\delta \Rightarrow (\mathbf{U}_0\mathbf{x})_4 - \mathbf{x} \cdot \mathbf{U}_0\mathbf{x} < 0 \tag{23}$$

so that x_4 decreases strictly as long as $\mathbf{x} \in C_\delta$ and $x_4 > 0$. Since, by (22), the set C_δ is forward invariant, it follows that for any initial condition in C_δ, strategy 4 is eliminated.

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Now let $\nabla V_0(\mathbf{x}) = (\partial V_0/\partial x_i)_{1\le i\le n}(\mathbf{x})$ denote the gradient of V_0 at $\mathbf{x}$. It is easy to see that V_0 is C^1. Therefore it follows from (22), $\dot{v}_0(t) = \nabla V_0(\mathbf{x}(t)) \cdot \dot{\mathbf{x}}(t)$ and compactness of $\{\mathbf{x} \in S_4 : V_0(\mathbf{x}) = \delta\}$ that

$$\exists \gamma > 0, \quad [v_0(t) = \delta \Rightarrow \dot{v}_0(t) \ge \gamma > 0]. \tag{24}$$

Similarly, since C_δ is compact, it follows from (23) that there exists $\gamma' > 0$ such that

$$\mathbf{x} \in C_\delta \Rightarrow (\mathbf{U}_0\mathbf{x})_4 - \mathbf{x} \cdot \mathbf{U}_0\mathbf{x} \le -\gamma' < 0. \tag{25}$$

Since $\dot{\mathbf{x}}$ is Lipschitz in the payoff matrix, it follows from (24) that for $\mathbf{U}$ close enough to $\mathbf{U}_0$, we still have $v_0(t) = \delta \Rightarrow \dot{v_0} > 0$ under the perturbed dynamics. Similarly, due to (25), we still have $\mathbf{x} \in C_\delta \Rightarrow (\mathbf{U}\mathbf{x})_4 - \mathbf{x} \cdot \mathbf{U}\mathbf{x} < 0$. Therefore the above reasoning applies and for every initial condition in C_δ, strategy 4 is eliminated.

Note that δ can be chosen arbitrarily small (see (21)). Therefore, to complete the proof of Proposition 5.1, it suffices to show that if C is a compact set disjoint from E_0 then, for δ sufficiently small, $C \subset C_\delta$. But since V_0 is positive on $S_4 \setminus E_0$, and since C is compact and disjoint from E_0, it follows that there exists $\delta' > 0$ such that, for all $\mathbf{x}$ in C, $V_0(\mathbf{x}) \ge \delta'$; hence, for all $\delta \le \delta'$, $C \subset C_\delta$. This completes the proof. ■

Hofbauer (2000, section 6) considers the following generalization of the BNN dynamics:

$$\dot{x}_i = f(k_i) - x_i \sum_{j=1}^{n} f(k_j) \tag{26}$$

where $f : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function with $f(0) = 0$ and $f(u) > 0$ for $u > 0$, and where k_i is defined as in (15). The results of this section generalize to any such dynamics:

Proposition 5.3. *Consider a dynamics of type (26). If C is a closed subset of S_4 disjoint from E_0, then there exists a neighborhood of G_0 such that, for every game in this neighborhood and every initial condition in C, $x_4(t) \to 0$ as $t \to +\infty$.*

Proof. Replace $V_0(\mathbf{x})$ by $W_0(\mathbf{x}) := \sum_i F(k_i(\mathbf{x}))$, where F is an anti-derivative of f, and replace k_i by $f(k_i)$. Let $\bar{f} = \sum_i f(k_i)$, $\tilde{f}_i = f(k_i)/\bar{f}$, and $\tilde{\mathbf{f}} = \left(\tilde{f}_i\right)_{1\le i\le N}$. Finally, let $w_0(t) = W_0(\mathbf{x}(t))$. As shown by Hofbauer (2000),

$$\dot{w}_0 = \bar{f}^2 \left[(\tilde{\mathbf{f}} - \mathbf{x}) \cdot \mathbf{U}_0(\tilde{\mathbf{f}} - \mathbf{x}) - (\tilde{\mathbf{f}} - \mathbf{x}) \cdot \mathbf{U}_0\mathbf{x}\right]$$

which is the analogue of (19). Then apply exactly the same proof as for BNN. ■

6. Robustness to the addition of mixed strategies as new pure strategies

We showed that for many dynamics, there exists an open set of symmetric 4×4 games for which, from an open set of initial conditions, the unique strategy used in correlated equilibrium is eliminated. Since we might not want to rule out the possibility that individuals use mixed strategies, and that mixed strategies be heritable, it is important to check whether our results change if we explicitly introduce mixed strategies as new pure strategies of the game. The paradigm is the following (Hofbauer and Sigmund, 1998, section 7.2): there is an underlying normal-form game, called the base game, and a finite number of types of agents. Each type plays a pure or mixed strategy of the base game. We assume that each pure strategy of the base game is

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played (as a pure strategy) by at least one type of agent, but otherwise we make no assumptions on the agents' types. The question is whether we can nonetheless be sure that, for an open set of initial conditions, all strategies used in correlated equilibrium are eliminated. This section shows that the answer is positive, at least for the best-response dynamics and the replicator dynamics. We first need some notations and vocabulary.

Let G be a finite game with strategy set $I = \{1, \ldots, N\}$ and payoff matrix $\mathbf{U}$. A finite game G' is *built on G by adding mixed strategies as new pure strategies* if:

First, letting $I' = \{1, \ldots, N, N+1, \ldots, N'\}$ be the set of pure strategies of G' and $\mathbf{U}'$ its payoff matrix, we may associate to each pure strategy i in I' a mixed strategy $\mathbf{p}^i$ in S_N in such a way that:

$$\forall i \in I', \forall j \in I', \quad \mathbf{e}'_i \cdot \mathbf{U}'\mathbf{e}'_j = \mathbf{p}^i \cdot \mathbf{U}\mathbf{p}^j \tag{27}$$

where $\mathbf{e}'_i$ is the unit vector in $S_{N'}$ corresponding to the pure strategy i.

Second, if $1 \le i \le N$, the pure strategy i in the game G' corresponds to the pure strategy i in the base game G:

$$1 \le i \le N \Rightarrow \mathbf{p}^i = \mathbf{e}_i. \tag{28}$$

If $\mu' = (\mu'(k,l))_{1 \le k,l \le N'}$ is a probability distribution over $I' \times I'$, then it induces the probability distribution μ on $I \times I$ given by:

$$\mu(i,j) = \sum_{1 \le k,l \le N'} \mu'(k,l) p_i^k p_j^l \quad \forall (i,j) \in I \times I.$$

It follows from a version of the revelation principle (see Myerson (1994)) that, if G' is built on G by adding mixed strategies as new pure strategies, then for any correlated equilibrium μ' of G', the induced probability distribution on $I \times I$ is a correlated equilibrium of G. Thus, if G is a 4×4 symmetric game with $\mathbf{e}_4 \otimes \mathbf{e}_4$ as unique correlated equilibrium, then μ' is a correlated equilibrium of G' if and only if, for every k, l in I' such that $\mu'(k,l)$ is positive, $\mathbf{p}^k = \mathbf{p}^l = \mathbf{e}_4$. Thus, the unique strategy of G used in correlated equilibria of G' is strategy 4. We show below that:

Proposition 6.1. *For the replicator dynamics and for the best-response dynamics, there exists an open set of* 4×4 *symmetric games such that, for any game G in this set:*

(i) $\mathbf{e}_4 \otimes \mathbf{e}_4$ *is the unique correlated equilibrium of G*

(ii) *For any game G' built on G by adding mixed strategies as new pure strategies and for an open set of initial conditions, every pure strategy k in I' such that $p_4^k > 0$ is eliminated.*

(the open set of initial conditions in property (ii) is a subset of $S_{N'}$, the simplex of mixed strategies of G', and may depend on G')

6.1. Proof for the best-response dynamics

Let G be a finite game and let G' be a finite game built on G by adding mixed strategies of G as new pure strategies. Associate to each mixed strategy $\mathbf{x}'$ in $S_{N'}$ the induced mixed strategy $\mathbf{x}$ in S_N defined by:

$$\mathbf{x} := \sum_{k=1}^{N'} x'_k \mathbf{p}^k. \tag{29}$$

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Let $\mathbf{x}'(\cdot)$ be a solution of the best-response dynamics in G' and $\mathbf{x}(\cdot)$ the induced mapping from $\mathbb{R}_+$ to S_N.

Proposition 6.2. $\mathbf{x}(\cdot)$ *is a solution of the best-response dynamics in* G.

Proof. For almost all $t \geq 0$, there exists a vector $\mathbf{b}' \in BR(\mathbf{x}'(t))$ such that $\dot{\mathbf{x}}'(t) = \mathbf{b}' - \mathbf{x}'(t)$. Let $\mathbf{b} := \sum_{k \in I'} b'_k \mathbf{p}^k \in S_N$. It follows from (29) that:

$$\dot{\mathbf{x}}(t) = \sum_{k=1}^{N'} \dot{\mathbf{x}}'_k \mathbf{p}^k = \sum_{k=1}^{N'} (b'_k - x'_k)\mathbf{p}^k = \mathbf{b} - \mathbf{x}(t). \tag{30}$$

Furthermore, since $\mathbf{b}'$ is a best response to $\mathbf{x}'(t)$, it follows from (27) and (28) that $\mathbf{b}$ is a best response to $\mathbf{x}(t)$. Together with (30), this implies that, for almost all t, $\dot{\mathbf{x}} \in BR(\mathbf{x}) - \mathbf{x}$. The result follows. ■

Since

$$x_i(t) \to 0 \Rightarrow \left(\forall k \in N', \left[p_i^k > 0 \Rightarrow x'_k(t) \to 0\right]\right),$$

Proposition 6.1 follows from Propositions 4.1 and 6.2.

6.2. Proof for the replicator dynamics

Recall that G_0 denotes game (1) with $\alpha = 0$. Since, as already mentioned, every neighborhood of G_0 contains an open set of games with $\mathbf{e}_4 \otimes \mathbf{e}_4$ as unique correlated equilibrium, it suffices to show that every game close enough to G_0 satisfies property (ii) of Proposition 6.1. This is done in the Appendix.

The intuition is the following: first note that, for a game G close to G_0, the set Γ defined in (2) is an attractor, close to which strategy 4 earns less than average. Now consider a game G' built on G by adding mixed strategies as new pure strategies and let Γ' denote the subset of $S_{N'}$ corresponding to Γ:

$$\Gamma' = \{\mathbf{x}' \in S_{N'} : x'_1 + x'_2 + x'_3 = 1 \text{ and } x'_1 x'_2 x'_3 = 0\}.$$

For an initial condition close to Γ': (a) as long as the share of strategies $k \geq 4$ remains low, the solution remains close to Γ'; (b) as long as the solution is close to Γ', strategy 4 earns less than average and its share x_4 decreases; (c) as long as the share of strategy 4 does not increase, the share of strategies $k \geq 5$ remains low; moreover, if the share of strategy 4 decreases, so does, on average, the share of each added mixed strategy in which strategy 4 is played with positive probability.

Putting (a), (b) and (c) together gives the result.

7. Discussion

We showed that elimination of all strategies used in correlated equilibrium is a robust phenomenon, in that it occurs for many dynamics, an open set of games and an open set of initial conditions. Furthermore, at least for some of the leading dynamics, the results are robust to the addition of mixed strategies as new pure strategies. Under the replicator dynamics, the best-response dynamics or the Brown–von Neumann–Nash dynamics, the basin of attraction of the Nash equilibrium of (1) can be made arbitrarily small. In particular, the minimal distance

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from the cyclic attractor to the basin of attraction of the Nash equilibrium can be made much larger than the minimal distance from the Nash equilibrium to the basin of attraction of the cyclic attractor. The latter would thus be stochastically stable in a model à la Kandori et al. (1993).[6] These results show a sharp difference between evolutionary dynamics and "adaptive heuristics" such as no-regret dynamics (Hart and Mas-Colell, 2003; Hart, 2005) or hypothesis testing (Young, 2004, chapter 8).

Some limitations of our results should however be stressed. First, our results have been shown here only for single-population dynamics. They imply that for *some* games and *some* interior initial conditions, two-population dynamics eliminate all strategies used in correlated equilibrium[7]; but maybe not for an open set of games nor for an open set of initial conditions.

Second, the monotonic and weakly sign-preserving dynamics of Section 3 are non-innovative: strategies initially absent do not appear. This has the effect that, even when focusing on interior initial conditions, the growth of the share of the population playing strategy i is limited by the current value of this share. This is appropriate if we assume that agents have to meet an agent playing strategy i to become aware of the possibility of playing strategy i; but in general, as discussed by e.g. Swinkels (1993, p. 459), this seems more appropriate in biology than in economics. While our results hold also for some important innovative dynamics, such as the best-response dynamics and a family of dynamics including the Brown–von Neumann–Nash dynamics, more general results would be welcome.

Third, in the games we considered, the unique correlated equilibrium is a strict Nash equilibrium, and is thus asymptotically stable under most reasonable dynamics, including all those we studied. Thus, there is still an important connection between equilibria and the outcome of evolutionary dynamics.

For Nash equilibrium, these three limitations can be overcome, at least partially: there are wide classes of multi-population innovative dynamics for which there exists an open set of games such that, for an open set of initial conditions, all strategies belonging to the support of at least one Nash equilibrium are eliminated (Viossat, 2005, chapter 11). Moreover, for the single-population replicator dynamics or the single-population best-response dynamics, there are games for which, for almost all initial conditions, all strategies used in Nash equilibrium are eliminated (Viossat, 2005, chapter 12). Whether these results extend to correlated equilibrium is an open question.

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[6] The (unperturbed) dynamics used by Kandori et al. (1993) is a discrete-time version of the best-response dynamics, but it could be replaced by a discrete-time version of another dynamics.

[7] This is because for symmetric two-player games with symmetric initial conditions, two-population dynamics reduce to single-population dynamics, at least for the replicator dynamics, the best-response dynamics and the Brown–von Neumann–Nash dynamics.

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Appendix. Proof of Proposition 6.1 for the replicator dynamics

We need to show that for every game G close enough to G_0, property (ii) of Proposition 6.1 is satisfied. As in Section 5, let $E_0 = [\mathbf{n}, \mathbf{e}_4]$, with $\mathbf{n} = (1/3, 1/3, 1/3, 0)$. For $\mathbf{x}$ in $S_4 \setminus \{\mathbf{e}_4\}$, let

$$V(\mathbf{x}) := 3\frac{(x_1 x_2 x_3)^{1/3}}{x_1 + x_2 + x_3}.$$

The function V takes its maximal value 1 on $E_0 \setminus \{\mathbf{e}_4\}$ and its minimal value 0 on the set $\{\mathbf{x} \in S_4 \setminus \{\mathbf{e}_4\} : x_1 x_2 x_3 = 0\}$. Fix δ in $]0, 1[$. If $V(\mathbf{x}) \leq \delta$ then $\mathbf{x} \notin E_0$, hence it follows from (20) that, at $\mathbf{x}$, strategy 4 earns strictly less than average. Together with a compactness argument, this implies that there exists $\gamma_1 > 0$ such that:

$$V(\mathbf{x}) \leq \delta \Rightarrow [(\mathbf{U}_0\mathbf{x})_4 - \mathbf{x} \cdot \mathbf{U}_0\mathbf{x} \leq -\gamma_1]. \tag{31}$$

Furthermore, it is shown in Viossat (2007) that in G_0, under the replicator dynamics, the function $V(\mathbf{x})$ decreases strictly along interior trajectories (except those starting in E_0). More precisely, for every interior initial condition $\mathbf{x}(0) \notin E_0$ and every t in $\mathbb{R}$, the function $v_0(t) := V(\mathbf{x}(t))$ satisfies $\dot{v}_0(t) < 0$. Together with the compactness of $\{\mathbf{x} \in S_4 \setminus \{\mathbf{e}_4\}, V(\mathbf{x}) = \delta\}$, this implies that there exists $\gamma_2 > 0$ such that

$$v_0(t) = \delta \Rightarrow \dot{v}_0(t) \leq -\gamma_2. \tag{32}$$

Fix a 4×4 matrix $\mathbf{U}$ and a solution $\mathbf{x}(\cdot)$ of the replicator dynamics with payoff matrix $\mathbf{U}$, with $\mathbf{x}(0) \neq \mathbf{e}_4$. Let $v(t) := V(\mathbf{x}(t))$. Thus, the difference between v_0 and v is that, in the definition of v, $\mathbf{x}(\cdot)$ is a solution of the replicator dynamics for the payoff matrix $\mathbf{U}$ and not for $\mathbf{U}_0$. Since $(\mathbf{U}\mathbf{x})_4 - \mathbf{x} \cdot \mathbf{U}\mathbf{x}$ and $\dot{\mathbf{x}}$ are Lipschitz in $\mathbf{U}$, it follows from (31) and (32) that there exists $\gamma > 0$ such that, if $\|\mathbf{U} - \mathbf{U}_0\| < \gamma$:

$$V(\mathbf{x}) \leq \delta \Rightarrow [(\mathbf{U}\mathbf{x})_4 - \mathbf{x} \cdot \mathbf{U}\mathbf{x} \leq -\gamma] \tag{33}$$

and

$$v(t) = \delta \Rightarrow \dot{v}(t) \leq -\gamma. \tag{34}$$

Fix a game G with payoff matrix $\mathbf{U}$ such that $\|\mathbf{U} - \mathbf{U}_0\| < \gamma$. Let G' be a game built on G by adding mixed strategies of G as new pure strategies, and let $\mathbf{U}'$ be its payoff matrix. For $\mathbf{x}'$ in $S_{N'}$ such that $x'_1 + x'_2 + x'_3 > 0$, let

$$V'(\mathbf{x}') := 3\frac{(x'_1 x'_2 x'_3)^{1/3}}{x'_1 + x'_2 + x'_3}.$$

Consider a solution $\mathbf{x}'(\cdot)$ of the replicator dynamics in G' (with $\sum_{1 \leq i \leq 3} x'_i(0) > 0$) and let $v'(t) = V'(\mathbf{x}'(t))$. On the face of $S_{N'}$ spanned by the strategies of the original game:

$$\left\{\mathbf{x}' \in S_{N'} : \sum_{1 \leq i \leq 4} x'_i = 1\right\},$$

the replicator dynamics behaves just as in the base game. Therefore, (33) and (34) imply trivially that:

$$\left[\sum_{1 \leq i \leq 4} x'_i = 1 \text{ and } V'(\mathbf{x}') \leq \delta\right] \Rightarrow \left[(\mathbf{U}'\mathbf{x}')_4 - \mathbf{x}' \cdot \mathbf{U}'\mathbf{x}' \leq -\gamma\right] \tag{35}$$

and

$$\left[\sum_{1\le i\le 4} x'_i = 1 \text{ and } v'(t) = \delta\right] \Rightarrow \dot{v}'(t) \le -\gamma. \tag{36}$$

Now define $\bar{\mathbf{x}}' \in S_{N'}$ as the projection of $\mathbf{x}'$ on the face of $S_{N'}$ spanned by the strategies of the original game. That is,

$$\bar{\mathbf{x}}'_i = \frac{x'_i}{\sum\limits_{1\le j\le 4} x'_j} \quad \text{if } 1 \le i \le 4, \text{ and } \bar{\mathbf{x}}'_i = 0 \text{ otherwise.}$$

Note that $V'(\mathbf{x}') = V'(\bar{\mathbf{x}}')$. Furthermore, a simple computation shows that

$$\max_{1\le i\le N'} |x'_i - \bar{x}'_i| \le N' \max_{5\le k\le N'} x'_k.$$

Therefore, since $(\mathbf{U}'\mathbf{x}')_4 - \mathbf{x}' \cdot \mathbf{U}'\mathbf{x}'$ and the vector field $\dot{\mathbf{x}}'$ are Lipschitz in $\mathbf{x}'$, it follows from (35) and (36) that there exist positive constants η and γ' such that

$$\left[\max_{5\le k\le N'} x'_k \le \eta \text{ and } V'(\mathbf{x}') \le \delta\right] \Rightarrow (\mathbf{U}'\mathbf{x}')_4 - \mathbf{x}' \cdot \mathbf{U}'\mathbf{x}' \le -\gamma' \tag{37}$$

and

$$\left[\max_{5\le k\le N'} x'_k \le \eta \text{ and } v'(t) = \delta\right] \Rightarrow \dot{v}'(t) \le -\gamma'. \tag{38}$$

Fix $\mathbf{y}' \in S_{N'}$ such that

$$\sum_{1\le i\le 4} y'_i = 1, \qquad V'(\mathbf{y}') < \delta \quad \text{and} \quad C := \min_{1\le i\le 3} y'_i > 0.$$

There exists an open neighborhood Ω of $\mathbf{y}'$ in $S_{N'}$ such that

$$\forall \mathbf{x}' \in \Omega, \quad \left[\min_{1\le i\le 3} x'_i > C/2, \max_{5\le k\le N'} x'_k < C\eta/2, \text{ and } V'(\mathbf{x}') < \delta\right].$$

Consider an interior solution $\mathbf{x}'(\cdot)$ of the replicator dynamics in G' with initial condition in Ω. Recall that $\mathbf{p}^k$ denotes the mixed strategy of G associated with the pure strategy k of G'. To prove Proposition 6.1 for the replicator dynamics, it suffices to show that:

Proposition A.1. *For all k in $\{4, \dots, N'\}$ such that $p^k_4 > 0$, $x'_k(t) \to_{t\to+\infty} 0$.*

Proof. We begin with two lemmas:

Lemma A.2. *Let $T > 0$ and $k \in \{5, \dots, N'\}$. If $x'_4(T) \le x'_4(0)$ then $x'_k(T) < \eta$.*

Proof. By construction of G', strategy $k \in I'$ earns the same payoff as the mixed strategy $\sum_{1\le i\le 4} p^k_i \mathbf{e}'_i$:

$$(\mathbf{U}'\mathbf{x}')_k = \sum_{1\le i\le 4} p^k_i (\mathbf{U}'\mathbf{x}')_i \quad \forall \mathbf{x}' \in S_{N'}.$$

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Therefore, it follows from the definition of the replicator dynamics that:

$$\frac{\dot{x}'_k}{x'_k} = \sum_{1\le i\le 4} p_i^k \frac{\dot{x}'_i}{x'_i}.$$

Integrating between 0 and T and taking the exponential of both sides leads to:

$$x'_k(T) = x'_k(0) \prod_{1\le i\le 4} \left(\frac{x'_i(T)}{x'_i(0)}\right)^{p_i^k}. \tag{39}$$

Noting that for $1 \le i \le 3$, we have $x'_i(T) \le 1$, $1/x'_i(0) \le 2/C$ and $1 \le 2/C$, we get:

$$\prod_{1\le i\le 3} \left(\frac{x'_i(T)}{x'_i(0)}\right)^{p_i^k} \le \prod_{1\le i\le 3} \left(\frac{2}{C}\right)^{p_i^k} = \left(\frac{2}{C}\right)^{1-p_4^k} \le \frac{2}{C}. \tag{40}$$

Since furthermore $x'_k(0) < C\eta/2$, we obtain from (39) and (40):

$$x'_k(T) < \frac{C\eta}{2}\frac{2}{C}\left(\frac{x'_4(T)}{x'_4(0)}\right)^{p_4^k} = \eta\left(\frac{x'_4(T)}{x'_4(0)}\right)^{p_4^k}. \tag{41}$$

The result follows. ■

Lemma A.3. *For all $t > 0$, $\max_{k\in\{5,\dots,N'\}} x'_k(t) < \eta$ and $v'(t) < \delta$.*

Proof. Otherwise there is a first time $T > 0$ such that $\max_{k\in\{5,\dots,N'\}} x'_k(T) = \eta$ or $v'(T) = \delta$ (or both). It follows from (37) and the definition of the replicator dynamics that if $0 \le t \le T$ then $\frac{\dot{x}'_4}{x'_4}(t) \le -\gamma' < 0$. Therefore $x'_4(T) \le x'_4(0)$. By Lemma A.2, this implies that $\max_{k\in\{5,\dots,N'\}} x'_k(T) < \eta$. Therefore, $v'(T) = \delta$. Due to (38), this implies that $\dot{v}'(T) < 0$. Therefore, there exists a time T_1 with $0 < T_1 < T$ such that $v'(T_1) > \delta$, hence a time T_2 with $0 < T_2 < T_1 < T$ such that $v'(T_2) = \delta$, contradicting the minimality of T. ■

We now conclude: it follows from Lemma A.3, Eq. (37) and the definition of the replicator dynamics that for all $t \ge 0$, $x'_4(t) \le \exp(-\gamma' t)x'_4(0)$. By (41) this implies that for every k in $\{5, \dots, N'\}$,

$$\forall t \ge 0, \quad x'_k(t) < \eta \exp(-p_4^k \gamma' t).$$

Therefore, if $p_4^k > 0$ then $x'_k(t) \to 0$ as $t \to +\infty$. ■

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