Postdoctoral position: Nonconvex optimization on measure spaces

General information

Host institution: Université Paris Dauphine-PSL (Paris, France). The successful applicant will be hosted by the mathematics institute (CEREMADE) and will collaborate with the computer science institute (LAMSADE).
Advisors: Clément W. Royer (webpage) and Irène Waldspurger (webpage)

Duration: One year.
Intended starting date: Early 2024 with significant flexibility.
Funding source: PEPR « PDE-AI » program on numerical analysis, optimal control and optimal transport for artificoal intelligence. More information about this research program can be found at https://pde-ai.math.cnrs.fr/).

Profile: The successful applicant will have a PhD in applied mathematics or a related field, with strong expertise in optimization and numerical analysis. Familiarity with imaging problems is not required but would be a plus. Knowledge of French is not required.
Research topic: The successful applicant will be expected to conduct research in the area of optimization over measure spaces. We have a specific topic in mind revolving on Frank-Wolfe methods described thereafter. However, upon demand of the candidate, other research directions may be pursued, provided that they fit within both the frame of the PEPR and our areas of expertise.

Focused research proposal

Overview: This project belongs to the domain of nonconvex optimization. The goal is to understand the properties of a specific nonconvex algorithm, designed for solving an optimization problem where the unknown lives in a set of measures. Optimization problems on the space of measures and associated nonconvex algorithms have raised significant interest in the last years, notably because they provide a useful reference point to understand the dynamics of network training [Chizat and Bach, 2018; Chizat, 2022]. In this project, the problem we consider is simpler than neural network training, and interesting in its own right. Indeed, it belongs to the class of so-called *super-resolution* problems, which enjoy both a rich mathematical theory and practical applications in imaging [Denoyelle, Duval, Peyré, and Soubies, 2019].

Technical description: We consider a super-resolution problem, where the goal is to recover a measure μ_* supported on some open set $\Omega \subset \mathbb{R}^d$. The recovery is performed from linear measurements

$$y_i = \int_\Omega \phi_i(x) d\mu_*(x),$$

where $\phi_1, \dots, \phi_m : \Omega \to \mathbb{R}$ are known. We are particularly interested in the case of a sparse measure μ_* (i.e. a linear combination of few Dirac delta functions).

Rather than enforcing a (nonconvex) sparsity constraint in the optimization problem, we consider instead the problem with (convex) total variation penalty [de Castro and Gamboa, 2012]:

$$\begin{aligned} &\text{minimize } \frac{1}{2}\sum_{i=1}^m \left| y_i - \int_\Omega \phi_i(x) d\mu(x) \right|^2 + \lambda ||\mu||_{TV}, \\ &\text{over all } \mu \in \mathcal{M}(\Omega), \end{aligned} \tag{B-LASSO}$$

where $||.||_{TV}$ denotes the total variation norm, λ is a positive real number, and $\mathcal{M}(\Omega)$ is the set of Radon measures on Ω with bounded total variation. This convexification approach is largely reminiscent of the LASSO approach in compressed sensing. Numerous works have been devoted to establishing conditions under which the solution of (B-LASSO) is the same, or close to the ground true μ_*; see [Candès and Fernandez-Granda, 2014] for one of the most emblematic of these works.

On the numerical side, Problem (B-LASSO) remains difficult to solve despite its convex nature, due to the unknown living in the infinite dimensional space $\mathcal{M}(\Omega)$. A recent algorithmic proposal, based on the Frank-Wolfe algorithm, recovers an approximation of μ_* by constructing a careful combination of Dirac delta functions one at a time[Bredies and Pikkarainen, 2013], but this method demonstrates slow convergence in practice.
A version of this algorithm using nonconvex steps, also called *Sliding Frank-Wolfe*, has been shown to behave extremely well on this problem [Denoyelle, Duval, Peyré, and Soubies, 2019]. However, convergence guarantees for that method are quite far from its impressive numerical performance. In particular, the method can be shown to reach the solution in a finite number of steps [Denoyelle, Duval, Peyré, and Soubies, 2019], but no upper bound on the number of steps exists. Recent results have examined the behavior of nonconvex steps [Traonmilin and Aujol, 2020], but do not provide a quantitative guarantee for the whole *Sliding Frank-Wolfe* algorithm.
The goal of this project is to improve on existing work by providing a convergence rate for *Sliding Frank-Wolfe* more consistent with its empirical performance. To attack this problem, we will first consider specific measurement functions in Problem (B-LASSO), such as the Fourier transform, a case of primary importance in super-resolution. Our objective will then be to extend these results to other measurement functions.

References

K. Bredies and H. K. Pikkarainen. Inverse problems in spaces of measures. *ESAIM: Control, Optimisation and Calculus of Variations*, 19(1):190–218, 2013.

E. J. Candès and C. Fernandez-Granda. Towards a mathematical theory of super-resolution. *Communications on Pure and Applied Mathematics*, 67 (6):906–956, 2014.

L. Chizat. Sparse optimization on measures with over-parameterized gradient descent. *Mathematical Programming*, 194(1):487–532, 2022.

Lenaic Chizat and Francis Bach. On the global convergence of gradient descent for over-parameterized models using optimal transport. *Advances in neural information processing systems*, 31, 2018.

Y. de Castro and F. Gamboa. Exact reconstruction using beurling minimal extrapolation. *Journal of Mathematical Analysis and Applications*, 395: 336–354, 2012.

Q. Denoyelle, V. Duval, G. Peyré, and E. Soubies. The sliding Frank–Wolfe algorithm and its application to super-resolution microscopy. *Inverse Problems*, 36(1), 2019.

Yann Traonmilin and Jean-François Aujol. The basins of attraction of the global minimizers of the non-convex sparse spike estimation problem. *Inverse Problems*, 36(4):045003, 2020.