Phase retrieval with random Gaussian sensing vectors by alternating projections

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Abstract

We consider a phase retrieval problem, where we want to reconstruct a n-dimensional vector from its phaseless scalar products with m sensing vectors. We assume the sensing vectors to be independently sampled from complex normal distributions. We propose to solve this problem with the classical non-convex method of alternating projections. We show that, when $m \geq Cn$ for C large enough, alternating projections succeed with high probability, provided that they are carefully initialized. We also show that there is a regime in which the stagnation points of the alternating projections method disappear, and the initialization procedure becomes useless. However, in this regime, m needs to be of the order of n^2. Finally, we conjecture from our numerical experiments that, in the regime $m = O(n)$, there are stagnation points, but the size of their attraction basin is small if m/n is large enough, so alternating projections can succeed with probability close to 1 even with no special initialization.

1 Introduction

The problem of reconstructing a low-rank matrix from linear observations appears under many forms in the fields of inverse problems and machine learning. An important amount of work has thus been devoted to the design of reconstruction algorithms coming with provable reconstruction guarantees. The first algorithms of this kind relied mostly on convexification techniques. They tended to have a high recovery rate, but a possibly prohibitive computational complexity. As a result, a need has emerged to prove similar guarantees for algorithms based on non-convex formulations, which are generally much faster.

In this article, we consider a subclass of low-rank recovery problems: *phase retrieval problems*. In the finite-dimensional setting, phase retrieval consists in recovering an unknown vector $x_0 \in$

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$\mathbb{C}^n$ from m phaseless linear measurements, of the form

$$b_k = |\langle a_k, x_0 \rangle|, \qquad k = 1, \dots, m,$$

where the *sensing vectors* $a_k \in \mathbb{C}^n$ are known. Phaseless measurements do not allow to distinguish x_0 from ux_0, for $u \in \mathbb{C}, |u| = 1$, so the goal is only to recover x_0 *up to a global phase*. Motivations for studying these problems come in particular from optical imaging; see [Schechtman, Eldar, Cohen, Chapman, Miao, and Segev, 2015] for a recent review. Phase retrieval problems can be seen as low-rank matrix recovery problems, because knowing $|\langle a_k, x_0 \rangle|$ amounts to knowing

$$|\langle a_k, x_0 \rangle|^2 = \mathrm{Tr}(a_k a_k^* x_0 x_0^*),$$

so reconstructing x_0 is equivalent to:

$$\text{Reconstruct } X_0 \in \mathcal{S}_n(\mathbb{C}) \text{ from } \{\mathrm{Tr}(a_k a_k^* X_0)\}_{k=1,\dots,m} \\ \text{such that } \mathrm{rank}(X_0) = 1. \tag{1}$$

The vector x_0 is uniquely determined by the m phaseless measurements as soon as $m \gtrsim 4n$ [Balan, Casazza, and Edidin, 2006]; however, reconstructing it is a priori NP-hard [Fickus, Mixon, Nelson, and Wang, 2014]. The oldest reconstruction algorithms [Gerchberg and Saxton, 1972; Fienup, 1982] were iterative: they started from a random initial guess of x_0, and tried to iteratively refine it by various heuristics. Although these algorithms are empirically seen to succeed in a number of cases, they can also get stuck in stagnation points, whose existence is due to the non-convexity of the problem.

To overcome these convergence problems, convexification methods have been introduced [Chai, Moscoso, and Papanicolaou, 2011; Candès, Strohmer, and Voroninski, 2013]. These methods consider the matricial formulation (1), but replace the non-convex rank constraint by a more favorable convex constraint. They provably reconstruct the unknown vector x_0 with high probability if the sensing vectors a_k are "random enough" [Candès and Li, 2014; Candès, Li, and Soltanolkotabi, 2015; Gross, Krahmer, and Kueng, 2015]. Numerical experiments show that they also perform well on more structured, non-random phase retrieval problems [Waldspurger, d'Aspremont, and Mallat, 2015; Sun and Smith, 2012].

Unfortunately, this good precision comes at a high computational cost: optimizing the $n \times n$ matrix X_0 is much slower that directly reconstructing the n-dimensional vector x_0. Consequently, convexification techniques are impractical when the dimension of x_0 exceeds a few hundred. Authors have thus recently begun to design fast non-convex algorithms, for which it is possible to establish similar reconstruction guarantees as for convexified algorithms. The methods that have been developed rely on the following two-step scheme:

(1) an initialization step, that returns a point close to the solution;

(2) a gradient descent (possibly with additional refinements) over a well-chosen non-convex cost function.

The intuitive reason why this scheme works is that the cost function, although globally non-convex, enjoys some good geometrical property in a neighborhood of the solution (like convexity or a weak form of it [White, Sanghavi, and Ward, 2015]). So, if the point returned by the initialization step belongs to this neighborhood, gradient descent converges to the true solution.

A preliminary form of this scheme appears in [Netrapalli, Jain, and Sanghavi, 2013], with an alternating minimization in step (2) instead of a gradient descent. Then, considering the cost function

$$L_1(x) = \sum_{k=1}^{m} \left(b_k^2 - | \langle a_k, x \rangle |^2 \right)^2 , \tag{2}$$

[Candès, Li, and Soltanolkotabi, 2015] proved the correctness of the two-step scheme, with high probability, in the regime $m = O(n \log n)$, for random independent Gaussian sensing vectors. In [Chen and Candès, 2015; Kolte and Özgür, 2016], the same result was shown in the regime $m = O(n)$ for a slightly different cost function, with additional truncation steps. In [Zhang and Liang, 2016], it was extended to the following non-smooth cost function:

$$L_2(x) = \sum_{k=1}^{m} \left(b_k - | \langle a_k, x \rangle | \right)^2 .$$

Additionally, Sun, Qu, and Wright [2016] have shown that, in the regime $m = O(n \log^3 n)$, the cost function (2) actually has no "bad critical point", and the initialization step is not necessary: the gradient descent in step (2) converges to the global minimum of L_1, almost whatever initial point it starts from. These authors have also numerically observed that, in the regime $m = O(n)$, despite the potential presence of bad critical points, the gradient descent succeeds, with at least constant probability, starting from a random initialization.

For other low-rank recovery problems than phase retrieval, we refer for example to [Sun and Luo, 2015; Ge, Lee, and Ma, 2016] for matrix completion, to [Tu, Boczar, Simchowitz, Soltanolkotabi, and Recht, 2016; Bhojanapalli, Neyshabur, and Srebo, 2016] for the case where the measurement scheme obeys a Restricted Isometry Property, and to [Bandeira, Boumal, and Voroninski, 2016] for $\mathbb{Z}_2$ synchronization problems.

In the case of phase retrieval, the most recently introduced non-convex algorithms are optimal in terms of both statistical and computational complexity, up to multiplicative constants. However, there is still a need to understand whether their theoretical reconstruction guarantees can be extended to more general classes of algorithms, that would not exactly follow the above two-step scheme, but would be closer to the algorithms that are actually used in applications. This in particular implies to answer the following two questions:

- In Step (2), can we replace the explicit minimization of a cost function by a “less local” search, like alternating projections [Gerchberg and Saxton, 1972] or Douglas-Rachford [Bauschke, Combettes, and Luke, 2002]?
- Is the initialization step (1) necessary, or can Step (2) converge to the global optimum even starting from a random initialization, at least in certain cases?

In this article, we answer the first question: we show that, in the optimal regime of $m = O(n)$ random independent Gaussian sensing vectors, replacing gradient descent with alternating projections yields exact recovery with high probability, and convergence occurs at a linear rate.

Théorème (See Corollary 3.7)**.** *There exist absolute constants $C_1, C_2, M > 0$, $\delta \in]0;1[$ such that, if $m \geq Mn$ and the sensing vectors are independently chosen according to complex normal distributions, the sequence of iterates $(z_t)_{t\in\mathbb{N}}$ produced by the alternating projections method satisfies*

$$\forall t \in \mathbb{N}^*, \qquad \inf_{\phi\in\mathbb{R}} ||e^{i\phi}x_0 - z_t|| \leq \delta^t ||x_0||,$$

with probability at least

$$1 - C_1 \exp(-C_2 m),$$

provided that alternating projections are correctly initialized, for example with the method described in [Chen and Candès, 2015].

Alternating projections, introduced by Gerchberg and Saxton [1972], is the most ancient algorithm for phase retrieval. It is an intuitive method, whose implementation is extremely simple, and with no parameter to choose or tune; it is thus widely used. In terms of complexity, it is slower, for general measurements, than the best non-convex methods by only a logarithmic factor in the precision. For more “structured” measurements (as in all applications that we know of), it is as fast (see Paragraph 3.3).

We believe that the second question, about the necessity of the initialization step, is also important. In addition to being a natural theoretical question, it has practical consequences: the initialization procedure depends on the probability distribution of the sensing vectors, and, for some families of sensing vectors appearing in applications, we do not (yet) have a valid initialization procedure. We partially answer it in the case where the sensing vectors are independent and Gaussian, and reconstruction is done with alternating projections. We propose a description of when this method globally converges to the true solution, depending on the number of measurements and the initialization procedure. This description is summarized in Figure 1.

As shown in the figure, there is a regime in which the stagnation points of the alternating projections routine disappear (except possibly on a “small” set that we define), and, with high probability, alternating projections converge starting from any initialization outside the small

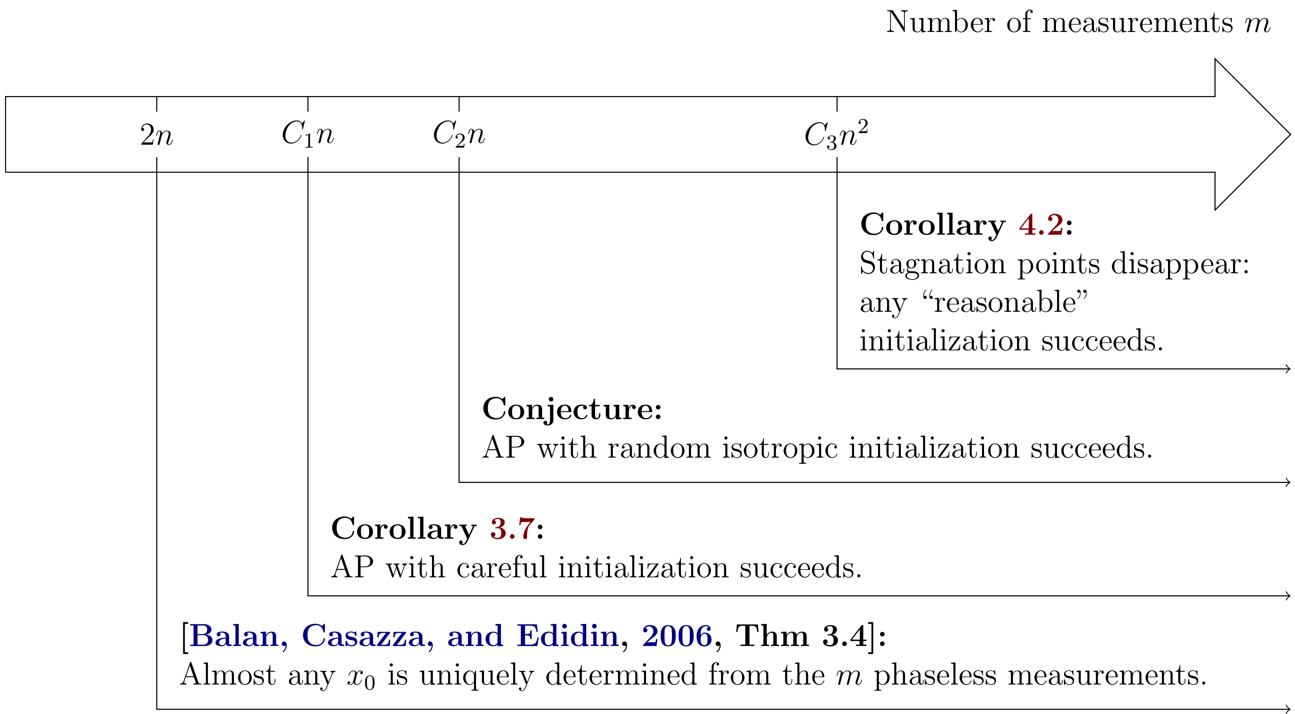

Figure 1: Schematic representation of the behavior of the alternating projections (AP) algorithm, as a function of the number of measurements m. All events happen only "with high probability".

set. This regime is $m = O(n^2)$. Our numerical experiments clearly indicate that, below this regime, there are stagnation points. It is however possible that the attraction basin of the stagnation points is small: even in the regime $m = O(n)$, we numerically see that alternating projections, starting from a random isotropic initialization[1], succeed with probability close to 1 despite the presence of stagnation points. We leave this assertion as a conjecture.

Théorème (Informal, see Corollary 4.2)**.** *There exist $C_1, C_2, \gamma, M > 0$, $\delta \in]0;1[$ such that, if $m \geq Mn^2$ and the sensing vectors are independently chosen according to complex normal distributions, with probability at least*

$$1 - C_1 \exp(-C_2 n),$$

the sequence of iterates $(z_t)_{t\in\mathbb{N}}$ produced by the alternating projections method satisfies

$$\forall t \geq \gamma \log n, \qquad \inf_{\phi\in\mathbb{R}} ||e^{i\phi}x_0 - z_t|| \leq \delta^{t-\gamma\log n}||x_0||,$$

starting from any initial point that does not belong to a small "bad set".

[1]By "isotropic", we mean that the law of the initial vector is invariant under linear unitary transformations.

Conjecture (See Conjecture 5.1). *Let any $\epsilon > 0$ be fixed. When $m \geq Cn$, for $C > 0$ large enough, alternating projections, starting from a random isotropic initialization, converge to the true solution with probability at least $1 - \epsilon$.*

These theorem and conjecture are the parallels for alternating projections of the results and numerical observations obtained by Sun, Qu, and Wright [2016] for gradient descent over the cost function (2). The "no stagnation point" regime is much less favorable in the case of alternating projections than in the case of gradient descent: $m = O(n^2) \gg O(n \log^3 n)$. It could be due to the discontinuity of the alternating projections operator, but we have no evidence to support this fact.

On the side of proof techniques, there has been a lot of work on the convergence of alternating projections in non-convex settings. Transversality arguments can be shown to prove, in certain cases, local convergence guarantees ("if the initial point is sufficiently close to the correct solution, alternating projections converge to this solution"). See for example [Lewis, Luke, and Malick, 2009; Drusvyatskiy, Ioffe, and Lewis, 2015]. These arguments can be used in phase retrieval, and yield local convergence results for relatively general families of sensing vectors (not necessarily random) [Noll and Rondepierre, 2016; Chen, Fannjiang, and Liu, 2016]. Unfortunately, they give no control on the convergence radius of the algorithm, so the obtained results have a mainly theoretical interest.

Bounding the convergence radius requires using the statistical properties of the sensing vectors. This was first attempted in [Netrapalli, Jain, and Sanghavi, 2013], where the authors proved the global convergence of a resampled version of the alternating projections algorithm. For a non resampled version, a preliminary result was given in [Soltanolkotabi, 2014]. However, the bound on the convergence radius that underlies this result is small. As a consequence, global convergence is only proven for a suboptimal number of measurements ($m = O(n \log^2 n)$), and with a complex initialization procedure.

A difficulty that we encounter is the fact that the alternating projections operator is not continuous. This difficulty also appears in the two recent articles [Zhang and Liang, 2016; Wang, Giannakis, and Eldar, 2016], where the authors consider a gradient descent over a function whose gradient is not continuous. The proof that we give for our Theorem 4.1 follows a different path as theirs (it does not use a regularity condition); the statistical tools are however similar.

The article is organized as follows. Section 2 precisely defines phase retrieval problems and the alternating projections algorithm. Section 3 states and proves the first main result: the global convergence of alternating projections, with proper initialization, for $m = O(n)$ independent Gaussian measurements. Section 4 proves the second main result: stagnation points disappear in the regime $m = O(n^2)$, making the initialization step useless. Finally, Section 5 presents numerical results, and conjectures that the alternating projections algorithm can succeed without special initialization in the regime $m = O(n)$, despite the presence of stagnation points. All technical lemmas are deferred to the appendices.

1.1 Notations

For any $z \in \mathbb{C}$, $|z|$ is the modulus of z. We extend this notation to vectors: if $z \in \mathbb{C}^k$ for some $k \in \mathbb{N}^*$, then $|z|$ is the vector of $(\mathbb{R}^+)^k$ such that

$$|z|_i = |z_i|, \qquad \forall i = 1, \dots, k.$$

For any $z \in \mathbb{C}$, we set $E_{\text{phase}}(z)$ to be the following subset of $\mathbb{C}$:

$$\begin{aligned} E_{\text{phase}}(z) &= \left\{ \tfrac{z}{|z|} \right\} && \text{if } z \in \mathbb{C} - \{0\}; \\ &= \{ e^{i\phi}, \phi \in \mathbb{R} \} && \text{if } z = 0. \end{aligned}$$

We extend this definition to vectors $z \in \mathbb{C}^k$:

$$E_{\text{phase}}(z) = \prod_{i=1}^{k} E_{\text{phase}}(z_i).$$

For any $z \in \mathbb{C}$, we define phase(z) by

$$\begin{aligned} \text{phase}(z) &= \tfrac{z}{|z|} && \text{if } z \in \mathbb{C} - \{0\}; \\ &= 1 && \text{if } z = 0, \end{aligned}$$

and extend this definition to vectors $z \in \mathbb{C}^k$, as for the modulus.

We denote by $\odot$ the pointwise product of vectors: for all $a, b \in \mathbb{C}^k$, $(a \odot b)$ is the vector of $\mathbb{C}^k$ such that

$$(a \odot b)_i = a_i b_i, \qquad \forall i = 1, \dots, k.$$

We define the operator norm of any matrix $A \in \mathbb{C}^{n_1 \times n_2}$ by

$$|||A||| = \sup_{v \in \mathbb{C}^{n_2}, ||v|| = 1} ||Av||.$$

We denote by $A^\dagger$ its Moore-Penrose pseudo-inverse. We note that $AA^\dagger$ is the orthogonal projection onto Range(A).

2 Problem setup

2.1 Phase retrieval problem

Les n, m be positive integers. The goal of a phase retrieval problem is to reconstruct an unknown vector $x_0 \in \mathbb{C}^n$ from m measurements with a specific form.

We assume $a_1, \dots, a_m \in \mathbb{C}^n$ are given; they are called the *sensing vectors*. We define a matrix $A \in \mathbb{C}^{m \times n}$ by

$$A = \begin{pmatrix} a_1^* \\ \vdots \\ a_m^* \end{pmatrix}.$$

This matrix is called the *measurement matrix*. The associated *phase retrieval* problem is:

$$\text{reconstruct } x_0 \text{ from } b \stackrel{def}{=} |Ax_0|. \tag{3}$$

As the modulus is invariant to multiplication by unitary complex numbers, we can never hope to reconstruct x_0 better than *up to multiplication by a global phase*. So, instead of exactly reconstructing x_0, we want to reconstruct x_1 such that

$$x_1 = e^{i\phi} x_0, \qquad \text{for some } \phi \in \mathbb{R}.$$

In all this article, we assume the sensing vectors to be independent realizations of centered Gaussian variables with identity covariance:

$$(a_i)_j \sim \mathcal{N}\left(0, \frac{1}{2}\right) + \mathcal{N}\left(0, \frac{1}{2}\right) i, \qquad \forall 1 \leq i \leq m, 1 \leq j \leq n. \tag{4}$$

The measurement matrix is in particular independent from x_0.

Balan, Casazza, and Edidin [2006] and Conca, Edidin, Hering, and Vinzant [2015] have proved that, for *generic* measurement matrices A, Problem (3) always has a unique solution, up to a global phase, provided that $m \geq 4n - 4$. In particular, with our measurement model (4), the reconstruction is guaranteed to be unique, with probability 1, when $m \geq 4n - 4$.

2.2 Alternating projections

The alternating projections method has been introduced for phase retrieval problems by Gerchberg and Saxton [1972]. It focuses on the reconstruction of Ax_0; if A is injective, this then allows to recover x_0.

To reconstruct Ax_0, it is enough to find $z \in \mathbb{C}^m$ in the intersection of the following two sets.

(1) $z \in \{z' \in \mathbb{C}^m, |z'| = b\}$;

(2) $z \in \text{Range}(A)$.

Indeed, when the solution to Problem (3) is unique, Ax_0 is the only element of $\mathbb{C}^m$ that simultaneously satisfies these two conditions (up to a global phase).

A natural heuristic to find such a z is to pick any initial guess z_0, then to alternatively project it on the two constraint sets. In this context, we call *projection* on a closed set $E \subset \mathbb{C}^m$ a function $P : \mathbb{C}^m \to E$ such that, for any $x \in \mathbb{C}^m$,

$$||x - P(x)|| = \inf_{e \in E} ||x - e||.$$

The two sets defining constraints (1) and (2) admit projections with simple analytical expressions, which leads to the following formulas:

$$y'_k = b \odot \text{phase}(y_k); \qquad \text{(Projection onto set (1))} \qquad (5a)$$

$$y_{k+1} = (AA^\dagger)y'_k. \qquad \text{(Projection onto set (2))} \qquad (5b)$$

If we define z_k as the unique vector such that $y_k = Az_k$, an equivalent form of these equations is:

$$z_{k+1} = A^\dagger(b \odot \text{phase}(Az_k)).$$

The hope is that the sequence $(y_k)_{k\in\mathbb{N}}$ converges towards Ax_0. Unfortunately, it can get stuck in *stagnation points*. The following proposition (proven in Appendix A) characterizes these stagnation points.

Proposition 2.1. *For any y_0, the sequence $(y_k)_{k\in\mathbb{N}}$ is bounded. Any accumulation point y_∞ of $(y_k)_{k\in\mathbb{N}}$ satisfies the following property:*

$$\exists u \in E_{\text{phase}}(y_\infty), \qquad (AA^\dagger)(b \odot u) = y_\infty.$$

In particular, if y_∞ has no zero entry,

$$(AA^\dagger)(b \odot \text{phase}(y_\infty)) = y_\infty.$$

Despite the relative simplicity of this characteristic property, it is extremely difficult to exactly compute the stagnation points, determine their attraction basin or avoid them when the algorithm happens to run into them.

The goal of this article is to show that, in certain settings, there are no stagnation points, or they can be avoided with a careful initialization procedure of the alternating projection routine.

3 Alternating projections with good initialization

In this section, we prove the first of our two main results: in the regime $m = O(n)$, the method of alternating projections converges to the correct solution with high probability, if it is carefully initialized.

3.1 Local convergence of alternating projections

This paragraph proves the key result that we will need to establish our statement. This result is a local contraction property of the alternating projections operator $x \to A^\dagger(b \odot \text{phase}(Ax))$.

Théorème 3.1. *There exist $\epsilon, C_1, C_2, M > 0$, and $\delta \in]0;1[$ such that, if $m \geq Mn$, then, with probability at least*

$$1 - C_1 \exp(-C_2 m),$$

the following property holds: for any $x \in \mathbb{C}^n$ such that

$$\inf_{\phi \in \mathbb{R}} ||e^{i\phi} x_0 - x|| \leq \epsilon ||x_0||,$$

we have

$$\inf_{\phi \in \mathbb{R}} ||e^{i\phi} x_0 - A^\dagger(b \odot \text{phase}(Ax))|| \leq \delta ||x_0 - x||. \tag{6}$$

Proof. For any $x \in \mathbb{C}^n$, we can write Ax as

$$Ax = \lambda_x (Ax_0) + \mu_x v^x, \tag{7}$$

where $\lambda_x \in \mathbb{C}, \mu_x \in \mathbb{R}^+$, and $v^x \in \text{Range}(A)$ is a unitary vector orthogonal to Ax_0.

The following lemma is proven in Paragraph B.1.

Lemme 3.2. *For any $z_0, z \in \mathbb{C}$,*

$$|\,\text{phase}(z_0 + z) - \text{phase}(z_0)| \leq 2.1_{|z| \geq |z_0|/6} + \frac{6}{5} \left| \text{Im} \left(\frac{z}{z_0} \right) \right|.$$

So, for any $x \in \mathbb{C}^n$,

$$\begin{aligned} &|\,\text{phase}(\lambda_x)(Ax_0)_i - (b \odot \text{phase}(Ax))_i| \\ &\qquad = |\text{phase}(\lambda_x)(Ax_0)_i - |Ax_0|_i \,\text{phase}((Ax)_i)| \\ &\qquad = |\text{phase}(\lambda_x)(Ax_0)_i - |Ax_0|_i \,\text{phase}(\lambda_x (Ax_0)_i + \mu_x (v^x)_i)| \\ &\qquad = |Ax_0|_i \left| \text{phase}(Ax_0)_i - \text{phase} \left((Ax_0)_i + \frac{\mu_x}{\lambda_x} (v^x)_i \right) \right| \end{aligned}$$

$$\leq 2.|Ax_0|_i 1_{|\mu_x/\lambda_x||v^x|_i \geq |Ax_0|_i/6} + \frac{6}{5}\left|\mathrm{Im}\left(\frac{\frac{\mu_x}{\lambda_x}v_i^x}{\mathrm{phase}((Ax_0)_i)}\right)\right|.$$

As a consequence,

$$\begin{aligned}
&||\,\mathrm{phase}(\lambda_x)(Ax_0) - b \odot \mathrm{phase}(Ax)|| \\
&\qquad\leq \left|\left|2.|Ax_0| \odot 1_{|\mu_x/\lambda_x||v^x| \geq |Ax_0|/6} + \frac{6}{5}\left|\mathrm{Im}\left(\left(\frac{\mu_x}{\lambda_x}v^x\right) \odot \overline{\mathrm{phase}(Ax_0)}\right)\right|\right|\right| \\
&\qquad\leq 2\left|\left||Ax_0| \odot 1_{6|\mu_x/\lambda_x||v^x| \geq |Ax_0|}\right|\right| + \frac{6}{5}\left|\left|\mathrm{Im}\left(\left(\frac{\mu_x}{\lambda_x}v^x\right) \odot \overline{\mathrm{phase}(Ax_0)}\right)\right|\right|. \qquad (8)
\end{aligned}$$

Two technical lemmas allow us to upper bound the terms of this sum. The first one is proved in Paragraph B.2, the second one in Paragraph B.3.

Lemme 3.3. *For any $\eta > 0$, there exists $C_1, C_2, M, \gamma > 0$ such that the inequality*

$$||\,|Ax_0| \odot 1_{|v| \geq |Ax_0|}|| \leq \eta ||v||$$

holds for any $v \in \mathrm{Range}(A)$ such that $||v|| < \gamma ||Ax_0||$, with probability at least

$$1 - C_1 \exp(-C_2 m),$$

when $m \geq Mn$.

Lemme 3.4. *For $M, C_1 > 0$ large enough, and $C_2 > 0$ small enough, when $m \geq Mn$, the property*

$$||\mathrm{Im}\,(v \odot \overline{\mathrm{phase}(Ax_0)})|| \leq \frac{4}{5}||v||$$

holds for any $v \in \mathrm{Range}(A) \cap \{Ax_0\}^\perp$, with probability at least

$$1 - C_1 \exp(-C_2 m).$$

Let us choose $\eta > 0$ such that

$$12\eta + \frac{24}{25} < 1.$$

We define $\gamma > 0$ as in Lemma 3.3. The events described in Lemmas 3.3 and 3.4 hold with probability at least

$$1 - 2C_1 \exp(-C_2 m).$$

When this happens, for all x such that

$$\left|\frac{\mu_x}{\lambda_x}\right| < \frac{\gamma}{6}\,||Ax_0||,$$

the terms in Equation (8) can be bounded as in the lemmas, because

$$\left|\left|6\frac{\mu_x}{\lambda_x}v^x\right|\right| = 6\left|\frac{\mu_x}{\lambda_x}\right| < \gamma||Ax_0||,$$

and $\frac{\mu_x}{\lambda_x}v^x \in \text{Range}(A) \cap \{Ax_0\}^\perp$. So the following inequality holds:

$$||\,\text{phase}(\lambda_x)(Ax_0) - b \odot \text{phase}(Ax)|| \le \left(12\eta + \frac{24}{25}\right)\left|\frac{\mu_x}{\lambda_x}\right|. \tag{9}$$

For any x such that $\inf_{\phi\in\mathbb{R}} ||e^{i\phi}x_0 - x|| \le \epsilon||x_0||$, if we set $\epsilon^x = \inf_{\phi\in\mathbb{R}} \frac{||e^{i\phi}x_0 - x||}{||x_0||} \le \epsilon$,

$$\inf_{\phi\in\mathbb{R}} ||e^{i\phi}Ax_0 - Ax|| \le \epsilon^x|||A|||\,||x_0||,$$

so, using Equation (7),

$$\inf_{\phi\in\mathbb{R}} |e^{i\phi} - \lambda_x|^2||Ax_0||^2 + |\mu_x|^2 \le (\epsilon^x)^2|||A|||^2||x_0||^2,$$

which implies

$$|\mu_x| \le \epsilon^x|||A|||\,||x_0||;$$
$$|\lambda_x| \ge 1 - \epsilon^x\frac{|||A|||\,||x_0||}{||Ax_0||}.$$

We can thus deduce from Equation (9) that, on an event of probability at least $1-2C_1\exp(-C_2m)$, as soon as $\inf_{\phi\in\mathbb{R}} ||e^{i\phi}x_0 - x|| \le \epsilon||x_0||$,

$$||\,\text{phase}(\lambda_x)(Ax_0) - b \odot \text{phase}(Ax)|| \le \left(12\eta + \frac{24}{25}\right)\frac{\epsilon^x}{1 - \epsilon^x\frac{|||A|||\,||x_0||}{||Ax_0||}}|||A|||\,||x_0|| \tag{10}$$

if

$$\frac{\epsilon^x}{1 - \epsilon^x\frac{|||A|||\,||x_0||}{||Ax_0||}}\frac{|||A|||\,||x_0||}{||Ax_0||} < \frac{\gamma}{6}. \tag{11}$$

Equation (10) implies in particular that, if Condition (11) holds,

$$||\,\text{phase}(\lambda_x)x_0 - A^\dagger(b \odot \text{phase}(Ax))|| \le \left(12\eta + \frac{24}{25}\right)\frac{\epsilon^x}{1 - \epsilon^x\frac{|||A|||\,||x_0||}{||Ax_0||}}|||A^\dagger|||\,|||A|||\,||x_0||. \tag{12}$$

To conclude, it is enough to control the norms of A and $A^\dagger$ with the following classical result.

Proposition 3.5 (Davidson and Szarek [2001], Thm II.13)**.** *If A is chosen according to Equation (4), then, for any t, with probability at least*

$$1 - 2\exp\left(-mt^2\right),$$

we have, for any $x \in \mathbb{C}^n$,

$$\sqrt{m}\left(1 - \sqrt{\frac{n}{m}} - t\right)||x|| \leq ||Ax|| \leq \sqrt{m}\left(1 + \sqrt{\frac{n}{m}} + t\right)||x||.$$

From this proposition, if we choose δ, M, t such that

$$12\eta + \frac{24}{25} < \delta < 1;$$
$$\epsilon < \min\left(\frac{1}{4}, \frac{\gamma}{24}, \frac{1}{2\delta}\left(\delta - 12\eta - \frac{24}{25}\right)\right);$$
$$\frac{1 + \sqrt{\frac{1}{M}} + t}{1 - \sqrt{\frac{1}{M}} - t} \leq \min\left(2, \frac{(1-2\epsilon)\delta}{12\eta + \frac{24}{25}}\right).$$

we have, for $m \geq Mn$, with probability at least $1 - 2e^{-mt^2}$, as soon as $\epsilon^x \leq \epsilon$,

$$\begin{aligned}
\frac{\epsilon^x}{1 - \epsilon^x \frac{|||A|||\,||x_0||}{||Ax_0||}} \frac{|||A|||\,||x_0||}{||Ax_0||} &\leq \frac{\epsilon}{1 - \epsilon \frac{|||A|||\,||x_0||}{||Ax_0||}} \frac{1 + \sqrt{\frac{1}{M}} + t}{1 - \sqrt{\frac{1}{M}} - t} \\
&\leq \frac{2\epsilon}{1 - \frac{1}{4}\frac{|||A|||\,||x_0||}{||Ax_0||}} \\
&\leq \frac{2\epsilon}{1 - \frac{1}{4}\frac{1+\sqrt{\frac{1}{M}}+t}{1-\sqrt{\frac{1}{M}}-t}} \\
&\leq 4\epsilon \\
&< \frac{\gamma}{6},
\end{aligned}$$

and

$$\begin{aligned}
&\left(12\eta + \frac{24}{25}\right)\frac{\epsilon^x}{1 - \epsilon^x \frac{|||A|||\,||x_0||}{||Ax_0||}} |||A^\dagger|||\,|||A|||\,||x_0|| \\
&\qquad \leq \left(12\eta + \frac{24}{25}\right)\frac{\epsilon^x}{1 - 2\epsilon} |||A^\dagger|||\,|||A|||\,||x_0||
\end{aligned}$$

$$\leq \left(12\eta + \frac{24}{25}\right) \frac{\epsilon^x}{1-2\epsilon} \frac{1+\sqrt{\frac{1}{M}}+t}{1-\sqrt{\frac{1}{M}}-t} ||x_0||$$
$$\leq \delta \epsilon^x ||x_0||.$$

We now combine this with Equation (12): with probability at least

$$1 - 2C_1 \exp(-C_2 m) - 2\exp\left(-mt^2\right),$$

we have, for all x such that $\inf_{\phi\in\mathbb{R}} ||e^{i\phi}x_0 - x|| \leq \epsilon ||x_0||$,

$$||\,\mathrm{phase}(\lambda_x)x_0 - A^\dagger(b \odot \mathrm{phase}(Ax))|| \leq \delta\epsilon^x ||x_0|| = \delta \inf_{\phi\in\mathbb{R}} ||e^{i\phi}x_0 - x||.$$

□

3.2 Global convergence

In the last paragraph, we have seen that the alternating projections operator is contractive, with high probability, in an $\epsilon||x_0||$-neighborhood of the solution x_0. This implies that, if the starting point of alternating projections is at distance at most $\epsilon||x_0||$ from x_0, alternating projections converge to x_0. So if we have a way to find such an initial point, we obtain a globally convergent algorithm.

Several initialization methods have been proposed that achieve the precision we need with an optimal number of measurements, that is $m = O(n)$. Let us mention the truncated spectral initialization by Chen and Candès [2015] (improving upon the slightly suboptimal spectral initializations introduced by Netrapalli, Jain, and Sanghavi [2013] and Candès, Li, and Soltanolkotabi [2015]), the null initialization by Chen, Fannjiang, and Liu [2016] and the method described by Gao and Xu [2016]. All these methods consist in computing the largest or smallest eigenvector of

$$\sum_{i=1}^m \alpha_i a_i a_i^*,$$

where the $\alpha_1, \dots, \alpha_m$ are carefully chosen coefficients, that depend only on b.

The method of [Chen and Candès, 2015], for example, has the following guarantees.

Théorème 3.6 (Proposition 3 of [Chen and Candès, 2015])**.** *Let $\epsilon > 0$ be fixed.*

We define z as the main eigenvector of

$$\frac{1}{m}\sum_{i=1}^m |a_i^* x_0|^2 a_i a_i^* 1_{|a_i^* x_0|^2 \leq \frac{9}{m}\sum_{j=1}^m |a_i^* x_0|^2}. \tag{13}$$

There exist $C_1, C_2, M > 0$ such that, with probability at least

$$1 - C_1 \exp(-C_2 m),$$

the vector z obeys

$$\inf_{\phi\in\mathbb{R},\lambda\in\mathbb{R}_+^*} ||e^{i\phi}x_0 - \lambda z|| \leq \epsilon ||x_0||,$$

provided that $m \geq Mn$.

Combining this initialization procedure with alternating projections, we get Algorithm 1. As shown by the following corollary, it converges towards the correct solution, at a linear rate, with high probability, for $m = O(n)$.

Input : $A \in \mathbb{C}^{m\times n}, b = |Ax_0| \in \mathbb{R}^m, T \in \mathbb{N}^*$.

Initialization: set z_0 to be the main eigenvector of the matrix in Equation (13).

for $t = 1$ **to** T **do**

| Set $z_t \leftarrow A^\dagger(b \odot \text{phase}(Az_{t-1}))$.

end

Output: z_T.

Algorithm 1: Alternating projections with truncated spectral initialization

Corollaire 3.7. *There exist $C_1, C_2, M > 0, \delta \in]0;1[$ such that, with probability at least*

$$1 - C_1 \exp(-C_2 m),$$

Algorithm 1 satisfies

$$\forall t \in \mathbb{N}^*, \qquad \inf_{\phi\in\mathbb{R}} ||e^{i\phi}x_0 - z_t|| \leq \delta^t ||x_0||, \tag{14}$$

provided that $m \geq Mn$.

Proof. Let us fix $\epsilon, \delta \in]0;1[$ as in Theorem 3.1. Let us assume that the properties described in Theorems 3.1 and 3.6 hold; it happens on an event of probability at least

$$1 - C_1 \exp(-C_2 m),$$

provided that $m \geq Mn$, for some constants $C_1, C_2, M > 0$.

Let us prove that, on this event, Equation (14) also holds.

We proceed by recursion. From Theorem 3.6, there exist $\phi \in \mathbb{R}, \lambda \in \mathbb{R}_+^*$ such that

$$||e^{i\phi}x_0 - \lambda z_0|| \leq \epsilon ||x_0||.$$

So, from Theorem 3.1, applied to $x = \lambda z_0$,

$$\inf_{\phi\in\mathbb{R}} ||e^{i\phi}x_0 - z_1|| = \inf_{\phi\in\mathbb{R}} ||e^{i\phi}x_0 - A^\dagger(b\odot \text{phase}(z_0))||$$

$$= \inf_{\phi\in\mathbb{R}} ||e^{i\phi}x_0 - A^\dagger(b\odot \text{phase}(\lambda z_0))||$$

$$\leq \delta \inf_{\phi\in\mathbb{R}} ||e^{i\phi}x_0 - \lambda z_0||$$

$$\leq \epsilon\delta||x_0||.$$

This proves Equation (14) for $t = 1$.

The same reasoning can be reapplied to also prove the equation for $t = 2, 3, \ldots$. □

3.3 Complexity

Let $\eta > 0$ be the relative precision that we want to achieve:

$$\inf_{\phi\in\mathbb{R}} ||e^{i\phi}x_0 - z_T|| \leq \eta||x_0||.$$

Let us compute the number of operations that Algorithm 1 requires to reach this precision.

The main eigenvector of the matrix defined in Equation (13) can be computed - up to precision η - in approximately $O(\log(1/\eta) + \log(n))$ power iterations. Each power iteration is essentially a matrix-vector multiplication, and thus requires $O(mn)$ operations.[2] As a consequence, the complexity of the initialization is

$$O(mn\,(\log(1/\eta) + \log(n))).$$

Then, at each step of the `for` loop, the most costly operation is the multiplication by $A^\dagger$. When performed with the conjugate gradient method, it requires $O(mn\log(1/\eta))$ operations. To reach a precision equal to η, we need to perform $O(\log(1/\eta))$ iterations of the loop. So the total complexity of Algorithm 1 is

$$O(mn\left(\log^2(1/\eta) + \log(n)\right)).$$

Let us mention that, when A has a special structure, there may exist fast algorithms for the multiplication by A and the orthogonal projection onto Range(A). In the case of masked

[2]These matrix-vector multiplications can be computed without forming the whole matrix (which would require $O(mn^2)$ operations), because this matrix factorizes as

$$\frac{1}{m}A^*\text{Diag}(|Ax_0|^2\odot I)A,$$

where $I \in \mathbb{R}^m$ is such that $\forall i \leq m, I_i = 1_{|A_ix_0|^2\leq \frac{9}{m}\sum_{j=1}^m |A_ix_0|^2}$.

	Alternating projections	Truncated Wirtinger flow
Unstructured case	$O\left(mn\left(\log^2(1/\eta)+\log(n)\right)\right)$	$O\left(mn\left(\log(1/\eta)+\log(n)\right)\right)$
Fourier masks	$O\left(m\log(n)\left(\log(1/\eta)+\log(n)\right)\right)$	$O\left(m\log(n)\left(\log(1/\eta)+\log(n)\right)\right)$

Figure 2: Complexity of alternating projections with initialization, and truncated Wirtinger flow.

Fourier measurements considered in [Candès, Li, and Soltanolkotabi, 2015], for example, assuming that our convergence theorem still holds, despite the non-Gaussianity of the measurements, the complexity of each of these operations reduces to $O(m \log n)$, yielding a global complexity of

$$O(m\log(n)(\log(1/\eta)+\log(n))).$$

The complexity is then almost linear in the number of measurements.

As a comparison, Truncated Wirtinger flow, which is currently the most efficient known method for phase retrieval from Gaussian measurements, has an identical complexity, up to a $\log(1/\eta)$ factor in the unstructured case (see Figure 2).

4 Alternating projections without good initialization

4.1 Main result

In this section, we assume that the number of measurements is quadratic in n instead of linear (that is $m \geq Mn^2$, for M large enough). In this setting, we show that any initialization vector x, unless it is almost orthogonal to the ground truth x_0, yields perfect recovery when provided to the alternating projection routine. This in particular proves that, in this regime, there is no stagnation point (unless possibly among the vectors almost orthogonal to x_0).

The convergence rate is almost as good as in the case where a good initialization is provided: after $O(\log n)$ iterations, it becomes linear.

We say that a vector $x \in \mathbb{C}^n$ is *not almost orthogonal* to x_0 if

$$\mu\frac{||x_0||\,||x||}{\sqrt{n}} \leq |\langle x_0, x\rangle|,$$

for some fixed constant $\mu > 0$. In what follows, we assume $\mu = 1$, but it is only to simplify the notations; the same result would hold for any value of μ.

We remark that, in the unit sphere, the proportion (in terms of volume) of vectors that are almost orthogonal to x_0 goes to a constant depending on μ when n goes to $+\infty$. This constant can be arbitrarily small if μ is small. As a consequence, if we choose $x \in \mathbb{C}^n$ according to an isotropic probability law, the probability that it is almost orthogonal to x_0 can be arbitrarily small.

To prove global convergence, we first need to understand what happens when we apply one iteration of the alternating projections routine to some vector x. We only consider vectors x that are not almost orthogonal to x_0. We also do not consider vectors that are very close to x_0: these vectors are already taken care of by Theorem 3.1.

Théorème 4.1. *For any $\epsilon > 0$, there exist $C_1, C_2, M, \delta > 0$ such that, if $m \geq Mn^2$, then, with probability at least*

$$1 - C_1 \exp(-C_2 m^{1/8}),$$

the following property holds: for any $x \in \mathbb{C}^n$ such that

$$\frac{||x_0||\,||x||}{\sqrt{n}} \leq |\langle x_0, x \rangle| \leq (1-\epsilon)||x_0||\,||x||, \tag{15}$$

we have

$$\frac{|\left\langle x_0, A^\dagger(b \odot \text{phase}(Ax))\right\rangle|}{||x_0||\,||A^\dagger(b \odot \text{phase}(Ax))||} \geq (1+\delta)\frac{|\langle x_0, x\rangle|}{||x_0||\,||x||}. \tag{16}$$

Before proving this theorem, let us establish its main consequence : the global convergence of alternating projections starting from any initial point that is not almost orthogonal to x_0. The algorithm is summarized in Algorithm 2 and global convergence is proven in Corollary 4.2.

Input : $A \in \mathbb{C}^{m \times n}, b = |Ax_0| \in \mathbb{R}^m, T \in \mathbb{N}^*$, any $x \in \mathbb{C}^n$ not almost orthogonal to x_0.

Initialization: set $z_0 = x$.

for $t = 1$ **to** T **do**

| Set $z_t \leftarrow A^\dagger(b \odot \text{phase}(Az_{t-1}))$.

end

Output: z_T.

Algorithm 2: Alternating projections without good initialization

Corollaire 4.2. *There exist $C_1, C_2, \gamma, M > 0, \Delta \in]0;1[$ such that, with probability at least*

$$1 - C_1 \exp(-C_2 n),$$

Algorithm 2 satisfies:

$$\forall t \geq \gamma \log n, \qquad \inf_{\phi \in \mathbb{R}} ||e^{i\phi}x_0 - z_t|| \leq \Delta^{t-\gamma \log n}||x_0||, \tag{17}$$

provided that $m \geq Mn^2$.

Proof. From Theorem 3.1, there exist $C_1^{(1)}, C_2^{(1)}, \epsilon^{(1)}, M^{(1)} > 0$ such that, if $m \geq M^{(1)} n$, then, with probability at least

$$1 - C_1^{(1)} \exp(-C_2^{(1)} m),$$

the following property holds: any $z \in \mathbb{C}^n$ such that $\inf_{\phi \in \mathbb{R}} ||e^{i\phi} x_0 - z|| \leq \epsilon^{(1)} ||x_0||$ satisfies

$$\inf_{\phi \in \mathbb{R}} ||e^{i\phi} x_0 - A^\dagger (b \odot \text{phase}(Az))|| \leq \delta^{(1)} \inf_{\phi \in \mathbb{R}} ||e^{i\phi} x_0 - z||, \tag{18}$$

for some absolute constant $\delta^{(1)} \in]0; 1[$. In the following, we assume that this event is realized.

We now use Theorem 4.1, for $\epsilon = {\epsilon^{(1)}}^2/2$. Let $C_1, C_2, M, \delta > 0$ be defined as in this theorem. We assume that the event described in the theorem is realized, which happens with probability at least $1 - C_1 \exp(-C_2 n)$.

We consider the sequence $(z_t)_{t \geq 0}$ defined in Algorithm 2, and distinguish two cases.

First, if the initial point $z_0 = x$ is such that

$$|\langle x_0, x \rangle| > (1 - \epsilon) ||x_0|| \, ||x||,$$

then, setting $x' = \frac{||x_0||}{||x||} x$,

$$\begin{aligned} \inf_{\phi \in \mathbb{R}} ||e^{i\phi} x_0 - x'|| &= \sqrt{||x_0||^2 + ||x'||^2 - 2 |\langle x_0, x' \rangle|} \\ &< ||x_0|| \sqrt{2\epsilon} \\ &= \epsilon^{(1)} ||x_0||. \end{aligned}$$

We can thus proceed by recursion, as in the proof of Corollary 3.7, to show that:

$$\forall t \in \mathbb{N}^*, \qquad \inf_{\phi \in \mathbb{R}} ||e^{i\phi} x_0 - z_t|| \leq (\delta^{(1)})^t \epsilon^{(1)} ||x_0||. \tag{19}$$

So Equation (17) is satisfied, provided that we have chosen $\Delta \geq \delta^{(1)}$.

Second, we consider the case where the initial point $z_0 = x$ is such that

$$|\langle x_0, x \rangle| \leq (1 - \epsilon) ||x_0|| \, ||x||.$$

Let then $\mathcal{T}$ be the smallest index t such that the following inequality is not satisfied:

$$\frac{||x_0|| \, ||z_t||}{\sqrt{n}} \leq |\langle x_0, z_t \rangle| \leq (1 - \epsilon) ||x_0|| \, ||z_t||. \tag{20}$$

As $z_0 = x$ is not almost orthogonal to x_0, we must have $\mathcal{T} \geq 1$. For any $t = 0, \ldots, \mathcal{T} - 1$, Equation (16) of Theorem 4.1 ensures that

$$\frac{|\langle x_0, z_{t+1} \rangle|}{||x_0|| \, ||z_{t+1}||} \geq (1 + \delta) \frac{|\langle x_0, z_t \rangle|}{||x_0|| \, ||z_t||}. \tag{21}$$

In particular,

$$\frac{|\langle x_0, z_{\mathcal{T}}\rangle|}{||x_0||\,||z_{\mathcal{T}}||} \geq \frac{|\langle x_0, z_0\rangle|}{||x_0||\,||z_0||} \geq \frac{1}{\sqrt{n}}.$$

As Equation (20) is not satisfied, it means that

$$|\langle x_0, z_{\mathcal{T}}\rangle| > (1-\epsilon)||x_0||\,||z_{\mathcal{T}}||.$$

We can now apply the same reasoning as the one that led to Equation (19), and get

$$\forall t \geq \mathcal{T}+1, \qquad \inf_{\phi\in\mathbb{R}} ||e^{i\phi}x_0 - z_t|| \leq (\delta^{(1)})^{t-\mathcal{T}}\epsilon^{(1)}||x_0||.$$

This implies Equation (17), provided that $\mathcal{T} \leq \gamma \log n$ for some absolute constant γ. From Equation (21) and the fact that z_0 is not almost orthogonal to x_0,

$$\frac{|\langle x_0, z_{\mathcal{T}-1}\rangle|}{||x_0||\,||z_{\mathcal{T}-1}||} \geq \frac{(1+\delta)^{\mathcal{T}-1}}{\sqrt{n}}.$$

As $\mathcal{T}-1$ satisfies Equation (20), we must have

$$\frac{(1+\delta)^{\mathcal{T}-1}}{\sqrt{n}} \leq 1-\epsilon \leq 1;$$

$$\Rightarrow \qquad \mathcal{T} \leq 1 + \frac{\log n}{2\log(1+\delta)}.$$

And this expression can be bounded by $\gamma \log n$, for some $\gamma > 0$ independent from n.

So we have shown that Equation (17) holds when the events described in Theorems 3.1 and 4.1 happen. When $m \geq \max(M, M^{(1)})n^2$, this occurs with probability at least

$$1 - C_1^{(1)}\exp(-C_2^{(1)}m) - C_1\exp(-C_2 n) \geq 1 - (C_1 + C_1^{(1)})\exp(-\min(C_2^{(1)}, C_2)n).$$

□

4.2 Proof of Theorem 4.1

Proof of Theorem 4.1. We will actually consider a variant of Equation (16), in the "image domain", that is in $\mathbb{C}^m$ instead of $\mathbb{C}^n$. This variant is easier to analyze and, according to the following lemma (proven in Paragraph C.1), it implies Equation (16).

Lemme 4.3. *To prove Theorem 4.1, it is enough to prove that there exist $C_1, C_2, M, \delta > 0$ such that, if $m \geq Mn^2$, then, with probability at least $1 - C_1\exp(-C_2 m^{1/8})$, the property*

$$|\langle Ax_0, b \odot \text{phase}(Ax)\rangle| \geq (1+\delta)m\frac{||x_0||}{||x||}|\langle x_0, x\rangle| \tag{22}$$

holds for any $x \in \mathbb{C}^n$ verifying Condition (15).

The proof of Equation (22) is in two parts. We first prove (Lemma 4.4) that this equation holds (with high probability) for all x belonging to a net with very small spacing. This part is the most technical: a direct union bound, that does not take advantage of the correlation between the vectors of the net, is not sufficient. We use a chaining argument instead. The detailed proof is in Paragraph C.2.

In a second part (Lemma 4.5), we prove that, with high probability, for any x and y very close, $|\langle Ax_0, b\odot \text{phase}(Ax)\rangle - \langle Ax_0, b\odot\text{phase}(y)\rangle|$ is small. This allows us to extend the inequality proven for vectors of the net to all vectors. This result is a consequence of two facts: first, the phase is a Lipschitz function outside any neighborhood of zero. Second, with high probability, for any x and y, the vectors Ax and Ay have few entries that are close to zero. The detailed proof is in Paragraph C.3.

Lemme 4.4. *For any $n\in\mathbb{N}^*$, we set*

$$\mathcal{E}_n = \left\{ x\in\mathbb{C}^n, ||x|| = 1 \text{ and } \frac{||x_0||\,||x||}{\sqrt{n}} \leq |\langle x_0, x\rangle| \leq (1-\epsilon)||x_0||\,||x|| \right\}.$$

Let α be any positive number.

There exist $c, C_1, C_2, M, \delta > 0$ and, for any $n\in\mathbb{N}^*$, a $cm^{-\alpha}$-net $\mathcal{N}_n$ of $\mathcal{E}_n$ such that, when $m\geq Mn^2$, with probability at least

$$1 - C_1\exp(-C_2 m^{1/2}),$$

the following property holds: for any $x\in\mathcal{N}_n$,

$$|\langle Ax_0, b\odot\text{phase}(Ax)\rangle| \geq (1+\delta) m \frac{||x_0||}{||x||} |\langle x_0, x\rangle|.$$

Lemme 4.5. *For any $c>0$, there exist $C_1, C_2, C_3 > 0$ such that, with probability at least*

$$1 - C_1\exp(-C_2 m^{1/8}),$$

the following property holds for any unit-normed $x, y\in\mathbb{C}^n$, when $m\geq 2n^2$:

$$|\langle Ax_0, b\odot\text{phase}(Ax)\rangle - \langle Ax_0, b\odot\text{phase}(Ay)\rangle| \leq C_3 ||x_0||^2 n m^{1/4} \quad \text{if } ||x-y||\leq cm^{-7/2}.$$

To conclude, we apply Lemma 4.4 with $\alpha = 7/2$. We define $c, C_1, C_2, M, \delta > 0$, the set $\mathcal{E}_n$ and the $cm^{-7/2}$-net $\mathcal{N}_n$ as in the statement of this lemma. With probability at least

$$1 - C_1\exp(-C_2 m^{1/2}) - C_1\exp(-C_2 m^{1/8}),$$

the events described in both Lemmas 4.4 and 4.5 happen. In this case, for any $x \in \mathbb{C}^n$ verifying Condition (15), the normalized vector $x' = x/||x||$ belongs to $\mathcal{E}_n$. As $\mathcal{N}_n$ is a $cm^{-7/2}$-net of $\mathcal{E}_n$, there exists $y \in \mathcal{N}_n$ such that

$$||x' - y|| \leq cm^{-7/2}.$$

By triangular inequality, and using Lemmas 4.4 and 4.5,

$$\begin{aligned}
|\langle Ax_0, b \odot \text{phase}(Ax')\rangle| &\geq |\langle Ax_0, b \odot \text{phase}(Ay)\rangle| \\
&\qquad - |\langle Ax_0, b \odot \text{phase}(Ay)\rangle - \langle Ax_0, b \odot \text{phase}(Ax')\rangle| \\
&\geq (1+\delta)m\frac{||x_0||}{||y||}|\,\langle x_0, y\rangle\,| - C_3||x_0||^2 nm^{1/4} \\
&= (1+\delta)m||x_0||\,|\,\langle x_0, y\rangle\,| - C_3||x_0||^2 nm^{1/4} \\
&\geq (1+\delta)m||x_0||\,|\,\langle x_0, x'\rangle\,| - (1+\delta)m||x_0||^2||x'-y|| - C_3||x_0||^2 nm^{1/4} \\
&\geq (1+\delta)m||x_0||\,|\,\langle x_0, x'\rangle\,| - ||x_0||^2\left((1+\delta)cm^{-5/2} + C_3 nm^{1/4}\right).
\end{aligned}$$

As x' belongs to $\mathcal{E}_n$, if $m \geq Mn^2$ and m is large enough,

$$\begin{aligned}
||x_0||\left((1+\delta)cm^{-5/2} + C_3 nm^{1/4}\right) &\leq ||x_0||\frac{\delta m n^{-1/2}}{2} \\
&\leq \frac{\delta m}{2}|\,\langle x_0, x'\rangle\,|.
\end{aligned}$$

So we deduce from this and the inequality immediately before:

$$\begin{aligned}
&|\langle Ax_0, b \odot \text{phase}(Ax')\rangle| \geq \left(1 + \frac{\delta}{2}\right) m||x_0||\,|\,\langle x_0, x'\rangle\,|; \\
\Rightarrow\qquad &|\langle Ax_0, b \odot \text{phase}(Ax)\rangle| \geq \left(1 + \frac{\delta}{2}\right) m\frac{||x_0||}{||x||}\,|\,\langle x_0, x\rangle\,|.
\end{aligned}$$

By Lemma 4.3, this is what we had to prove.

□

5 Numerical experiments

In this section, we numerically validate the results obtained in Corollaries 3.7 and 4.2. We formulate a conjecture about the convergence of alternating projections with random initialization, in the regime $m = O(n)$.

The code used to generate Figures 3, 4 and 6 is available at `http://www-math.mit.edu/~waldspur/code/alternating_projections_code.zip`.

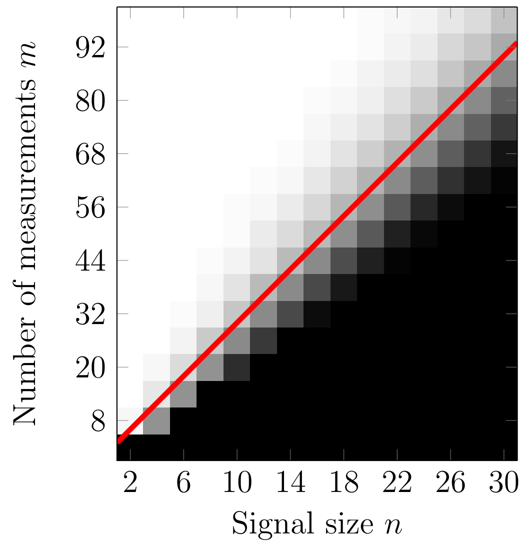

Figure 3: Probability of success for Algorithm 1, as a function of n and m. Black points indicate a probability equal to 0, and white points a probability equal to 1. The red line serves as a reference: it is the line $m = 3n$.

5.1 Alternating projections with initialization

Our first experiment consists in a numerical validation of Corollary 3.7: alternating projections succeed with high probability, when they start from a good initial point, in the regime where the number of measurements is linear in the problem dimension ($m = O(n)$).

We use the initialization method described in [Chen and Candès, 2015], as presented in Algorithm 1. We run the algorithm for various choices of n and m, 3000 times for each choice. This allows us to compute an empirical probability of success, for each value of (n, m).

The results are presented in Figure 3. They confirm that, when $m = Cn$, for a sufficiently large constant $C > 0$, the success probability can be arbitrarily close to 1.

5.2 Alternating projections without good initialization

5.2.1 Disappearing of stagnation points

Next, we investigate Corollary 4.2: if $m \geq Cn^2$, for $C > 0$ large enough, the method of alternating projections succeeds, with high probability, starting from any initialization (that is not almost orthogonal to the true solution). In particular, there is no stagnation point, unless possibly among vectors that are *almost orthogonal* to the true solution.

To numerically validate this result, we have generated vectors x_0 of size n and measurements

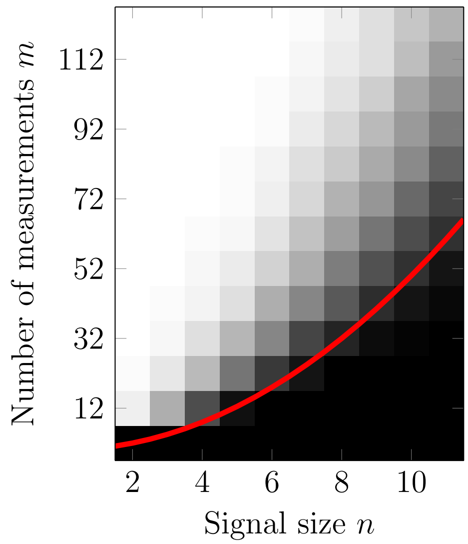

Figure 4: Probability that there is no stagnation point, as a function of n and m. Black points indicate a probability equal to 0, and white points a probability equal to 1. The red line serves as a reference: it is the line $m = \frac{1}{2}n^2$.

matrices A of size $m \times n$ for various choices of n and m. For each (x_0, A), we have randomly chosen 10000 initializations that were not almost orthogonal to x_0, and we have recorded whether alternating projections, starting from these initializations, always succeeded in reconstructing x_0 from $|Ax_0|$. When at least one of these initializations failed, it proved that there was at least one stagnation point. Otherwise, we have considered it as a sign of absence of stagnation points.

We could thus compute, for each choice of (n, m), the probability of absence of stagnation point. The result is displayed on Figure 4. As foreseen by Corollary 4.2, the probability becomes arbitrarily close to 1 when $m \geq Cn^2$ for $C > 0$ large enough.

The same results are presented in Figure 5 under a different form. The graph on the left hand side shows, for each n, the number M_n of measurements above which the probability that there is at least one stagnation point drops under 0.5. The curve has a clear quadratic shape.

The plot on the right hand side represents M_n/n^2 as a function of n. It is clearly upper bounded by a constant. It also seems to be lower bounded by a positive constant (or possibly by a very slowly decaying function, like $(\log\log)^{-1}$), which indicates that the number of measurements $m = O(n^2)$ that appears in Corollary 4.2 is probably optimal: when $m \ll n^2$, the probability that there are no stagnation points is small.

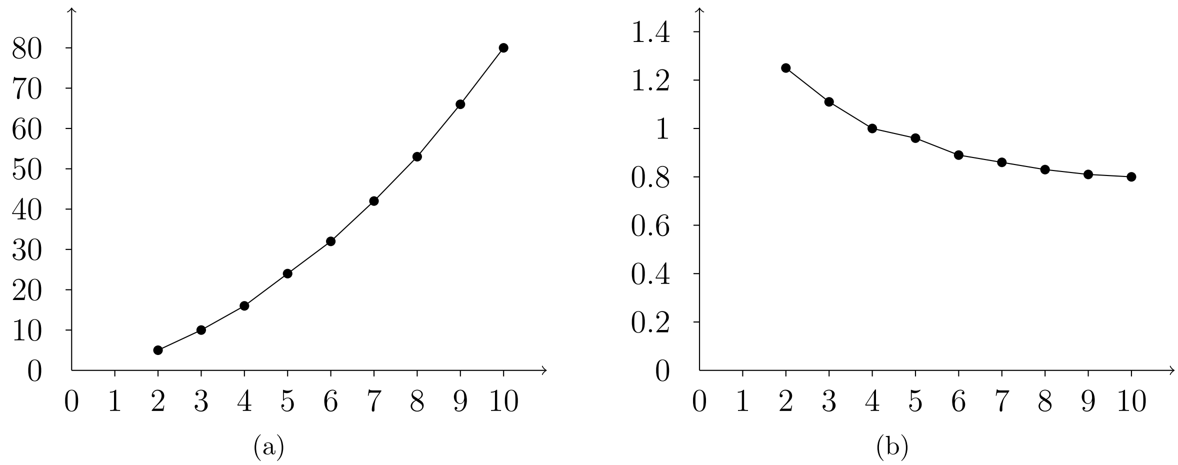

Figure 5: (a) For each signal size n, the smallest number of measurements m for which the probability that there exist stagnation points is under 0.5. (b) The same curve, renormalized by division by n^2.

5.2.2 Random initialization

Our last experiment consists in measuring the probability that alternating projections succeed, when started from a random initial point (sampled from the unit sphere with uniform probability).

The results are presented in Figure 6. They lead to the following conjecture.

Conjecture 5.1. *Let any $\epsilon > 0$ be fixed. When $m \geq Cn$, for $C > 0$ large enough, alternating projections with a random isotropic initialization succeed with probability at least $1 - \epsilon$.*

As we have seen in Paragraph 5.2.1, in the regime $m = O(n)$, there are (attractive) stagnation points, so there are initializations for which alternating projections fail. However, it seems that these bad initializations occupy a very small volume in the space of all possible initial points. Therefore, a random initialization leads to success with high probability.

Unfortunately, proving this conjecture a priori requires to evaluate in some way the size of the attraction basin of stagnation points, which seems difficult.

Acknowledgments

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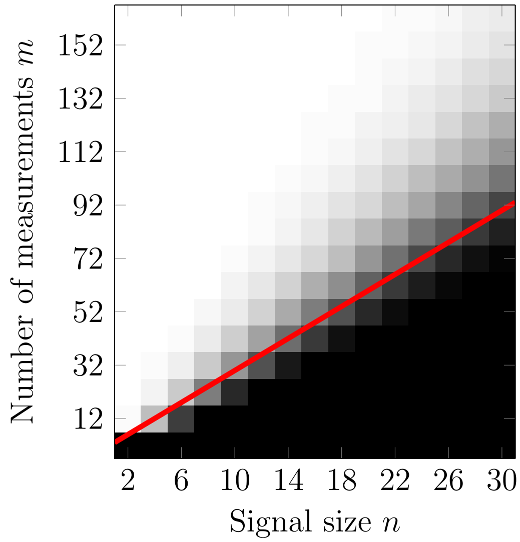

Figure 6: Probability of success for alternating projections with a random Gaussian initialization, as a function of n and m. Black points indicate a probability equal to 0, and white points a probability equal to 1. The red line serves as a reference: it is the line $m = 3n$.

A Proposition 2.1

Proposition (Proposition 2.1)**.** *For any y_0, the sequence $(y_k)_{k\in\mathbb{N}}$ is bounded. Any accumulation point y_∞ of $(y_k)_{k\in\mathbb{N}}$ satisfies the following property:*

$$\exists u \in E_{\text{phase}}(y_\infty), \qquad (AA^\dagger)(b \odot u) = y_\infty.$$

In particular, if y_∞ has no zero entry,

$$(AA^\dagger)(b \odot \text{phase}(y_\infty)) = y_\infty.$$

Proof of Proposition 2.1. The boundedness of $(y_k)_{k\in\mathbb{N}}$ is a consequence of the fact that $||y'_k|| = ||b||$ for all k, so $||y_{k+1}|| \le |||AA^\dagger|||\, ||b||$.

Let us show the second part of the statement. Let y_∞ be an accumulation point of $(y_k)_{k\in\mathbb{N}}$, and let $\phi : \mathbb{N} \to \mathbb{N}$ be an extraction such that

$$y_{\phi(n)} \to y_\infty \quad \text{when} \quad n \to +\infty.$$

By compacity, as $(y'_{\phi(n)})_{n\in\mathbb{N}}$ and $(y_{\phi(n)+1})_{n\in\mathbb{N}}$ are bounded sequences, we can assume, even if we have to consider replace ϕ by a subextraction, that they also converge. We denote by y'_∞ and y^{+1}_∞ their limits:

$$y'_{\phi(n)} \to y'_\infty \quad \text{and} \quad y_{\phi(n)+1} \to y^{+1}_\infty \quad \text{when } n \to +\infty.$$

Let us define

$$E_b = \{y' \in \mathbb{C}^m, |y'| = b\}.$$

We observe that, for any k,

$$d(y'_{k-1}, \text{Range}(A)) \geq d(y_k, E_b) \geq d(y'_k, \text{Range}(A)).$$

Indeed, because the operators $y \to b \odot \text{phase}(y)$ and $y \to (AA^\dagger)y$ are projections,

$$\begin{aligned} d(y'_{k-1}, \text{Range}(A)) = &\quad d(y'_{k-1}, y_k) &\geq d(y_k, E_b); \\ d(y_k, E_b) = &\quad d(y_k, y'_k) &\geq d(y'_k, \text{Range}(A)). \end{aligned}$$

So the sequences $(d(y_k, E_b))_{k\in\mathbb{N}}$ and $(d(y'_k, \text{Range}(A)))_{k\in\mathbb{N}}$ converge to the same non-negative limit, that we denote by δ. In particular,

$$d(y_\infty, E_b) = \delta = d(y'_\infty, \text{Range}(A)).$$

If we pass to the limit the equalities $d(y_{\phi(n)}, E_b) = ||y_{\phi(n)} - y'_{\phi(n)}||$ and $d(y'_{\phi(n)}, \text{Range}(A)) = ||y'_{\phi(n)} - y_{\phi(n)+1}||$, we get

$$||y_\infty - y'_\infty|| = ||y'_\infty - y^{+1}_\infty|| = \delta = d(y'_\infty, \text{Range}(A)).$$

As Range(A) is convex, the projection of y'_∞ onto it is uniquely defined. This implies

$$y_\infty = y^{+1}_\infty,$$

and, because $\forall n, y_{\phi(n)+1} = (AA^\dagger)y'_{\phi(n)}$,

$$y_\infty = y^{+1}_\infty = (AA^\dagger)y'_\infty.$$

To conclude, we now have to show that $y'_\infty = b \odot u$ for some $u \in E_{\text{phase}}(y_\infty)$. We use the fact that, for all n, $y'_{\phi(n)} = b \odot \text{phase}(y_{\phi(n)})$.

For any $i \in \{1, \dots, m\}$, if $(y_\infty)_i \neq 0$, phase is continuous around $(y_\infty)_i$, so $(y'_\infty)_i = b_i \,\text{phase}((y_\infty)_i)$. We then set $u_i = \text{phase}((y_\infty)_i)$, and we have $(y'_\infty)_i = b_i u_i$.

If $(y_\infty)_i = 0$, we set $u_i = \text{phase}((y'_\infty)_i) \in E_{\text{phase}}(0) = E_{\text{phase}}((y_\infty)_i)$. We then have $y'_\infty = |y'_\infty| u_i = b_i u_i$.

With this definition of u, we have, as claimed, $y'_\infty = b \odot u$ and $u \in E_{\text{phase}}(y_\infty)$.

□

B Technical lemmas for Section 3

B.1 Proof of Lemma 3.2

Lemme (Lemma 3.2). *For any $z_0, z \in \mathbb{C}$,*

$$|\operatorname{phase}(z_0 + z) - \operatorname{phase}(z_0)| \leq 2.1_{|z| \geq |z_0|/6} + \frac{6}{5}\left|\operatorname{Im}\left(\frac{z}{z_0}\right)\right|.$$

Proof. The inequality holds if $z_0 = 0$, so we can assume $z_0 \neq 0$. We remark that, in this case,

$$|\operatorname{phase}(z_0 + z) - \operatorname{phase}(z_0)| = |\operatorname{phase}(1 + z/z_0) - 1|.$$

It is thus enough to prove the lemma for $z_0 = 1$, so we make this assumption.

When $|z| \geq 1/6$, the inequality is valid. Let us now assume that $|z| < 1/6$. Let $\theta \in \left]-\frac{\pi}{2}; \frac{\pi}{2}\right[$ be such that

$$e^{i\theta} = \operatorname{phase}(1 + z).$$

Then

$$\begin{aligned}
|\operatorname{phase}(1 + z) - 1| &= |e^{i\theta} - 1| \\
&= 2|\sin(\theta/2)| \\
&\leq |\tan\theta| \\
&= \frac{|\operatorname{Im}(1 + z)|}{|\operatorname{Re}(1 + z)|} \\
&\leq \frac{|\operatorname{Im}(z)|}{1 - |z|} \\
&\leq \frac{6}{5}|\operatorname{Im}(z)|.
\end{aligned}$$

So the inequality is also valid. □

B.2 Proof of Lemma 3.3

Lemme (Lemma 3.3). *For any $\eta > 0$, there exists $C_1, C_2, M, \gamma > 0$ such that the inequality*

$$||\,|Ax_0| \odot 1_{|v| \geq |Ax_0|}|| \leq \eta||v||$$

holds for any $v \in \operatorname{Range}(A)$ such that $||v|| < \gamma||Ax_0||$, with probability at least

$$1 - C_1 \exp(-C_2 m),$$

when $m \geq Mn$.

Proof. For any $S \subset \{1, \ldots, m\}$, we denote by 1_S the vector of $\mathbb{C}^m$ such that

$$(1_S)_j = 1 \text{ if } j \in S$$
$$= 0 \text{ if } j \notin S.$$

We use the following two lemmas, proven in Paragraphs B.2.1 and B.2.2.

Lemme B.1. *Let $\beta \in]0; 1/2[$ be fixed. There exist $C_1 > 0$ such that, with probability at least*

$$1 - C_1 \exp(-\beta^3 m/e),$$

the following property holds: for any $S \subset \{1, \ldots, m\}$ such that $\mathrm{Card}(S) \geq \beta m$*,*

$$|| \, |Ax_0| \odot 1_S|| \geq \beta^{3/2} e^{-1/2} ||Ax_0||. \tag{23}$$

Lemme B.2. *Let $\beta \in \left]0; \frac{1}{100}\right]$ be fixed. There exist $M, C_1, C_2 > 0$ such that, if $m \geq Mn$, then, with probability at least*

$$1 - C_1 \exp(-C_2 m),$$

the following property holds: for any $S \subset \{1, \ldots, m\}$ such that $\mathrm{Card}(S) < \beta m$ *and for any $y \in$* $\mathrm{Range}(A)$*,*

$$||y \odot 1_S|| \leq 10\sqrt{\beta \log(1/\beta)} ||y||. \tag{24}$$

Let $\beta > 0$ be such that $10\sqrt{\beta \log(1/\beta)} \leq \eta$. Let M be as in Lemma B.2. We set

$$\gamma = \beta^{3/2} e^{-1/2}.$$

We assume that Equations (23) and (24) hold; from the lemmas, this occurs with probability at least

$$1 - C_1' \exp(-C_2' m),$$

for some constants $C_1', C_2' > 0$, provided that $m \geq Mn$.

On this event, for any $v \in \mathrm{Range}(A)$ such that $||v|| < \gamma ||Ax_0||$, if we set $S_v = \{i \text{ s.t. } |v_i| \geq |Ax_0|_i\}$, we have that

$$\mathrm{Card}\, S_v < \beta m.$$

Indeed, if it was not the case, we would have, by Equation (23),

$$\begin{aligned} ||v|| &\geq ||v \odot 1_{S_v}|| \\ &\geq || \, |Ax_0| \odot 1_{S_v}|| \\ &\geq \beta^{3/2} e^{-1/2} ||Ax_0|| \\ &= \gamma ||Ax_0||, \end{aligned}$$

which is in contradiction with the way we have chosen v.

So we can apply Equation (24), and we get

$$\begin{aligned} || \, |Ax_0| \odot 1_{|v| \geq |Ax_0|} || &\leq ||v \odot 1_S|| \\ &\leq 10\sqrt{\beta \log(1/\beta)} ||v|| \\ &\leq \eta ||v||. \end{aligned}$$

□

B.2.1 Proof of Lemma B.1

Proof of Lemma B.1. If we choose C_1 large enough, it is enough to show the property for m larger than some fixed constant.

We first assume S fixed, with cardinality $\text{Card}\, S \geq \beta m$. We use the following lemma.

Lemme B.3 (Dasgupta and Gupta [2003], Lemma 2.2)**.** *Let $k_1 < k_2$ be natural numbers. Let $X \in \mathbb{C}^{k_2}$ be a random vector whose coordinates are independent, Gaussian, of variance 1. Let Y be the projection of X onto its k_1 first coordinates. Then, for any $t > 0$,*

$$\text{Proba}\left(\frac{||Y||}{||X||} \leq \sqrt{\frac{tk_1}{k_2}} \right) \leq \exp\left(k_1(1 - t + \log t)\right) \qquad \textit{if } t < 1;$$

$$\text{Proba}\left(\frac{||Y||}{||X||} \geq \sqrt{\frac{tk_1}{k_2}} \right) \leq \exp\left(k_1(1 - t + \log t)\right) \qquad \textit{if } t > 1.$$

From this lemma, for any $t \in]0; 1[$, because Ax_0 has independent Gaussian coordinates,

$$P\left(\frac{|| \, |Ax_0| \odot 1_S||}{||Ax_0||} \leq \sqrt{t\beta} \right) \leq \exp\left(-\beta m(t - 1 - \ln t)\right).$$

In particular, for $t = \frac{\beta^2}{e}$,

$$P\left(\frac{|| \, |Ax_0| \odot 1_S||}{||Ax_0||} \leq \beta^{3/2} e^{-1/2} \right) \leq \exp\left(-\beta m \left(\frac{\beta^2}{e} - 2 \ln \beta \right) \right). \tag{25}$$

As soon as m is large enough, the number of subsets S of $\{1, \dots, m\}$ with cardinality $\lceil \beta m \rceil$ satisfies

$$\binom{m}{\lceil \beta m \rceil} \leq \left(\frac{em}{\lceil \beta m \rceil} \right)^{\lceil \beta m \rceil}$$

$$\leq \exp\left(2m\beta \log \frac{1}{\beta}\right). \tag{26}$$

(The first inequality is a classical result regarding binomial coefficients.)

We combine Equations (25) and (26): Property (23) is satisfied for any S of cardinality $\lceil \beta m \rceil$ with probability at least

$$1 - \exp\left(-\frac{\beta^3}{e} m\right),$$

provided that m is larger that some constant which depends on β.

If it is satisfied for any S of cardinality $\lceil \beta m \rceil$, then it is satisfied for any S of cardinality larger than βm, which implies the result. □

B.2.2 Proof of Lemma B.2

Proof of Lemma B.2. We first assume S to be fixed, of cardinality exactly $\lceil \beta m \rceil$.

Any vector $y \in \text{Range}(A)$ is of the form $y = Av$, for some $v \in \mathbb{C}^n$. Inequality (24) can then be rewritten as:

$$||A_S v|| = ||\text{Diag}(1_S) Av|| \leq 10\sqrt{\beta \log(1/\beta)}||Av||, \tag{27}$$

where A_S, by definition, is the submatrix obtained from A by extracting the rows whose indexes are in S.

We apply Proposition 3.5 to A and A_S, respectively for $t = \frac{1}{2}$ and $t = 3\sqrt{\log(1/\beta)}$. It guarantees that the following properties hold:

$$\inf_{v \in \mathbb{C}^n} \frac{||Av||}{||v||} \geq \sqrt{m}\left(\frac{1}{2} - \sqrt{\frac{n}{m}}\right);$$
$$\sup_{v \in \mathbb{C}^n} \frac{||A_S v||}{||v||} \leq \sqrt{\text{Card}\, S}\left(1 + \sqrt{\frac{n}{\text{Card}\, S}} + 3\sqrt{\log(1/\beta)}\right),$$

with respective probabilities at least

$$1 - 2\exp\left(-\frac{m}{4}\right);$$
$$\text{and } 1 - 2\exp\left(-9(\text{Card}\, S)\log(1/\beta)\right) \geq 1 - 2\exp\left(-9\beta\log(1/\beta)m\right).$$

Assuming $m \geq Mn$ for some $M > 0$, we deduce from these inequalities that

$$\forall v \in \mathbb{C}^n, \qquad ||A_S v|| \leq \sqrt{\frac{\text{Card}\, S}{m}}\left(\frac{1 + \sqrt{\frac{n}{\text{Card}\, S}} + 3\sqrt{\log(1/\beta)}}{\frac{1}{2} - \sqrt{\frac{n}{m}}}\right)||Av||$$

$$\leq \sqrt{\beta + \frac{1}{m}} \left(\frac{1 + \sqrt{\frac{1}{\beta M}} + 3\sqrt{\log(1/\beta)}}{\frac{1}{2} - \sqrt{\frac{1}{M}}} \right) ||Av||, \tag{28}$$

with probability at least

$$1 - 2\exp\left(-9\beta\log(1/\beta)m\right) - 2\exp\left(-\frac{m}{4}\right).$$

If we choose M large enough, we can upper bound Equation (28) by $(\epsilon + 2\sqrt{\beta}(1+3\sqrt{\log(1/\beta)}))||Av|| \leq (\epsilon + 8\sqrt{\beta}\sqrt{\log(1/\beta)})$ for any fixed $\epsilon > 0$. So this inequality implies Equation (27).

As in the proof of Lemma B.1, there are at most

$$\exp\left(2m\beta\log\frac{1}{\beta}\right)$$

subsets of $\{1, \dots, m\}$ with cardinality $\lceil \beta m \rceil$, as soon as m is large enough. As a consequence, Equation (27) holds for any $v \in \mathbb{C}^n$ and S of cardinality $\lceil \beta m \rceil$ with probability at least

$$1 - 2\exp\left(-7\beta\log(1/\beta)m\right) - 2\exp\left(-\left(\frac{1}{4} - 2\beta\log\frac{1}{\beta}\right)m\right).$$

When $\beta \leq \frac{1}{100}$, we have

$$\frac{1}{4} - 2\beta\log\frac{1}{\beta} > 0,$$

so the resulting probability is larger than

$$1 - C_1\exp(-C_2 m),$$

for some well-chosen constants $C_1, C_2 > 0$.

This ends the proof. Indeed, if Equation (27) holds for any set of cardinality $\lceil \beta m \rceil$, it also holds for any set of cardinality $\mathrm{Card}\, S < \beta m$, because $||A_{S'}v|| \leq ||A_S v||$ whenever $S' \subset S$. This implies Equation (24). □

B.3 Proof of Lemma 3.4

Lemme (Lemma 3.4)**.** *For $M, C_1 > 0$ large enough, and $C_2 > 0$ small enough, when $m \geq Mn$, the property*

$$||\mathrm{Im}\,(v \odot \overline{\mathrm{phase}(Ax_0)})|| \leq \frac{4}{5}||v|| \tag{29}$$

holds for any $v \in \mathrm{Range}(A) \cap \{Ax_0\}^\perp$, with probability at least

$$1 - C_1\exp(-C_2 m).$$

Proof. If we multiply x_0 by a positive real number, we can assume $||x_0|| = 1$. Moreover, as the law of A is invariant under right multiplication by a unitary matrix, we can assume that

$$x_0 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Then, if we write A_1 the first column of A, and $A_{2:n}$ the submatrix of A obtained by removing this first column,

$$\text{Range}(A) \cap \{Ax_0\}^\perp = \left\{ w - \frac{\langle w, A_1 \rangle}{||A_1||^2} A_1, w \in \text{Range}(A_{2:n}) \right\}. \tag{30}$$

We first observe that

$$\sup_{w \in \text{Range}(A_{2:n}) - \{0\}} \frac{| \langle w, A_1 \rangle |}{||w||}$$

is the norm of the orthogonal projection of A_1 onto $\text{Range}(A_{2:n})$. The $(n-1)$-dimensional subspace $\text{Range}(A_{2:n})$ has an isotropic distribution in $\mathbb{C}^m$, and is independent of A_1. Thus, from Lemma B.3 coming from [Dasgupta and Gupta, 2003], for any $t > 1$,

$$\sup_{w \in \text{Range}(A_{2:n}) - \{0\}} \frac{| \langle w, A_1 \rangle |}{||w|| \, ||A_1||} < \sqrt{\frac{t(n-1)}{m}},$$

with probability at least

$$1 - \exp\left(-(n-1)(t - 1 - \ln t)\right).$$

We take $t = \frac{m}{n-1}(0.04)^2$ (which is larger than 1 when $m \geq Mn$ with $M > 0$ large enough), and it implies that

$$\sup_{w \in \text{Range}(A_{2:n}) - \{0\}} \frac{| \langle w, A_1 \rangle |}{||w|| \, ||A_1||} < 0.04 \tag{31}$$

with probability at least

$$1 - \exp(-c_2 m)$$

for some constant $c_2 > 0$, provided that $m \geq Mn$ with M large enough.

Second, as $A_{2:n}$ is a random matrix of size $m \times (n-1)$, whose entries are independent and distributed according to the law $\mathcal{N}(0, 1/2) + \mathcal{N}(0, 1/2)i$, we deduce from Proposition 3.5 applied with $t = 0.01$ that, with probability at least

$$1 - 2\exp\left(-10^{-4} m\right),$$

we have, for any $x \in \mathbb{C}^{n-1}$,

$$||A_{2:n}x|| \geq \sqrt{m}\left(1 - \sqrt{\frac{(n-1)}{m}} - 0.01\right)||x|| \geq 0.98\sqrt{m}||x||, \tag{32}$$

provided that $m \geq 10000n$.

We now set

$$C = \operatorname{Diag}(\overline{\operatorname{phase}(A_1)})A_{2:n}.$$

The matrix $\begin{pmatrix}\operatorname{Im} C & \operatorname{Re} C\end{pmatrix}$ has size $m \times (2(n-1))$; its entries are independent and distributed according to the law $\mathcal{N}(0, 1/2)$. So by [Davidson and Szarek, 2001, Thm II.13] (applied with $t = 0.01$), with probability at least

$$1 - \exp(-5.10^{-5}m),$$

we have, for any $x \in \mathbb{R}^{2(n-1)}$,

$$\left|\left|\begin{pmatrix}\operatorname{Im} C & \operatorname{Re} C\end{pmatrix} x\right|\right| \leq \sqrt{\frac{m}{2}}\left(1 + \sqrt{\frac{2(n-1)}{m}} + 0.01\right)||x|| \leq 1.02\sqrt{\frac{m}{2}}||x||, \tag{33}$$

provided that $m \geq 20000n$.

When Equations (32) and (33) are simultaneously valid, any $w = A_{2:n}w'$ belonging to $\operatorname{Range}(A_{2:n})$ satisfies:

$$\begin{aligned}
\left|\left|\operatorname{Im}\left(w \odot \overline{\operatorname{phase}(Ax_0)}\right)\right|\right| &= ||\operatorname{Im}(Cw')|| \\
&= \left|\left|\begin{pmatrix}\operatorname{Im} C & \operatorname{Re} C\end{pmatrix}\begin{pmatrix}\operatorname{Re} w' \\ \operatorname{Im} w'\end{pmatrix}\right|\right| \\
&\leq 1.02\sqrt{\frac{m}{2}}\left|\left|\begin{pmatrix}\operatorname{Re} w' \\ \operatorname{Im} w'\end{pmatrix}\right|\right| \\
&= 1.02\sqrt{\frac{m}{2}}||w'|| \\
&\leq \frac{1.02}{0.98\sqrt{2}}||A_{2:n}w'|| \\
&= \frac{1.02}{0.98\sqrt{2}}||w|| \\
&\leq 0.75||w||. \qquad (34)
\end{aligned}$$

We now conclude. Equations (31), (32) and (33) hold simultaneously with probability at least

$$1 - C_1\exp(-C_2 m)$$

for any C_1 large enough and C_2 small enough, provided that $m \geq Mn$ with M large enough. Let us show that, on this event, Equation (29) also holds. Any $v \in \mathrm{Range}(A) \cap \{Ax_0\}^\perp$, from Equality (30), can be written as

$$v = w - \frac{\langle w, A_1 \rangle}{||A_1||^2} A_1,$$

for some $w \in \mathrm{Range}(A_{2:n})$. Using Equation (31), then Equation (34), we get:

$$\begin{aligned}\left|\left|\mathrm{Im}\left(v \odot \overline{\mathrm{phase}(Ax_0)}\right)\right|\right| &\leq \left|\left|\mathrm{Im}\left(w \odot \overline{\mathrm{phase}(Ax_0)}\right)\right|\right| + \left|\left|\frac{\langle w, A_1 \rangle}{||A_1||^2} A_1\right|\right| \\ &\leq \left|\left|\mathrm{Im}\left(w \odot \overline{\mathrm{phase}(Ax_0)}\right)\right|\right| + 0.04||w|| \\ &\leq 0.79||w||.\end{aligned}$$

But then, by Equation (31) again,

$$||v||^2 = ||w||^2 - \frac{\langle w, A_1 \rangle^2}{||A_1||^2} \geq (1 - (0.04)^2)||w||^2.$$

So

$$\begin{aligned}\left|\left|\mathrm{Im}\left(v \odot \overline{\mathrm{phase}(Ax_0)}\right)\right|\right| &\leq 0.79||w|| \\ &\leq \frac{0.79}{\sqrt{1-(0.04)^2}}||v|| \\ &\leq \frac{4}{5}||v||.\end{aligned}$$

□

C Technical lemmas for Section 4

C.1 Proof of Lemma 4.3

Lemme (Lemma 4.3)**.** *To prove Theorem 4.1, it is enough to prove that there exist $C_1, C_2, M, \delta > 0$ such that, if $m \geq Mn^2$, then, with probability at least $1 - C_1 \exp(-C_2 m^{1/8})$, the property*

$$|\langle Ax_0, b \odot \mathrm{phase}(Ax) \rangle| \geq (1+\delta) m \frac{||x_0||}{||x||} |\langle x_0, x \rangle| \tag{22}$$

holds for any $x \in \mathbb{C}^n$ verifying Condition (15).

Proof of Lemma 4.3. Let us define $\lambda_1(A) \geq \cdots \geq \lambda_n(A)$ to be the n singular values of A. From Proposition 3.5, setting $t = \delta'/\sqrt{n}$ for δ' small enough, if M is high enough, we have with probability larger than $1 - C_1 \exp(-C_2 m/n) \geq 1 - C_1 \exp(-C_2 m^{1/2})$,

$$\frac{\lambda_1^2(A)}{\lambda_n^2(A)} - 1 \leq \frac{\delta}{3}\frac{1}{\sqrt{n}},$$
$$\text{and } \lambda_1(A)\lambda_n(A) \leq \left(\frac{1+\delta}{1+2\delta/3}\right) m,$$

when $m \geq Mn^2$.

In this case, we have in particular, for any x satisfying Equation (15),

$$\frac{\lambda_1^2(A)}{\lambda_n^2(A)} - 1 \leq \frac{\delta}{3}\frac{|\langle x_0, x\rangle|}{||x_0||\,||x||}.$$

For any x,

$$\begin{aligned}
|\langle Ax_0, b \odot \text{phase}(Ax)\rangle| &= |\left\langle Ax_0, (AA^\dagger)(b \odot \text{phase}(Ax))\right\rangle| \\
&= |\left\langle (A^*A)x_0, A^\dagger(b \odot \text{phase}(Ax))\right\rangle| \\
&\leq \lambda_n^2(A)|\left\langle x_0, A^\dagger(b \odot \text{phase}(Ax))\right\rangle| \\
&\qquad + |\left\langle (A^*A - \lambda_n^2(A)\text{Id})x_0, A^\dagger(b \odot \text{phase}(Ax))\right\rangle| \\
&\leq \lambda_n^2(A)|\left\langle x_0, A^\dagger(b \odot \text{phase}(Ax))\right\rangle| \\
&\quad + (\lambda_1^2(A) - \lambda_n^2(A))||x_0||\,||A^\dagger(b \odot \text{phase}(Ax))||.
\end{aligned}$$

So when x satisfies Equations (15) and (22),

$$\begin{aligned}
\frac{|\left\langle x_0, A^\dagger(b \odot \text{phase}(Ax))\right\rangle|}{||x_0||\,||A^\dagger(b \odot \text{phase}(Ax))||} &\geq \frac{1}{\lambda_n^2(A)}\frac{|\langle Ax_0, b \odot \text{phase}(Ax)\rangle|}{||x_0||\,||A^\dagger(b \odot \text{phase}(Ax))||} - \left(\frac{\lambda_1^2(A)}{\lambda_n^2(A)} - 1\right) \\
&\geq \frac{1}{\lambda_n^2(A)}\frac{|\langle Ax_0, b \odot \text{phase}(Ax)\rangle|}{||x_0||\,||A^\dagger(b \odot \text{phase}(Ax))||} - \frac{\delta}{3}\frac{|\langle x_0, x\rangle|}{||x_0||\,||x||} \\
&\geq (1+\delta)\frac{m}{\lambda_n^2(A)}\frac{|\langle x_0, x\rangle|}{||x||\,||A^\dagger(b \odot \text{phase}(Ax))||} - \frac{\delta}{3}\frac{|\langle x_0, x\rangle|}{||x_0||\,||x||} \\
&\geq (1+\delta)\frac{m}{\lambda_n(A)}\frac{|\langle x_0, x\rangle|}{||x||\,||b||} - \frac{\delta}{3}\frac{|\langle x_0, x\rangle|}{||x_0||\,||x||} \\
&= (1+\delta)\frac{m}{\lambda_n(A)}\frac{|\langle x_0, x\rangle|}{||x||\,||Ax_0||} - \frac{\delta}{3}\frac{|\langle x_0, x\rangle|}{||x_0||\,||x||} \\
&\geq (1+\delta)\frac{m}{\lambda_1(A)\lambda_n(A)}\frac{|\langle x_0, x\rangle|}{||x||\,||x_0||} - \frac{\delta}{3}\frac{|\langle x_0, x\rangle|}{||x_0||\,||x||}
\end{aligned}$$

$$\geq \left(1+\frac{\delta}{3}\right)\frac{|\langle x_0, x\rangle|}{||x_0||\,||x||}.$$

So Equation (16) is also satisfied (although for a smaller value of δ). □

C.2 Proof of Lemma 4.4

Lemme (Lemma 4.4)**.** *For any $n \in \mathbb{N}^*$, we set*

$$\mathcal{E}_n = \left\{ x \in \mathbb{C}^n, ||x|| = 1 \textit{ and } \frac{||x_0||\,||x||}{\sqrt{n}} \leq |\langle x_0, x\rangle| \leq (1-\epsilon)||x_0||\,||x|| \right\}.$$

Let α be any positive number.

There exist $c, C_1, C_2, M, \delta > 0$ and, for any $n \in \mathbb{N}^*$, a $cm^{-\alpha}$-net $\mathcal{N}_n$ of $\mathcal{E}_n$ such that, when $m \geq Mn^2$, with probability at least

$$1 - C_1 \exp(-C_2 m^{1/2}),$$

the following property holds: for any $x \in \mathcal{N}_n$,

$$|\langle Ax_0, b \odot \mathrm{phase}(Ax)\rangle| \geq (1+\delta) m \frac{||x_0||}{||x||} |\langle x_0, x\rangle|.$$

Proof. For any $n \in \mathbb{N}^*$, $k \in \mathbb{N}$, let $\mathcal{M}_n^k$ be a 2^{-k}-net of $\mathcal{E}_n$. As $\mathcal{E}_n$ is a closed subset of the complex unit sphere of dimension n, we can construct $\mathcal{M}_n^k$ as

$$\mathcal{M}_n^k = \{P_{\mathcal{E}_n}(y), y \in \mathcal{V}_n^k\},$$

where $\mathcal{V}_n^k$ is a $2^{-(k+1)}$-net of the unit sphere, and, for any y, $P_{\mathcal{E}_n}(y)$ is a point in $\mathcal{E}_n$ whose distance to y is minimal. From [Vershynin, 2012, Lemma 5.2], this implies that we can choose $\mathcal{M}_n^k$ such that

$$\mathrm{Card}\,\mathcal{M}_n^k \leq \left(1 + \frac{2}{2^{-(k+1)}}\right)^{2n} \leq 2^{2n(k+3)}. \tag{35}$$

For any $x \in \mathbb{C}^n$, we set

$$F(x) = \mathbb{E}\left(\langle Ax_0, b \odot \mathrm{phase}(Ax)\rangle\right)$$

(where the expectation denotes the expectation over A with x_0 and x fixed).

The main difficulty consists in showing that $\langle Ax_0, b \odot \mathrm{phase}(Ax)\rangle$ is close to its expectation for all $x \in \mathcal{M}_n^K$, with $K \in \mathbb{N}^*$ relatively large. This is what the following lemma does; it is proved in Paragraph C.2.1.

Lemme C.1. *For any $\eta, \mathcal{A} > 0$, there exist $c, C_1, C_2, M > 0$ such that, when $m \geq Mn^2$, for any $k \in \mathbb{N}$ such that $k \leq \mathcal{A} \log m - c$, with probability at least*

$$1 - C_1 \exp(-C_2 m^{1/2}),$$

the following property holds: for any $x \in \mathcal{M}_n^k, y \in \mathcal{M}_n^{k+1}$ such that $||x - y|| \leq 2^{-(k-1)}$,

$$|\,(\langle Ax_0, b \odot \text{phase}(Ax)\rangle - F(x)) - (\langle Ax_0, b \odot \text{phase}(Ay)\rangle - F(y))\,| \leq \frac{\eta}{(k+1)^2} \frac{m}{\sqrt{n}} ||x_0||^2.$$

In the case $k = 0$, we additionally have, with the same probability: for all $x \in \mathcal{M}_n^0$,

$$|\,(\langle Ax_0, b \odot \text{phase}(Ax)\rangle - F(x))\,| \leq \eta \frac{m}{\sqrt{n}} ||x_0||^2.$$

Let $\eta, \mathcal{A} > 0$ be temporarily fixed. We set $K = \lceil \mathcal{A} \log m - c \rceil$. The event described in the previous lemma holds for all $k \leq K - 1$ with probability at least $1 - KC_1 \exp(-C_2 m^{1/2})$.

For any $x \in \mathcal{M}_n^K$, there exists a sequence $(y_0, y_1, \ldots, y_{K-1}, y_K)$ such that

$$y_K = x;$$
$$\forall k \leq K, y_k \in \mathcal{M}_n^k;$$
$$\forall k \leq K-1, ||y_k - y_{k+1}|| \leq 2^{-k}.$$

So when the event of Lemma C.1 holds, we have, for any $x \in \mathcal{M}_n^K$,

$$\begin{aligned}
&|\langle Ax_0, b \odot \text{phase}(Ax)\rangle - F(x)| \\
&\quad \leq |\langle Ax_0, b \odot \text{phase}(Ay_0)\rangle - F(y_0)| \\
&\qquad + \sum_{k=0}^{K-1} |(\langle Ax_0, b \odot \text{phase}(Ay_k)\rangle - F(y_k)) - (\langle Ax_0, b \odot \text{phase}(Ay_{k+1})\rangle - F(y_{k+1}))| \\
&\quad \leq \eta \frac{m}{\sqrt{n}} \left(1 + \sum_{k=0}^{K-1} \frac{1}{(k+1)^2}\right) ||x_0||^2 \\
&\quad \leq \eta \left(1 + \frac{\pi^2}{6}\right) \frac{m}{\sqrt{n}} ||x_0||^2.
\end{aligned}$$

To conclude, we only have to evaluate F. This is done by the following lemma, proven in Paragraph C.2.2.

Lemme C.2. *There exist $\delta > 0$ such that, for any $x \in \mathcal{E}_n$,*

$$|F(x)| \geq (1 + \delta) m \frac{||x_0||}{||x||} |\,\langle x_0, x\rangle\,|.$$

We combine this lemma and the equation before the lemma: with probability at least $1 - KC_1 \exp(-C_2 m^{1/2})$, for any $x \in \mathcal{M}_n^K$,

$$\begin{aligned}
|\langle Ax_0, b \odot \mathrm{phase}(Ax)\rangle| &\geq |F(x)| - \eta\left(1 + \frac{\pi^2}{6}\right)\frac{m}{\sqrt{n}}||x_0||^2 \\
&\geq (1+\delta)m\frac{||x_0||}{||x||}|\langle x_0, x\rangle| - \eta\left(1 + \frac{\pi^2}{6}\right)\frac{m}{\sqrt{n}}||x_0||^2 \\
&\geq \left(1 + \delta - \eta\left(1 + \frac{\pi^2}{6}\right)\right)m\frac{||x_0||}{||x||}|\langle x_0, x\rangle|.
\end{aligned}$$

For the last inequality, we have used the fact that $x \in \mathcal{E}_n$, so $|\langle x_0, x\rangle| \geq ||x_0||\,||x||/\sqrt{n}$.

We can choose $\eta > 0$ sufficiently small so that $1 + \delta - \eta\left(1 + \frac{\pi^2}{6}\right) > 1 + \frac{\delta}{2}$. We fix $\mathcal{A}$ to be any real number larger than $\alpha/\log 2$. Then, from the definition of K,

$$2^{-K} \leq 2^{-\mathcal{A}\log m + c} = 2^c m^{-\mathcal{A}\log 2} \leq 2^c m^{-\alpha}.$$

As $K \leq \mathcal{A}\log m - c + 1$, we can upper bound $1 - KC_1\exp(-C_2 m^{1/2})$ by $1 - C_1'\exp(-C_2' m^{1/2})$, for $C_1', C_2' > 0$ well-chosen. If we summarize, we get that, with probability at least $1 - C_1'\exp(-C_2' m^{1/2})$,

$$\forall x \in \mathcal{M}_n^K, \qquad |\langle Ax_0, b \odot \mathrm{phase}(Ax)\rangle| \geq \left(1 + \frac{\delta}{2}\right)m\frac{||x_0||}{||x||}|\langle x_0, x\rangle|,$$

and $\mathcal{M}_n^K$ is a $2^c m^{-\alpha}$-net of $\mathcal{E}_n$. The lemma is proved. □

C.2.1 Proof of Lemma C.1

Lemme (Lemma C.1)**.** *For any $\eta, \mathcal{A} > 0$, there exist $c, C_1, C_2, M > 0$ such that, when $m \geq Mn^2$, for any $k \in \mathbb{N}$ such that $k \leq \mathcal{A}\log m - c$, with probability at least*

$$1 - C_1\exp(-C_2 m^{1/2}),$$

the following property holds: for any $x \in \mathcal{M}_n^k, y \in \mathcal{M}_n^{k+1}$ such that $||x - y|| \leq 2^{-(k-1)}$,

$$|(\langle Ax_0, b \odot \mathrm{phase}(Ax)\rangle - F(x)) - (\langle Ax_0, b \odot \mathrm{phase}(Ay)\rangle - F(y))| \leq \frac{\eta}{(k+1)^2}\frac{m}{\sqrt{n}}||x_0||^2.$$

In the case $k = 0$, we additionally have, with the same probability: for all $x \in \mathcal{M}_n^0$,

$$|(\langle Ax_0, b \odot \mathrm{phase}(Ax)\rangle - F(x))| \leq \eta\frac{m}{\sqrt{n}}||x_0||^2.$$

Proof of Lemma C.1. We only prove the first part of the lemma. The proof of the second one follows the same principle.

As our expressions are all homogeneous in x_0, we can assume that $||x_0|| = 1$.

For any $j = 1, \dots, m$, let us denote by a_j^* the j-th line of A. We have

$$\langle Ax_0, b \odot \text{phase}(Ax) \rangle = \sum_{j=1}^{m} |a_j^* x_0|^2 \, \text{phase}(a_j^* x) \, \text{phase}(\overline{a_j^* x_0}).$$

As all the a_j^* are identically distributed,

$$\forall j, \qquad \mathbb{E}\left(|a_j^* x_0|^2 \, \text{phase}(a_j^* x) \, \text{phase}(\overline{a_j^* x_0})\right) = \frac{1}{m} \mathbb{E} \langle Ax_0, b \odot \text{phase}(Ax) \rangle = \frac{1}{m} F(x).$$

So for any fixed x, y, we have

$$\begin{aligned} (\langle Ax_0, b \odot \text{phase}(Ax) \rangle - F(x)) - (\langle Ax_0, b \odot \text{phase}(Ay) \rangle - F(y)) \\ = \sum_{j=1}^{m} \left(|a_j^* x_0|^2 Z_j - \mathbb{E}(|a_j^* x_0|^2 Z_j)\right), \end{aligned} \tag{36}$$

with

$$Z_j = \text{phase}(a_j^* x) \, \text{phase}(\overline{a_j^* x_0}) - \text{phase}(a_j^* y) \, \text{phase}(\overline{a_j^* x_0}).$$

Were there no terms "$|a_j^* x_0|^2$" in Equation (36), we could apply Bennett's concentration inequality: the random variables Z_j are bounded by 2 in modulus, and, as we are going to see, their variance is small if x and y are close. Bennett's inequality would then guarantee that the term in Equation (36) is small with high probability. Unfortunately, the $|a_j^* x_0|^2$ are not almost surely bounded, so we cannot directly apply Bennett's inequality.

To overcome this problem, we first condition over Ax_0. When conditioned over Ax_0, the random variables $|a_j^* x_0|^2 Z_j$ are almost surely bounded; we will prove that they still have a small variance. We still cannot directly apply Bennett's inequality, because the bounds depend on j, but we can adapt its proof, and get a concentration inequality for the following sum:

$$\sum_{j=1}^{m} |a_j^* x_0|^2 Z_j - |a_j^* x_0|^2 \mathbb{E}(Z_j | Ax_0).$$

After that, we will also need to derive a concentration inequality for

$$\sum_{j=1}^{m} |a_j^* x_0|^2 \mathbb{E}(Z_j | Ax_0) - \mathbb{E}(|a_j^* x_0|^2 Z_j),$$

but it will be easier.

The first step is to control the distribution of the $|a_j^* x_0|$. The idea is that there are a few indexes j for which $|a_j^* x_0|$ is large, but these are sufficiently rare so that the sum $\sum_j |a_j^* x_0|^2 Z_j$, when conditioned over Ax_0, essentially behaves as if all random variables were bounded by the same constant.

The proof of the following lemma is in Paragraph C.2.3.

Lemme C.3. *For some constants $C_1, C_2 > 0$, the following event happens with probability at least $1 - C_1 e^{-C_2\sqrt{m}}$: for any $s \in \{1, \dots, \lfloor m^{1/4} \rfloor\}$,*

$$\mathrm{Card}\left\{j \in \{1, \dots, m\}, |a_j^* x_0| \geq s\right\} \leq \frac{m}{s^2} \max(m^{-1/2}, e^{-s^2/2})$$

and

$$\mathrm{Card}\left\{j \in \{1, \dots, m\}, |a_j^* x_0| > m^{1/4}\right\} = 0.$$

Let us denote by $\mathcal{E}_0$ the event described in the previous lemma:

$$\begin{aligned}\mathcal{E}_0 = \Big(\forall s \in \{1, \dots, \lfloor m^{1/4} \rfloor\}, \mathrm{Card}\left\{j \in \{1, \dots, m\}, |a_j^* x_0| \geq s\right\} \leq \frac{m}{s^2} \max(m^{-1/2}, e^{-s^2/2});\\ \text{and } \mathrm{Card}\left\{j \in \{1, \dots, m\}, |a_j^* x_0| > m^{1/4}\right\} = 0\Big).\end{aligned} \tag{37}$$

The second step is to get an upper bound on the variance of the Z_j, conditioned by Ax_0. The proof of the following lemma is in Paragraph C.2.4.

Lemme C.4. *There exists a constant $C > 0$ depending only on ϵ such that, for any fixed unit-normed x, y such that*

$$|\langle x_0, x\rangle| \leq (1-\epsilon)||x_0||\,||x|| \quad \textit{and} \quad |\langle x_0, y\rangle| \leq (1-\epsilon)||x_0||\,||y||, \tag{38}$$

we have, for any j,

$$\mathrm{Var}(Z_j|Ax_0) \leq C\left(1 + \frac{|a_j^* x_0|^2}{||x_0||^2}\right)||x-y||^2 \log\left(4||x-y||^{-1}\right).$$

From the previous lemma, we deduce that, if $x \in \mathcal{M}_n^k, y \in \mathcal{M}_n^{k+1}$ are fixed and satisfy $||x - y|| \leq 2^{-(k-1)}$, we have

$$\begin{aligned}\mathrm{Var}(\mathrm{Re}\, Z_j|Ax_0) \leq \mathrm{Var}(Z_j|Ax_0) \leq C'\left(1 + |a_j^* x_0|^2\right)\gamma^{-2k},\\ \mathrm{Var}(\mathrm{Im}\, Z_j|Ax_0) \leq \mathrm{Var}(Z_j|Ax_0) \leq C'\left(1 + |a_j^* x_0|^2\right)\gamma^{-2k}.\end{aligned} \tag{39}$$

where γ can be any real number in $]1;2[$, and $C' > 0$ is a large enough constant (depending on γ).

To follow the proof of Bennett's inequality, we now have to upper bound, for suitable values of $\lambda > 0$,

$$\mathbb{E}\left(\exp\left(\lambda\sum_{j=1}^{m}\mathrm{Re}\,\left(|a_j^*x_0|^2Z_j - |a_j^*x_0|^2\mathbb{E}(Z_j|Ax_0)\right)\right)\middle|Ax_0\right)$$
$$= \prod_{j=1}^{m}\mathbb{E}(e^{\lambda\mathrm{Re}\,(|a_j^*x_0|^2Z_j - |a_j^*x_0|^2\mathbb{E}(Z_j|Ax_0))}\Big|Ax_0).$$

We use here the fact that, even when conditioned on Ax_0, the Z_j are independent random variables.

The upper bound relies on the following lemma, proven in Paragraph C.2.5.

Lemme C.5. *Let Z be any real random variable such that $|Z| \le 2$ with probability 1. If we set $\sigma^2 = \mathrm{Var}(Z)$, then, for any $\lambda \in \mathbb{R}^+$,*

$$\mathbb{E}\left(e^{\lambda(Z-\mathbb{E}(Z))}\right) \le 1 + \frac{\sigma^2}{16}\left(e^{4\lambda} - 1 - 4\lambda\right).$$

From Equation (39) and the previous lemma, for any $\lambda \ge 0$,

$$\mathbb{E}\left(e^{\lambda|a_j^*x_0|^2\mathrm{Re}\,(Z_j-\mathbb{E}(Z_j|Ax_0))}\middle|Ax_0\right)$$
$$\le 1 + \frac{C'(1+|a_j^*x_0|^2)\gamma^{-2k}}{16}\left(e^{4|a_j^*x_0|^2\lambda} - 1 - 4|a_j^*x_0|^2\lambda\right).$$

So we can upper bound

$$\mathbb{E}\left(\exp\left(\lambda\sum_{j=1}^{m}\mathrm{Re}\,\left(|a_j^*x_0|^2Z_j - |a_j^*x_0|^2\mathbb{E}(Z_j|Ax_0)\right)\right)\middle|Ax_0\right)$$
$$\le \exp\left(\sum_{j=1}^{m}\log\left(1 + \frac{C'(1+|a_j^*x_0|^2)\gamma^{-2k}}{16}\left(e^{4|a_j^*x_0|^2\lambda} - 1 - 4|a_j^*x_0|^2\lambda\right)\right)\right). \quad (40)$$

On the event $\mathcal{E}_0$ defined in Equation (37), we can simplify the sum inside the exponential. Specifically, if we define the function

$$P : s \in \mathbb{R}^+ \to \frac{m}{\max(s,1)^2}\max(m^{-1/2}, e^{-s^2/2}),$$

we have that, on the event $\mathcal{E}_0$, for any non-decreasing function $f : \mathbb{R}^+ \to \mathbb{R}^+$,

$$\begin{aligned}
\sum_{j=1}^{m} f(|a_j^* x_0|) &\leq \sum_{s=1}^{+\infty} f(s)\Big(\operatorname{Card}\{j, |a_j^* x_0| \geq s-1\} - \operatorname{Card}\{j, |a_j^* x_0| \geq s\}\Big) \\
&= \sum_{s=0}^{+\infty} (f(s+1) - f(s)) \operatorname{Card}\{j, |a_j^* x_0| \geq s\} + m f(0) \\
&\leq \sum_{s=1}^{\lfloor m^{1/4} \rfloor} (f(s+1) - f(s)) P(s) + P(0) f(0) \\
&\leq \sum_{s=1}^{\lfloor m^{1/4} \rfloor} f(s) \left(P(s-1) - P(s)\right) + f(\lfloor m^{1/4} \rfloor + 1) P(\lfloor m^{1/4} \rfloor) \\
&= \sum_{s=1}^{\lfloor m^{1/4} \rfloor} f(s) \int_s^{s+1} (-P'(t-1)) dt + f(\lfloor m^{1/4} \rfloor + 1) P(\lfloor m^{1/4} \rfloor) \\
&\leq \int_1^{m^{1/4}+1} f(t)(-P'(t-1)) dt + f(\lfloor m^{1/4} \rfloor + 1) P(\lfloor m^{1/4} \rfloor).
\end{aligned}$$

By a direct computation, we see that, if $C > 0$ is properly chosen, we can bound:

$$\begin{aligned}
-P'(s-1) &\leq C \frac{m}{s^2} e^{-s^2/4} \text{ if } s \leq \sqrt{\log m} + 1, \\
&\leq C \frac{m^{1/2}}{s^3} \text{ if } \sqrt{\log m} + 1 < s \leq m^{1/4} + 1; \\
P(\lfloor m^{1/4} \rfloor) &\leq C.
\end{aligned}$$

So

$$\frac{1}{m} \sum_{j=1}^{m} f(|a_j^* x_0|) \leq C \left(\int_1^{\sqrt{\log m}+1} \frac{f(t)}{t^2} e^{-t^2/4} dt + m^{-1/2} \int_{\sqrt{\log m}+1}^{m^{1/4}+1} \frac{f(t)}{t^3} dt \right) + \frac{C}{m} f(m^{1/4} + 1).$$

We plug this inequality into Equation (40). For any $\lambda \geq 0$, we set

$$f_\lambda(x) = \log \left(1 + \frac{C'(1 + x^2)\gamma^{-2k}}{16} (e^{4\lambda x^2} - 1 - 4\lambda x^2) \right),$$

and, on the event $\mathcal{E}_0$, we have:

$$\mathbb{E} \left(\exp \left(\lambda \sum_{j=1}^{m} \operatorname{Re} \left(|a_j^* x_0|^2 Z_j - |a_j^* x_0|^2 \mathbb{E}(Z_j | A x_0) \right) \right) \middle| A x_0 \right)$$

$$\leq \exp\left(Cm\left(\int_1^{\sqrt{\log m}+1}\frac{f_\lambda(t)}{t^2}e^{-t^2/4}dt + m^{-1/2}\int_{\sqrt{\log m}+1}^{m^{1/4}+1}\frac{f_\lambda(t)}{t^3}dt + \frac{1}{m}f(m^{1/4}+1)\right)\right). \tag{41}$$

We upper bound the sum of the integrals, using standard analysis techniques. The detailed proof is in Paragraph C.2.6.

Lemme C.6. *There exists a constant $\tilde{C} > 0$ depending only on γ and $\epsilon > 0$ such that, for any $\lambda \in]0; \frac{1}{40}[$,*

$$\int_1^{\sqrt{\log m}+1}\frac{f_\lambda(t)}{t^2}e^{-t^2/4}dt + m^{-1/2}\int_{\sqrt{\log m}+1}^{m^{1/4}+1}\frac{f_\lambda(t)}{t^3}dt + \frac{1}{m}f_\lambda(m^{1/4}+1) \leq \tilde{C}\gamma^{-2k}\lambda^2,$$

provided that

$$\Big(\log(\max(1,\gamma^k/\lambda))+1\Big)\left(\frac{\gamma^k}{\lambda}\right)^{4/3} \leq m^{1/2}; \tag{42a}$$

$$\frac{m^{1/2}\lambda\gamma^{-2k}}{1+\log m} \geq 1. \tag{42b}$$

We apply this result with

$$\lambda = \frac{\eta\gamma^{2k}}{8C\tilde{C}(k+1)^2m^{1/4}},$$

where $\eta > 0$ is the fixed constant chosen in the statement of Lemma C.1, C is the constant of Equation (41) and $\tilde{C}$ is the one of Lemma C.6. We consider only the values of $k \in \mathbb{N}$ such that

$$\gamma^{2k} < \frac{C\tilde{C}}{5\eta}m^{1/4}, \tag{43}$$

which in particular ensures that

$$\lambda < \frac{1}{40}.$$

With this definition, Conditions (42a) and (42b) are satisfied. Indeed, as $\gamma > 1$,

$$\frac{\gamma^k}{\lambda} = \frac{8C\tilde{C}(k+1)^2m^{1/4}}{\eta\gamma^k} = O(m^{1/4});$$

$$\Rightarrow \qquad \big(\log(\max(1,\gamma^k/\lambda))+1\big)\left(\frac{\gamma^k}{\lambda}\right)^{4/3} = O(m^{1/3}\log m) \leq m^{1/2},$$

if m is large enough. For the second condition, because of Equation (43),

$$\begin{aligned}\frac{m^{1/2}\lambda\gamma^{-2k}}{1+\log m} &= \frac{m^{1/2}}{1+\log m}\frac{\eta}{8C\tilde{C}(k+1)^2 m^{1/4}} \\ &\geq \frac{m^{1/4}}{1+\log m}\frac{\eta}{8CC'\left(1+\log(C\tilde{C}m^{1/4}/(5\eta))/(2\log(\gamma))\right)^2} \\ &\geq c\frac{m^{1/4}}{(1+\log m)^3} \\ &\geq 1,\end{aligned}$$

if m is large enough. (In the second inequality, $c > 0$ is a positive constant.)

As the two conditions are satisfied, we can combine Lemma C.6 and Equation (41). We get that, on the event $\mathcal{E}_0$,

$$\begin{aligned}\mathbb{E}\left(\exp\left(\lambda\sum_{j=1}^m \mathrm{Re}\,\left(|a_j^*x_0|^2 Z_j - |a_j^*x_0|^2\mathbb{E}(Z_j|Ax_0)\right)\right)\middle|Ax_0\right) &\leq \exp\left(C\tilde{C}m\gamma^{-2k}\lambda^2\right) \\ &= \exp\left(\frac{\eta^2\gamma^{2k}m^{1/2}}{64C\tilde{C}(k+1)^4}\right).\end{aligned}$$

So, by Markov's inequality, on the event $\mathcal{E}_0$, if $m \geq n^2$,

$$\begin{aligned}&P\left(\sum_{j=1}^m \mathrm{Re}\,\left(|a_j^*x_0|^2 Z_j - |a_j^*x_0|^2\mathbb{E}(Z_j|Ax_0)\right) \geq \frac{\eta m}{4(k+1)^2\sqrt{n}}\middle|Ax_0\right) \\ &\leq P\left(\sum_{j=1}^m \mathrm{Re}\,\left(|a_j^*x_0|^2 Z_j - |a_j^*x_0|^2\mathbb{E}(Z_j|Ax_0)\right) \geq \frac{\eta m^{3/4}}{4(k+1)^2}\middle|Ax_0\right) \\ &\leq \exp\left(\frac{\eta^2\gamma^{2k}m^{1/2}}{64C\tilde{C}(k+1)^4}\right)\exp\left(-\frac{\lambda\eta m^{3/4}}{4(k+1)^2}\right) \\ &= \exp\left(-\frac{\eta^2\gamma^{2k}m^{1/2}}{64C\tilde{C}(k+1)^4}\right).\end{aligned}$$

We integrate over Ax_0, and obtain

$$P\left(\mathcal{E}_0 \cap \left\{\sum_{j=1}^m \mathrm{Re}\,\left(|a_j^*x_0|^2 Z_j - |a_j^*x_0|^2\mathbb{E}(Z_j|Ax_0)\right) \geq \frac{\eta m}{4(k+1)^2\sqrt{n}}\right\}\right)$$

$$\leq \exp\left(-\frac{\eta^2\gamma^{2k}m^{1/2}}{16C\tilde{C}(k+1)^4}\right).$$

We can apply the same reasoning to $-\mathrm{Re}\,(Z_j), \mathrm{Im}\,(Z_j)$ and $-\mathrm{Im}\,(Z_j)$. This yields:

$$P\left(\mathcal{E}_0 \cap \left\{\left|\sum_{j=1}^m \left(|a_j^* x_0|^2 Z_j - |a_j^* x_0|^2 \mathbb{E}(Z_j|Ax_0)\right)\right| \geq \frac{\eta m}{2(k+1)^2\sqrt{n}}\right\}\right)$$
$$\leq 4\exp\left(-\frac{\eta^2\gamma^{2k}m^{1/2}}{16C\tilde{C}(k+1)^4}\right). \tag{44}$$

Now that we have a bound for $\sum_{j=1}^m \left(|a_j^* x_0|^2 Z_j - |a_j^* x_0|^2 \mathbb{E}(Z_j|Ax_0)\right)$, we remember that we also have to bound

$$\sum_{j=1}^m \left(|a_j^* x_0|^2 \mathbb{E}(Z_j|Ax_0) - \mathbb{E}(|a_j^* x_0|^2 Z_j)\right).$$

We remark that, for all j, $\mathbb{E}(Z_j|Ax_0) = \mathbb{E}(Z_j|a_j^* x_0)$, so that the random variables $|a_j^* x_0|^2 \mathbb{E}(Z_j|Ax_0)$, for $j = 1, \dots, m$, are independent and identically distributed.

We begin with the following lemma, proven in Paragraph C.2.7.

Lemme C.7. *There exist a constant $C > 0$ depending only on ϵ such that, for any fixed unit-normed x, y such that*

$$|\langle x_0, x\rangle| \leq (1-\epsilon)||x_0||\,||x|| \quad \textit{and} \quad |\langle x_0, y\rangle| \leq (1-\epsilon)||x_0||\,||y||,$$

and any $j = 1, \dots, m$,

$$|\mathbb{E}(Z_j|a_j^* x_0)| \leq C \min\left(1, ||x-y||\left(1 + \frac{|a_j^* x_0|}{||x_0||}\right)\right).$$

To simplify the expressions, we still assume that $||x_0|| = 1$. If $||x-y|| \leq 2^{-(k-1)}$, the previous lemma guarantees that, for any j,

$$|\mathbb{E}(Z_j|a_j^* x_0)| \leq 2C \min\left(1, \gamma^{-k}(1 + |a_j^* x_0|)\right), \tag{45}$$

where γ is still our real number in $]1;2[$.

This inequality allows us to upper bound $\mathbb{E}\left(e^{\lambda\left(|a_j^* x_0|^2 \mathbb{E}(Z_j|Ax_0) - \mathbb{E}(|a_j^* x_0|^2 Z_j)\right)}\right)$, for λ small enough. The next lemma is proved in Paragraph C.2.8.

Lemme C.8. *There exist constants $c, C' > 0$, that depend only on γ and ϵ, such that, for any $\lambda \in [-c; c]$,*

$$\log\left(\mathbb{E}\left(e^{\lambda \mathrm{Re}\left(|a_j^* x_0|^2 \mathbb{E}(Z_j|Ax_0) - \mathbb{E}(|a_j^* x_0|^2 Z_j)\right)}\right)\right) \leq C' \lambda^2 \gamma^{-2k},$$
$$\textit{and } \log\left(\mathbb{E}\left(e^{\lambda \mathrm{Im}\left(|a_j^* x_0|^2 \mathbb{E}(Z_j|Ax_0) - \mathbb{E}(|a_j^* x_0|^2 Z_j)\right)}\right)\right) \leq C' \lambda^2 \gamma^{-2k}.$$

So by Markov's inequality, taking

$$\lambda = \frac{\eta \gamma^{2k}}{8C'(k+1)^2 m^{1/4}},$$

for k such that

$$\gamma^{2k} \leq \frac{8cC'}{\eta} m^{1/4}, \tag{46}$$

we have, when $m \geq n^2$,

$$\begin{aligned}
&P\left(\mathrm{Re}\left(\sum_{j=1}^m \left(|a_j^* x_0|^2 \mathbb{E}(Z_j|Ax_0) - \mathbb{E}(|a_j^* x_0|^2 Z_j)\right)\right) \geq \frac{\eta m}{4(k+1)^2 \sqrt{n}}\right) \\
&\quad \leq P\left(\mathrm{Re}\left(\sum_{j=1}^m \left(|a_j^* x_0|^2 \mathbb{E}(Z_j|Ax_0) - \mathbb{E}(|a_j^* x_0|^2 Z_j)\right)\right) \geq \frac{\eta m^{3/4}}{4(k+1)^2}\right) \\
&\quad \leq \mathbb{E}\left(\exp\left(\lambda \mathrm{Re}\left(\sum_{j=1}^m \left(|a_j^* x_0|^2 \mathbb{E}(Z_j|Ax_0) - \mathbb{E}(|a_j^* x_0|^2 Z_j)\right)\right)\right)\right) \exp\left(-\frac{\lambda \eta m^{3/4}}{4(k+1)^2}\right) \\
&\quad = \exp\left(m \log\left(\mathbb{E}\left(e^{\lambda \mathrm{Re}\left(|a_j^* x_0|^2 \mathbb{E}(Z_j|Ax_0) - \mathbb{E}(|a_j^* x_0|^2 Z_j)\right)}\right)\right) - \frac{\lambda \eta m^{3/4}}{4(k+1)^2}\right) \\
&\quad \leq \exp\left(m C' \lambda^2 \gamma^{-2k} - \frac{\lambda \eta m^{3/4}}{4(k+1)^2}\right) \\
&\quad = \exp\left(-\frac{m^{1/2} \eta^2 \gamma^{2k}}{64 C'(k+1)^4}\right).
\end{aligned}$$

The same inequality holds if we replace Re by $-\mathrm{Re}$, Im or $-\mathrm{Im}$, so we obtain:

$$P\left(\left|\sum_{j=1}^m \left(|a_j^* x_0|^2 \mathbb{E}(Z_j|Ax_0) - \mathbb{E}(|a_j^* x_0|^2 Z_j)\right)\right| \geq \frac{\eta m}{2(k+1)^2 \sqrt{n}}\right) \leq 4 \exp\left(-\frac{m^{1/2} \eta^2 \gamma^{2k}}{64 C'(k+1)^4}\right).$$

We are close to the end. The previous equation, combined with Equation (44) yields, by triangular inequality, that for any fixed $x \in \mathcal{M}_n^k, y \in \mathcal{M}_n^{k+1}$ such that $||x-y|| \le 2^{-(k-1)}$,

$$P\left(\mathcal{E}_0 \cap \left\{\left|\sum_{j=1}^m \left(|a_j^* x_0|^2 Z_j - \mathbb{E}(|a_j^* x_0|^2 Z_j)\right)\right| \ge \frac{\eta m}{(k+1)^2\sqrt{n}}\right\}\right)$$
$$\le 8\exp\left(-\mathcal{C}\frac{\gamma^{2k}}{(k+1)^4} m^{1/2}\right),$$

where $\mathcal{C}$ is a constant that depends only on η, ϵ and γ. We recall that Z_j depends on x and y, although it does not appear in the notation.

From Equation (35),

$$\text{Card}\,\mathcal{M}_n^k \le 2^{2n(k+3)} \quad \text{and} \quad \text{Card}\,\mathcal{M}_n^{k+1} \le 2^{2n(k+4)}.$$

The number of possible pairs $(x, y) \in \mathcal{M}_n^k \times \mathcal{M}_n^{k+1}$ is then bounded by

$$2^{2n(2k+7)} \le e^{10n(k+1)},$$

and by union bound,

$$P\left(\mathcal{E}_0 \cap \left\{\exists x, y \in \mathcal{M}_n^k \times \mathcal{M}_n^{k+1}, \left|\sum_{j=1}^m \left(|a_j^* x_0|^2 Z_j - \mathbb{E}(|a_j^* x_0|^2 Z_j)\right)\right| \ge \frac{\eta m}{(k+1)^2\sqrt{n}}\right\}\right)$$
$$\le 8\exp\left(-\mathcal{C}\frac{\gamma^{2k}}{(k+1)^4} m^{1/2} + 10n(k+1)\right).$$

From Lemma C.3, the probability of $\mathcal{E}_0$ is at least $1 - C_1 e^{-C_2 m^{1/2}}$ for some constants $C_1, C_2 > 0$, so

$$P\left(\forall x, y \in \mathcal{M}_n^k \times \mathcal{M}_n^{k+1}, \left|\sum_{j=1}^m \left(|a_j^* x_0|^2 Z_j - \mathbb{E}(|a_j^* x_0|^2 Z_j)\right)\right| < \frac{\eta m}{(k+1)^2\sqrt{n}}\right)$$
$$\ge 1 - 8\exp\left(-\mathcal{C}\frac{\gamma^{2k}}{(k+1)^4} m^{1/2} + 10n(k+1)\right) - C_1\exp(-C_2 m^{1/2}).$$

There exists a constant $\mathcal{C}'$ depending only on γ such that $\gamma^{2k} \ge \mathcal{C}'(k+1)^5$ for any $k \in \mathbb{N}$. If we assume that $m \ge Mn^2$ for some $M > 0$, we have

$$P\left(\forall x, y \in \mathcal{M}_n^k \times \mathcal{M}_n^{k+1}, \left|\sum_{j=1}^m \left(|a_j^* x_0|^2 Z_j - \mathbb{E}(|a_j^* x_0|^2 Z_j)\right)\right| < \frac{\eta m}{(k+1)^2\sqrt{n}}\right)$$

$$\geq 1 - 8\exp\left(-m^{1/2}(k+1)(\mathcal{C}\mathcal{C}' - 10M^{-1/2})\right) - C_1\exp(-C_2 m^{1/2})$$
$$\geq 1 - 8\exp\left(-(\mathcal{C}\mathcal{C}' - 10M^{-1/2})m^{1/2}\right) - C_1\exp(-C_2 m^{1/2}).$$

When $M > 0$ is large enough, this can be lower bounded by $1 - C_1\exp(-C_2 m^{1/2})$, where the constants $C_1, C_2 > 0$ depend on η, ϵ and γ but not on k, m or n.

We recall Equations (43) and (46): the reasoning holds only for the values of k such that

$$\gamma^{2k} < \alpha m^{1/4},$$

where, again, $\alpha > 0$ is a constant that depends only on η, ϵ and γ. This means that, if we have chosen $\gamma \in]1; 2[$ sufficiently close to 1, it holds for any k satisfying

$$k < \mathcal{A} \ln m - c,$$

where $c \in \mathbb{R}$ is a constant that does not depend on n or m.

□

C.2.2 Proof of Lemma C.2

Lemme (Lemma C.2)**.** *There exist $\delta > 0$ such that, for any $x \in \mathcal{E}_n$,*

$$|F(x)| \geq (1+\delta)m\frac{||x_0||}{||x||}|\langle x_0, x\rangle|.$$

Proof. We write

$$x = \alpha x_0 + \beta x',$$

with $\alpha, \beta \in \mathbb{C}$ and $x' \in \mathbb{C}^n$ such that $\langle x_0, x'\rangle = 0$ and $||x'|| = 1$.

$$\begin{aligned}
F(x) &= \mathbb{E}(\langle Ax_0, b \odot \mathrm{phase}(Ax)\rangle) \\
&= \sum_{j=1}^{m} \mathbb{E}\left(\overline{(Ax_0)_j}|(Ax_0)_j|\,\mathrm{phase}((Ax)_j)\right) \\
&= m\mathbb{E}\left(\overline{(Ax_0)_1}|(Ax_0)_1|\,\mathrm{phase}((Ax)_1)\right) \\
&= m\mathbb{E}\left(\overline{(Ax_0)_1}|(Ax_0)_1|\,\mathrm{phase}(\alpha(Ax_0)_1 + \beta(Ax')_1)\right) \\
&= m||x_0||^2\,\mathrm{phase}(\alpha)\mathbb{E}\left(\frac{\overline{(Ax_0)_1}}{||x_0||}\frac{|(Ax_0)_1|}{||x_0||}\,\mathrm{phase}\left(\frac{(Ax_0)_1}{||x_0||} + \frac{\beta}{\alpha||x_0||}(Ax')_1\right)\right)
\end{aligned}$$

$$= m||x_0||^2 \,\mathrm{phase}(\alpha)\mathbb{E}\left(\overline{Z_1}|Z_1|\,\mathrm{phase}\left(Z_1 + \frac{|\beta|}{|\alpha|\,||x_0||}Z_2\right)\right).$$

where $Z_1 = \frac{(Ax_0)_1}{||x_0||}$ and $Z_2 = \mathrm{phase}(\beta/\alpha)(Ax')_1$ are independent complex Gaussian variables with variance 1.

The expectation cannot be analytically computed, but it can be lower bounded by a simple function. The following lemma is proven in Paragraph C.2.9.

Lemme C.9. *For any $t \in \mathbb{R}^+$, we set*

$$f(t) = \mathbb{E}\left(\overline{Z_1}|Z_1|\,\mathrm{phase}\left(Z_1 + tZ_2\right)\right).$$

The function f is real-valued. For any $\gamma > 0$, there exist $\delta > 0$ such that

$$\forall t \in [\gamma; +\infty[, \qquad f(t) \geq \frac{1+\delta}{\sqrt{1+t^2}}.$$

As x belongs to $\mathcal{E}_n$, we have:

$$\begin{aligned}
\frac{|\beta|}{|\alpha|\,||x_0||} &= \frac{\sqrt{||x||^2 - |\alpha|^2||x_0||^2}}{|\alpha|\,||x_0||} \\
&= \sqrt{\frac{1}{|\alpha|^2||x_0||^2} - 1} \\
&= \sqrt{\frac{||x_0||^2}{|\langle x_0, x\rangle|^2} - 1} \\
&\geq \sqrt{\frac{1}{(1-\epsilon)^2} - 1}.
\end{aligned}$$

Consequently, we can apply the lemma with $\gamma = \sqrt{\frac{1}{(1-\epsilon)^2} - 1}$. It implies that, for some $\delta > 0$ that depends only on ϵ,

$$\begin{aligned}
|F(x)| &\geq m||x_0||^2(1+\delta)\frac{1}{\sqrt{1 + \left(\frac{|\beta|}{|\alpha|\,||x_0||}\right)^2}} \\
&= m||x_0||^2(1+\delta)|\alpha|\,||x_0|| \\
&= (1+\delta)m\frac{||x_0||}{||x||}|\langle x_0, x\rangle|.
\end{aligned}$$

□

C.2.3 Proof of Lemma C.3

Lemme (Lemma C.3)**.** *For some constants $C_1, C_2 > 0$, the following event happens with probability at least $1 - C_1 e^{-C_2\sqrt{m}}$: for any $s \in \{1, \ldots, \lfloor m^{1/4} \rfloor\}$,*

$$\mathrm{Card}\left\{j \in \{1, \ldots, m\}, |a_j^* x_0| \geq s\right\} \leq \frac{m}{s^2} \max(m^{-1/2}, e^{-s^2/2})$$

and

$$\mathrm{Card}\left\{j \in \{1, \ldots, m\}, |a_j^* x_0| > m^{1/4}\right\} = 0.$$

Proof of Lemma C.3. We recall that $A_1 x_0, \ldots, A_m x_0$ are independent complex Gaussian random variables with variance $||x_0||^2 = 1$. In particular, for any $s \in \mathbb{N}$,

$$\begin{aligned} P(|a_j^* x_0| \geq s) &= e^{-s^2}; \\ \mathbb{E}(1_{|a_j^* x_0| \geq s}) &= e^{-s^2}; \\ \mathrm{Var}(1_{|a_j^* x_0| \geq s}) &\leq e^{-s^2}. \end{aligned}$$

We first consider the values of s belonging to $\{1, \ldots, \lfloor \sqrt{\log m} \rfloor\}$. For any of these s, by Bennett's inequality, if we denote by h the function $h : x \in \mathbb{R}^+ \to (1 + x)\log(1 + x) - x$,

$$\begin{aligned} &P\left(\mathrm{Card}\left\{j \in \{1, \ldots, m\}, |a_j^* x_0| \geq s\right\} \geq \frac{m}{s^2} e^{-s^2/2}\right) \\ =&P\left(\sum_{j=1}^{m}\left(1_{|a_j^* x_0| \geq s} - \mathbb{E}\left(1_{|a_j^* x_0| \geq s}\right)\right) \geq m\left(\frac{e^{-s^2/2}}{s^2} - e^{-s^2}\right)\right) \\ \leq& \exp\left(-m e^{-s^2} h\left(\frac{e^{s^2/2}}{s^2} - 1\right)\right) \\ =& \exp\left(-m \frac{e^{-s^2/2}}{s^2}\left(\frac{s^2}{2} - 2\log(s) - 1 + s^2 e^{-s^2/2}\right)\right) \\ \leq& \exp\left(-c_1 m e^{-s^2/2}\right), \end{aligned}$$

for some absolute constant $c_1 > 0$. As $s \leq \sqrt{\log m}$, this yields:

$$P\left(\mathrm{Card}\left\{j \in \{1, \ldots, m\}, |a_j^* x_0| \geq s\right\} \geq \frac{m}{s^2} e^{-s^2/2}\right) \leq \exp(-c_1 m^{1/2}).$$

Second, we consider the values of s in $\{\left\lfloor \sqrt{\log m} \right\rfloor + 1, \ldots, \lfloor m^{1/4} + 1 \rfloor\}$.

$$P\left(\mathrm{Card}\left\{j \in \{1, \ldots, m\}, |a_j^* x_0| \geq s\right\} \geq \frac{m^{1/2}}{s^2}\right)$$

$$
\begin{aligned}
&= P\left(\sum_{j=1}^{m}\left(1_{|a_j^* x_0|\geq s} - \mathbb{E}\left(1_{|a_j^* x_0|\geq s}\right)\right) \geq m\left(\frac{m^{-1/2}}{s^2} - e^{-s^2}\right)\right)\\
&\leq \exp\left(-me^{-s^2} h\left(\frac{m^{-1/2}}{s^2}e^{s^2} - 1\right)\right)\\
&= \exp\left(-m^{1/2}\left(1 - \frac{\log m}{2s^2} - \frac{2\log s}{s^2} - \frac{1}{s^2} + m^{1/2}e^{-s^2}\right)\right)\\
&\overset{(a)}{\leq} \exp\left(-m^{1/2}\left(1 - \frac{1}{2} - \frac{\log(\log m)}{\log m} - \frac{1}{\log m}\right)\right)\\
&\leq \exp\left(-\frac{m^{1/2}}{4}\right).
\end{aligned}
$$

as soon as m is large enough. For (a), we have used the inequality $s \geq \sqrt{\log m}$.

To conclude, we observe that, if

$$
\mathrm{Card}\left\{j \in \{1,\dots,m\}, |a_j^* x_0| \geq s\right\} \leq \frac{m^{1/2}}{s^2}
$$

for $s = \lfloor m^{1/4} + 1 \rfloor > m^{1/4}$, we must have

$$
\mathrm{Card}\left\{j \in \{1,\dots,m\}, |a_j^* x_0| > m^{1/4}\right\} = 0.
$$

So we see that the desired event holds, for m large enough, with probability at least

$$
1 - \sqrt{\log m} e^{-c_1\sqrt{m}} - m^{1/4} e^{-\sqrt{m}/4},
$$

which can be bounded by $1 - C_1 e^{-C_2\sqrt{m}}$ for $C_1, C_2 > 0$ well-chosen. □

C.2.4 Proof of Lemma C.4

Lemme (Lemma C.4)**.** *There exists a constant $C > 0$ depending only on ϵ such that, for any fixed unit-normed x, y such that*

$$
|\langle x_0, x\rangle| \leq (1-\epsilon)||x_0||\,||x|| \quad \textit{and} \quad |\langle x_0, y\rangle| \leq (1-\epsilon)||x_0||\,||y||, \tag{47}
$$

we have, for any j,

$$
\mathrm{Var}(Z_j|Ax_0) \leq C\left(1 + \frac{|a_j^* x_0|^2}{||x_0||^2}\right)||x-y||^2 \log\left(4||x-y||^{-1}\right).
$$

Proof of Lemma C.4. By the definition of Z_j, it suffices to prove

$$\mathrm{Var}(\mathrm{phase}(a_j^* x) - \mathrm{phase}(a_j^* y)|Ax_0) \leq C \left(1 + \frac{|a_j^* x_0|^2}{||x_0||^2}\right) ||x-y||^2 \log\left(4||x-y||^{-1}\right). \tag{48}$$

We write

$$x = \alpha_x x_0 + x' \text{ and } y = \alpha_y x_0 + \beta x' + y'',$$

where $\alpha_x, \alpha_y, \beta$ are complex numbers and $x', y'' \in \mathbb{C}^n$ satisfy $\langle x', x_0 \rangle = \langle y'', x_0 \rangle = \langle x', y'' \rangle = 0$. Because of Equation (47), and because x, y are unit-normed,

$$||x'|| \geq \sqrt{\epsilon(2-\epsilon)} \geq \sqrt{\epsilon}; \tag{49a}$$

$$|\beta - 1| = \frac{|\langle y-x, x'\rangle|}{||x'||^2} \leq \frac{1}{\sqrt{\epsilon}}||y-x||; \tag{49b}$$

$$||\alpha_x x_0 - \beta\alpha_y x_0|| = \frac{|\langle x-y, x_0\rangle|}{||x_0||} \leq ||x-y||; \tag{49c}$$

$$||y''|| = \frac{|\langle y-x, y''\rangle|}{||y''||} \leq ||x-y||. \tag{49d}$$

As $|Z_j|$ is bounded (by 2), the desired inequality is true for $||x-y|| \geq \sqrt{\epsilon}/2$, provided that C is large enough, so we can assume $||x-y|| < \sqrt{\epsilon}/2$, which in particular guarantees that $|\beta| > 1/2$.

As

$$\begin{aligned}\mathrm{Var}(\mathrm{phase}(a_j^* x) - \mathrm{phase}(a_j^* y)|Ax_0) &\leq \mathbb{E}\left(\left|\mathrm{phase}(a_j^* x) - \mathrm{phase}(a_j^* y)\right|^2 \middle| Ax_0\right) \\ &= 2\left(1 - \mathrm{Re}\left(\mathbb{E}\left(\mathrm{phase}(\overline{a_j^* x})\,\mathrm{phase}(a_j^* y)|Ax_0\right)\right)\right),\end{aligned}$$

we only need, in order to prove Equation (48), to show that, for some constant $C > 0$,

$$1 - \mathrm{Re}\left(\mathbb{E}\left(\mathrm{phase}(\overline{a_j^* x})\,\mathrm{phase}(a_j^* y)|Ax_0\right)\right) \leq C\left(1 + \frac{|a_j^* x_0|^2}{||x_0||^2}\right)||x-y||^2 \log(4||x-y||^{-1}). \tag{50}$$

We have

$$\begin{aligned}\mathrm{phase}(a_j^* x) &= \mathrm{phase}\left(\frac{a_j^* x'}{||x'||} + \frac{\alpha_x}{||x'||} a_j^* x_0\right); \\ \mathrm{phase}(a_j^* y) &= \mathrm{phase}\left(\frac{a_j^* x'}{||x'||} + \frac{\alpha_y}{\beta||x'||} a_j^* x_0 + \frac{1}{\beta||x'||} a_j^* y''\right)\mathrm{phase}(\beta),\end{aligned}$$

and $\frac{a_j^* x'}{||x'||}$ is a complex Gaussian random variable with variance 1, independent from Ax_0 and $a_j^* y''$. So

$$1 - \mathrm{Re}\left(\mathbb{E}\left(\mathrm{phase}(\overline{a_j^* x})\,\mathrm{phase}(a_j^* y)|Ax_0, a_j^* y''\right)\right)$$

$$= 1 - \frac{1}{\pi}\mathrm{Re}\left(\mathrm{phase}(\beta)\int_{\mathbb{C}}\mathrm{phase}\left(\overline{z + \frac{\alpha_x}{||x'||}a_j^* x_0}\right)\mathrm{phase}\left(z + \frac{\alpha_y}{\beta||x'||}a_j^* x_0 + \frac{1}{\beta||x'||}a_j^* y''\right)e^{-|z|^2}d^2 z\right).$$

We upper bound this quantity with the following proposition, proven in Paragraph C.2.10.

Proposition C.10. *Let us define the function*

$$\begin{array}{rccc} G: & \mathbb{C}^2 & \to & \mathbb{C} \\ & (a,b) & \to & 1 - \frac{1}{\pi}\mathrm{Re}\int_{\mathbb{C}}\mathrm{phase}(\overline{z+a})\,\mathrm{phase}(z+b)e^{-|z|^2}d^2 z. \end{array}$$

For some constant $c_1 > 0$, the following inequalities are true:

$$\begin{array}{rl} \forall a, b \in \mathbb{C}, & |\mathrm{Re}\,G(a,b)| \le c_1 |a-b|^2 \max\left(1, \log\left(|a-b|^{-1}\right)\right), \\ & |\mathrm{Im}\,G(a,b)| \le c_1|a-b|. \end{array}$$

So

$$\begin{aligned}
& 1 - \mathrm{Re}\left(\mathbb{E}\left(\mathrm{phase}(\overline{a_j^* x})\,\mathrm{phase}(a_j^* y)|Ax_0, a_j^* y''\right)\right) \\
&= 1 - \mathrm{Re}\left(\mathrm{phase}(\beta)\right) + \mathrm{Re}\left(\mathrm{phase}(\beta)\right)\mathrm{Re}\,G\left(\frac{\alpha_x}{||x'||}a_j^* x_0, \frac{\alpha_y}{\beta||x'||}a_j^* x_0 + \frac{1}{\beta||x'||}a_j^* y''\right) \\
&\qquad - \mathrm{Im}\left(\mathrm{phase}(\beta)\right)\mathrm{Im}\,G\left(\frac{\alpha_x}{||x'||}a_j^* x_0, \frac{\alpha_y}{\beta||x'||}a_j^* x_0 + \frac{1}{\beta||x'||}a_j^* y''\right) \\
&\le c_1 \left|\frac{\alpha_y - \beta\alpha_x}{\beta||x'||}a_j^* x_0 + \frac{1}{\beta||x'||}a_j^* y''\right|^2 \max\left(1, \log\left(\left|\frac{\alpha_y - \beta\alpha_x}{\beta||x'||}a_j^* x_0 + \frac{1}{\beta||x'||}a_j^* y''\right|^{-1}\right)\right) \\
&\qquad + |1 - \mathrm{Re}\left(\mathrm{phase}\,\beta\right)| + |\mathrm{Im}\left(\mathrm{phase}(\beta)\right)|\left|\frac{\alpha_y - \beta\alpha_x}{\beta||x'||}a_j^* x_0 + \frac{1}{\beta||x'||}a_j^* y''\right| \\
&\le c_1 \left(\left|\frac{\alpha_y - \beta\alpha_x}{\beta||x'||}a_j^* x_0\right| + \left|\frac{1}{\beta||x'||}a_j^* y''\right|\right)^2 \max\left(1, \log\left(\left|\frac{\alpha_y - \beta\alpha_x}{\beta||x'||}a_j^* x_0\right| + \left|\frac{1}{\beta||x'||}a_j^* y''\right|\right)^{-1}\right) \\
&\qquad + 2|1-\beta|^2 + |\beta - 1|\left(\left|\frac{\alpha_y - \beta\alpha_x}{\beta||x'||}a_j^* x_0\right| + \left|\frac{1}{\beta||x'||}a_j^* y''\right|\right) \\
&\le 2c_1 \left|\frac{\alpha_y - \beta\alpha_x}{\beta||x'||}a_j^* x_0\right|^2 \max\left(1, \log\left(\left|\frac{\alpha_y - \beta\alpha_x}{\beta||x'||}a_j^* x_0\right|^{-1}\right)\right) \\
&\qquad + 2c_1 \left|\frac{1}{\beta||x'||}a_j^* y''\right|^2 \max\left(1, \log\left(\left|\frac{1}{\beta||x'||}a_j^* y''\right|^{-1}\right)\right) \\
&\qquad + 2|1-\beta|^2 + |\beta - 1|\left(\left|\frac{\alpha_y - \beta\alpha_x}{\beta||x'||}a_j^* x_0\right| + \left|\frac{1}{\beta||x'||}a_j^* y''\right|\right)
\end{aligned}$$

$$\begin{aligned}
&\overset{(*)}{\leq} 2c_1 \left|\frac{\alpha_y - \beta\alpha_x}{\beta||x'||}\right|^2 \max(||x_0||, |a_j^* x_0|)^2 \max\left(1, \log\left(\left|\frac{\alpha_y - \beta\alpha_x}{\beta||x'||}. \max(||x_0||, |a_j^* x_0|)\right|^{-1}\right)\right) \\
&\qquad + 2c_1 \left|\frac{1}{\beta||x'||} a_j^* y''\right|^2 \max\left(1, \log\left(\left|\frac{1}{\beta||x'||} a_j^* y''\right|^{-1}\right)\right) \\
&\qquad + 2|1-\beta|^2 + |\beta - 1| \left(\frac{|\alpha_y - \beta\alpha_x|\ ||x_0||}{\beta||x'||}\frac{|a_j^* x_0|}{||x_0||} + \left|\frac{1}{\beta||x'||} a_j^* y''\right|\right) \\
&\leq 2c_1 \left|\frac{\alpha_y - \beta\alpha_x}{\beta||x'||}\right|^2 \max(||x_0||, |a_j^* x_0|)^2 \max\left(1, \log\left(\left|\frac{\alpha_y - \beta\alpha_x}{\beta||x'||}||x_0||\right|^{-1}\right)\right) \\
&\qquad + 2c_1 \left|\frac{1}{\beta||x'||} a_j^* y''\right|^2 \max\left(1, \log\left(\left|\frac{1}{\beta||x'||} a_j^* y''\right|^{-1}\right)\right) \\
&\qquad + 2\frac{||x-y||^2}{\epsilon} + \frac{||x-y||}{\sqrt{\epsilon}}\left(2\frac{||x-y||}{\sqrt{\epsilon}}\frac{|a_j^* x_0|}{||x_0||} + \frac{2}{\sqrt{\epsilon}}\left|a_j^* y''\right|\right) \\
&\leq c_2 ||x-y||^2 \left(1 + \frac{|a_j^* x_0|}{||x_0||}\right)^2 \max(1, \log ||x-y||^{-1}) + c_2 |a_j^* y''|^2 \max(1, \log |a_j^* y''|^{-1}) \\
&\qquad + c_2 ||x-y||\, |a_j^* y''|.
\end{aligned}$$

For $(*)$, we have used the fact that $t \to t^2 \max(1, \log(1/t))$ is non-decreasing. For the last two lines, we have used this same fact and Equations (49a), (49b) and (49c).

The random variable $a_j^* y''$ is complex and Gaussian, has variance $||y''||^2$ and is independent from Ax_0, so, taking the expectation over $a_j^* y''$ then using Equation (49d), we get:

$$\begin{aligned}
&1 - \mathrm{Re}\left(\mathbb{E}\left(\mathrm{phase}(\overline{a_j^* x})\, \mathrm{phase}(a_j^* y) | Ax_0\right)\right) \\
&\leq c_2 ||x-y||^2 \left(1 + \frac{|a_j^* x_0|}{||x_0||}\right)^2 \max(1, \log ||x-y||^{-1}) + c_3 ||y''||^2 \max(1, \log ||y''||^{-1}) \\
&\qquad + c_3 ||y''||\, ||x-y|| \\
&\leq c_4 ||x-y||^2 \left(1 + \frac{|a_j^* x_0|}{||x_0||}\right)^2 \max(1, \log ||x-y||^{-1}).
\end{aligned}$$

As $||x - y|| \leq 2$ (because x and y are unit-normed), this implies Equation (50) and concludes. □

C.2.5 Proof of Lemma C.5

Lemme (Lemma C.5)**.** *Let Z be any real random variable such that $|Z| \leq 2$ with probability 1. If we set $\sigma^2 = \mathrm{Var}(Z)$, then, for any $\lambda \in \mathbb{R}^+$,*

$$\mathbb{E}\left(e^{\lambda(Z-\mathbb{E}(Z))}\right) \leq 1 + \frac{\sigma^2}{16}\left(e^{4\lambda} - 1 - 4\lambda\right).$$

Proof of Lemma C.5. Let us define $Z' = Z - \mathbb{E}(Z)$. We have $|Z'| \leq 4$ with probability 1, $\mathbb{E}(Z') = 0$ and $\mathbb{E}(Z'^2) = \sigma^2$. Then,

$$\begin{aligned}
\mathbb{E}\left(e^{\lambda Z'}\right) &= \mathbb{E}\left(1 + \lambda Z' + \sum_{k\geq 2} \frac{\lambda^k Z'^k}{k!}\right) \\
&\leq 1 + \sum_{k\geq 2} \mathbb{E}\left(\frac{\lambda^k Z'^2 4^{k-2}}{k!}\right) \\
&= 1 + \frac{\sigma^2}{16}\left(e^{4\lambda} - 1 - 4\lambda\right).
\end{aligned}$$

□

C.2.6 Proof of Lemma C.6

Lemme (Lemma C.6)**.** *There exists a constant $\tilde{C} > 0$ depending only on γ and $\epsilon > 0$ such that, for any $\lambda \in]0; \frac{1}{40}[$,*

$$\int_1^{\sqrt{\log m}+1} \frac{f_\lambda(t)}{t^2} e^{-t^2/4} dt + m^{-1/2} \int_{\sqrt{\log m}+1}^{m^{1/4}+1} \frac{f_\lambda(t)}{t^3} dt + \frac{1}{m} f_\lambda(m^{1/4}+1) \leq \tilde{C}\gamma^{-2k}\lambda^2,$$

provided that

$$\left(\log(\max(1, \gamma^k/\lambda)) + 1\right)\left(\frac{\gamma^k}{\lambda}\right)^{4/3} \leq m^{1/2}; \tag{42a}$$

$$\frac{m^{1/2}\lambda\gamma^{-2k}}{1 + \log m} \geq 1. \tag{42b}$$

Proof of Lemma C.6. As we only consider the function f_λ on $]1; +\infty[$, we can upper bound it by the slightly simpler expression

$$\tilde{f}_\lambda(x) = \log\left(1 + \frac{C'\gamma^{-2k}}{8} x^2 (e^{4\lambda x^2} - 1 - 4\lambda x^2)\right).$$

Let X_0 be the (unique) positive number such that

$$\frac{C'\gamma^{-2k}}{8} X_0^2 (e^{4\lambda X_0^2} - 1 - 4\lambda X_0^2) = 1.$$
$$\iff \qquad \lambda X_0^2 (e^{4\lambda X_0^2} - 1 - 4\lambda X_0^2) = \frac{8}{C'} \lambda \gamma^{2k}. \tag{52}$$

The function $\tilde{f}_\lambda$ satisfies the following inequalities:

$$\forall x \in \mathbb{R}^+, \qquad \tilde{f}_\lambda(x) \leq \frac{C'\gamma^{-2k}}{8} x^2 (e^{4\lambda x^2} - 1 - 4\lambda x^2);$$

$$\begin{aligned}
\forall x \geq X_0, \qquad \tilde{f}_\lambda(x) &\leq \log\left(\frac{C'\gamma^{-2k}}{8} x^2 (e^{4\lambda x^2} - 1 - 4\lambda x^2)\right) + \log 2 \\
&\leq \log\left(\frac{C'\gamma^{-2k}}{8} x^2 e^{4\lambda x^2}\right) + \log 2 \\
&\leq \log\left(\frac{C'}{4}\right) + 2\log x + 4\lambda x^2.
\end{aligned}$$

In particular, if $X_0 \leq 2m^{1/4}$,

$$\begin{aligned}
\frac{1}{m} f_\lambda(m^{1/4} + 1) &\leq \frac{1}{m} f_\lambda(2m^{1/4}) \\
&\leq \frac{1}{m}\left(\log\left(\frac{C'}{4}\right) + 2\log(2m^{1/4}) + 16\lambda m^{1/2}\right) \\
&\leq D\left(\frac{\log m + \lambda m^{1/2}}{m}\right) \\
&\overset{(42b)}{\leq} \frac{2D\lambda}{m^{1/2}} \\
&\overset{(42b)}{\leq} 2D\lambda^2\gamma^{-2k},
\end{aligned}$$

and if $X_0 > 2m^{1/4}$, from the definition of X_0, we see that

$$\begin{aligned}
\frac{1}{m} f_\lambda(m^{1/4} + 1) &\leq \frac{1}{m} \log(1 + \frac{C'\gamma^{-2k}}{8} X_0^2 (e^{4\lambda X_0^2} - 1 - 4\lambda X_0^2)) \\
&\leq \frac{\log 2}{m} \\
&\leq (\log 2) \frac{\lambda^2\gamma^{-2k}}{(m^{1/2}\lambda\gamma^{-2k})^2}
\end{aligned}$$

$$\overset{(42b)}{\leq} (\log 2)\lambda^2\gamma^{-2k}.$$

So $\frac{1}{m} f_\lambda(m^{1/4}+1)$ is bounded by $\tilde{C}\gamma^{-2k}\lambda^2$ and we only have to show the same bound for the integral terms.

Using the inequalities we have established over $\tilde{f}_\lambda$,

$$\int_1^{\sqrt{\log m}+1} \frac{f_\lambda(t)}{t^2} e^{-t^2/4} dt + m^{-1/2} \int_{\sqrt{\log m}+1}^{m^{1/4}+1} \frac{f_\lambda(t)}{t^3} dt$$

$$\leq \frac{C'\gamma^{-2k}}{8} \int_0^{+\infty} (e^{4\lambda t^2} - 1 - 4\lambda t^2) e^{-t^2/4} dt \tag{53}$$

$$+ m^{-1/2} \frac{C'\gamma^{-2k}}{8} \int_{\sqrt{\log m}+1}^{\max(X_0, \sqrt{\log m}+1)} \frac{1}{t}(e^{4\lambda t^2} - 1 - 4\lambda t^2) dt \tag{54}$$

$$+ m^{-1/2} \int_{\min(m^{1/4}+1, \max(X_0, \sqrt{\log m}+1))}^{m^{1/4}+1} \frac{1}{t^3} \left(\log\left(\frac{C'}{4}\right) + 2\log t + 4\lambda t^2 \right) dt. \tag{55}$$

We separately study each of the three right-side terms.

For Term (53), we can do an exact computation, taking into account the fact that $\lambda \leq 1/40$:

$$(53) = \frac{C'\sqrt{\pi}}{8} \gamma^{-2k} \left(\frac{1}{\sqrt{1-16\lambda}} - 1 - 8\lambda \right)$$
$$\leq \frac{C''\sqrt{\pi}}{8} \gamma^{-2k} \lambda^2.$$

For Term (54), if $X_0 < \sqrt{\log m} + 1$, then it is zero. Otherwise, $X_0 \geq \sqrt{\log m} + 1$ and

$$(54) \leq m^{-1/2} \frac{C'\gamma^{-2k}}{8} \int_0^{X_0} \frac{1}{t}(e^{4\lambda t^2} - 1 - 4\lambda t^2) dt$$

$$= m^{-1/2} \frac{C'\gamma^{-2k}}{8} \int_0^{2\sqrt{\lambda}X_0} \frac{1}{t}(e^{t^2} - 1 - t^2) dt. \tag{56}$$

When $\frac{8}{C'}\lambda\gamma^{2k} \leq 1$, we check from the definition of X_0 (Equation (52)) that $\lambda X_0^2 \leq 1$, so $2\sqrt{\lambda}X_0 \leq 2$ and

$$(54) \leq m^{-1/2} \frac{C''\gamma^{-2k}}{8} \int_0^{2\sqrt{\lambda}X_0} t^3 dt.$$
$$= m^{-1/2} \frac{C''\gamma^{-2k}}{2} \lambda^2 X_0^4.$$

From Equation (52) again, we see that, as $\lambda X_0^2 \leq 1$,

$$\frac{8}{C'}\lambda\gamma^{2k} \geq C'''(\lambda X_0^2)^3;$$

$$\Rightarrow \qquad \left(\frac{8}{C'C'''}\right)^{2/3}\frac{\gamma^{4k/3}}{\lambda^{4/3}} \geq X_0^4. \tag{57}$$

From Condition (42a), we know that $\left(\frac{\gamma^k}{\lambda}\right)^{4/3} \leq m^{1/2}$, so

$$(54) \leq m^{-1/2}C''''\gamma^{-2k}\lambda^2\left(\frac{\gamma^k}{\lambda}\right)^{4/3} \leq C''''\gamma^{-2k}\lambda^2.$$

On the other hand, when $\frac{8}{C'}\lambda\gamma^{2k} > 1$, $2\sqrt{\lambda}X_0$ is bounded away from zero. We evaluate the integral in Equation (56) by parts:

$$\begin{aligned}(54) &\leq m^{-1/2}\frac{C'\gamma^{-2k}}{8}\int_0^{2\sqrt{\lambda}X_0}\frac{1}{t}(e^{t^2}-1-t^2)dt \\ &\leq m^{-1/2}C''\gamma^{-2k}\frac{1}{\lambda X_0^2}e^{4\lambda X_0^2}.\end{aligned}$$

From Equation (52), we can compute that, when $2\sqrt{\lambda}X_0$ is bounded away from zero,

$$\frac{1}{X_0^2}e^{4\lambda X_0^2} \leq C'''\frac{\lambda^2\gamma^{2k}}{\left(1+\log\left(8\lambda\gamma^{2k}/C'\right)\right)^2},$$

which yields, together with Condition (42b):

$$(54) \leq \frac{m^{-1/2}C''C'''\lambda}{\left(1+\log\left(8\lambda\gamma^{2k}/C'\right)\right)^2} \leq m^{-1/2}C''C'''\lambda \leq C''C'''\lambda^2\gamma^{-2k}.$$

Finally, we consider the last term. When $X_0 \geq m^{1/4}+1$, it is zero, so we only have to consider the case where $X_0 < m^{1/4}+1$.

$$\begin{aligned}(55) &\leq m^{-1/2}C''\int_{\max(X_0,1)}^{m^{1/4}+1}\frac{1+\log t+\lambda t^2}{t^3}dt \\ &= m^{-1/2}C''\left[-\frac{3+2\log t}{4t^2}+\lambda\log t\right]_{\max(X_0,1)}^{m^{1/4}+1}\end{aligned}$$

$$\leq m^{-1/2}C''\left(\frac{1+\log(\max(1,X_0))}{X_0^2}+\lambda\log\left(\frac{m^{1/4}+1}{\max(X_0,1)}\right)\right)$$

$$\leq m^{-1/2}C''\left(\frac{1+\log(\max(1,X_0))}{X_0^2}+\lambda\log\left(m^{1/4}+1\right)\right). \tag{58}$$

The second part of Equation (58) can be upper bounded as desired, thanks to Condition (42b):

$$m^{-1/2}C''\lambda\log(m^{1/4}+1)\leq m^{-1/2}C''\lambda(1+\log m)$$
$$\leq C''\lambda^2\gamma^{-2k}. \tag{59}$$

For the first part, let us distinguish the cases $\frac{8}{C'}\lambda\gamma^{2k}\leq 1$ and $\frac{8}{C'}\lambda\gamma^{2k}>1$.

In the case where $\frac{8}{C'}\lambda\gamma^{2k}\leq 1$, we see (in a similar way as in Equation (57)) that

$$c_1\frac{\gamma^{k/3}}{\lambda^{1/3}}\leq X_0\leq c_2\frac{\gamma^{k/3}}{\lambda^{1/3}},$$

so

$$m^{-1/2}\left(\frac{1+\log(\max(1,X_0))}{X_0^2}\right)\leq m^{-1/2}C'''\left(1+\log(\max(1,\gamma^k/\lambda))\right)\frac{\lambda^{2/3}}{\gamma^{2k/3}}$$
$$= m^{-1/2}C'''\left(1+\log(\max(1,\gamma^k/\lambda))\right)\lambda^2\gamma^{-2k}\left(\frac{\gamma^k}{\lambda}\right)^{4/3}$$
$$\leq C'''\lambda^2\gamma^{-2k}. \tag{60}$$

For the last equality, we have used Condition (42a).

In the case where $\frac{8}{C'}\lambda\gamma^{2k}>1$, as we have already seen, $\sqrt{\lambda}X_0$ is bounded away from 0, so, for some constant $C'''>0$,

$$X_0\geq C'''\lambda^{-1/2},$$

which implies

$$m^{-1/2}\left(\frac{1+\log(\max(1,X_0))}{X_0^2}\right)\leq C''''m^{-1/2}\lambda\left(1+\log(\max(1,\lambda^{-1/2}))\right)$$
$$\leq m^{-1/2}C''''\lambda\left(1+\log\left(\max\left(1,\frac{1}{\lambda\gamma^{-2k}}\right)\right)\right). \tag{61}$$

From Condition (42b), we know that

$$\lambda\gamma^{-2k}\geq m^{-1/2}(1+\log m);$$

$$\Rightarrow\qquad \frac{1+\log\left(\max\left(1,\frac{1}{\lambda\gamma^{-2k}}\right)\right)}{\lambda\gamma^{-2k}}\leq m^{1/2}\frac{1+\log\left(\frac{m^{1/2}}{1+\log m}\right)}{1+\log m}\leq m^{1/2}.$$

We plug this into Equation (61) and get

$$m^{-1/2}\left(\frac{1+\log(\max(1,X_0))}{X_0^2}\right)\le C''''\lambda^2\gamma^{-2k}. \tag{62}$$

Finally, we combine Equations (59), (60) and (62). With Equation (58), they show that

$$(55)\le\mathcal{C}\lambda^2\gamma^{-2k},$$

for some constant $\mathcal{C}>0$. □

C.2.7 Proof of Lemma C.7

Lemme (Lemma C.7)**.** *There exist a constant $C>0$ depending only on ϵ such that, for any fixed unit-normed x,y such that*

$$|\langle x_0,x\rangle|\le(1-\epsilon)||x_0||\,||x|| \quad \textit{and} \quad |\langle x_0,y\rangle|\le(1-\epsilon)||x_0||\,||y||,$$

and any $j=1,\dots,m$,

$$|\mathbb{E}(Z_j|a_j^*x_0)|\le C\min\left(1,||x-y||\left(1+\frac{|a_j^*x_0|}{||x_0||}\right)\right).$$

Proof of Lemma C.7. As $Z_j=\operatorname{phase}(a_j^*x)\operatorname{phase}(\overline{a_j^*x_0})-\operatorname{phase}(a_j^*y)\operatorname{phase}(\overline{a_j^*x_0})$,

$$|\mathbb{E}(Z_j|a_j^*x_0)|=|\mathbb{E}(\operatorname{phase}(a_j^*x)|a_j^*x_0)-\mathbb{E}(\operatorname{phase}(a_j^*y)|a_j^*x_0)|.$$

As in the proof of Lemma C.4, we write

$$x=\alpha_xx_0+x' \text{ and } y=\alpha_yx_0+\beta x'+y'',$$

where α_x,α_y,β are complex numbers and $x',y''\in\mathbb{C}^n$ satisfy $\langle x',x_0\rangle=\langle y'',x_0\rangle=\langle x',y''\rangle=0$. We recall Equations (49a) to (49d):

$$||x'||\ge\sqrt{\epsilon(2-\epsilon)}\ge\sqrt{\epsilon}; \tag{49a}$$

$$|\beta-1|=\frac{|\langle y-x,x'\rangle|}{||x'||^2}\le\frac{1}{\sqrt{\epsilon}}||y-x||; \tag{49b}$$

$$||\alpha_xx_0-\alpha_yx_0||=\frac{|\langle x-y,x_0\rangle|}{||x_0||}\le||x-y||; \tag{49c}$$

$$||y''||=\frac{|\langle y-x,y''\rangle|}{||y''||}\le||x-y||. \tag{49d}$$

The variable Z_j is bounded in modulus by 2, so the desired inequality holds for $||x-y|| \geq \sqrt{\epsilon}/2$ if we choose $C \geq 4/\sqrt{\epsilon}$. In what follows, we assume that $||x-y|| < \sqrt{\epsilon}/2$, which notably guarantees that $|\beta| > 1/2$.

The random variables $a_j^* x_0, a_j^* x'$ and $a_j^* y''$ are independent complex Gaussians, with respective variances $||x_0||^2, ||x'||^2, ||y''||^2$. Thus,

$$\begin{aligned}\mathbb{E}(\text{phase}(a_j^* x)|a_j^* x_0) &= \mathbb{E}\left(\text{phase}\left(\frac{\alpha_x}{||x'||}a_j^* x_0 + \frac{a_j^* x'}{||x'||}\right)\middle| a_j^* x_0\right) \\ &= \frac{1}{\pi}\int_{\mathbb{C}} \text{phase}\left(\frac{\alpha_x}{||x'||}a_j^* x_0 + z\right) e^{-|z|^2} d^2 z, \end{aligned} \tag{63}$$

and similarly,

$$\mathbb{E}(\text{phase}(a_j^* y)|a_j^* x_0, a_j^* y'') = \frac{\text{phase}(\beta)}{\pi}\int_{\mathbb{C}} \text{phase}\left(\frac{\alpha_y}{\beta||x'||}a_j^* x_0 + \frac{a_j^* y''}{\beta||x'||} + z\right) e^{-|z|^2} d^2 z. \tag{64}$$

The function

$$a \in \mathbb{C} \to \frac{1}{\pi}\int_{\mathbb{C}} \text{phase}(a+z) e^{-|z|^2} d^2 z = \frac{1}{\pi}\int_{\mathbb{C}} \text{phase}(z) e^{-|z-a|^2} d^2 z$$

is Lipschitz (as can be seen by derivation under the integral sign). If we denote by $D > 0$ the Lipschitz constant, Equations (63) and (64) imply that

$$\begin{aligned}&\left|\mathbb{E}(\text{phase}(a_j^* x)|a_j^* x_0) - \overline{\text{phase}(\beta)}\mathbb{E}(\text{phase}(a_j^* y)|a_j^* x_0, a_j^* y'')\right| \\ &\leq D\left|\left|\frac{\alpha_x}{||x'||}a_j^* x_0 - \left(\frac{\alpha_y}{\beta||x'||}a_j^* x_0 + \frac{a_j^* y''}{\beta||x'||}\right)\right|\right| \\ &\leq D\left(||x-y||\frac{|a_j^* x_0|}{||x_0||}\left(\frac{1}{\sqrt{\epsilon}} + \frac{2}{\epsilon}\right) + \frac{2}{\sqrt{\epsilon}}|a_j^* y''|\right) \\ &\leq D||x-y||\left(\frac{|a_j^* x_0|}{||x_0||}\left(\frac{1}{\sqrt{\epsilon}} + \frac{2}{\epsilon}\right) + \frac{2}{\sqrt{\epsilon}}\frac{|a_j^* y''|}{||y''||}\right).\end{aligned}$$

For the last two inequalities, we have used Equations (49a) to (49d). We finally take the expectation over $a_j^* y''$; by triangular inequality,

$$\begin{aligned}&\left|\mathbb{E}(\text{phase}(a_j^* x)|a_j^* x_0) - \overline{\text{phase}(\beta)}\mathbb{E}(\text{phase}(a_j^* y)|a_j^* x_0)\right| \\ &\leq D||x-y||\left(\frac{|a_j^* x_0|}{||x_0||}\left(\frac{1}{\sqrt{\epsilon}} + \frac{2}{\epsilon}\right) + \sqrt{\frac{\pi}{\epsilon}}\right)\end{aligned}$$

$$\leq C||x-y||\left(1+\frac{|a_j^*x_0|}{||x_0||}\right),$$

when $C>0$ is large enough. Additionally,

$$\begin{aligned}
&\left|\mathbb{E}(\text{phase}(a_j^*y)|a_j^*x_0) - \overline{\text{phase}(\beta)}\mathbb{E}(\text{phase}(a_j^*y)|a_j^*x_0)\right| \\
&\quad\leq |1-\beta| \\
&\quad\leq 2\frac{|1-\beta|}{|\beta|} \\
&\quad\leq \frac{4}{\sqrt{\epsilon}}||y-x||.
\end{aligned}$$

So by triangular inequality,

$$\begin{aligned}
&\left|\mathbb{E}(\text{phase}(a_j^*x)|a_j^*x_0) - \mathbb{E}(\text{phase}(a_j^*y)|a_j^*x_0)\right| \\
&\quad\leq C'||x-y||\left(1+\frac{|a_j^*x_0|}{||x_0||}\right),
\end{aligned}$$

We also have

$$|\mathbb{E}(Z_j|a_j^*x_0)| \leq C,$$

for any constant $C \geq 2$, so

$$|\mathbb{E}(Z_j|a_j^*x_0)| \leq C\min\left(1, ||x-y||\left(1+\frac{|a_j^*x_0|}{||x_0||}\right)\right)$$

when $C>0$ is large enough. □

C.2.8 Proof of Lemma C.8

Lemme (Lemma C.8)**.** *There exist constants $c, C' > 0$, that depend only on γ and ϵ, such that, for any $\lambda \in [-c; c]$,*

$$\begin{aligned}
&\log\left(\mathbb{E}\left(e^{\lambda \text{Re}\left(|a_j^*x_0|^2\mathbb{E}(Z_j|Ax_0) - \mathbb{E}(|a_j^*x_0|^2 Z_j)\right)}\right)\right) \leq C'\lambda^2\gamma^{-2k}, \\
\textit{and }&\log\left(\mathbb{E}\left(e^{\lambda \text{Im}\left(|a_j^*x_0|^2\mathbb{E}(Z_j|Ax_0) - \mathbb{E}(|a_j^*x_0|^2 Z_j)\right)}\right)\right) \leq C'\lambda^2\gamma^{-2k}.
\end{aligned}$$

Proof of Lemma C.8. We only prove the first inequality; the proof of the second one is identical. We assume that λ is positive; the same reasoning holds with minor modifications when λ is negative.

To simplify the notations, we set

$$\mathcal{Z}_j = |a_j^* x_0|^2 \mathbb{E}(Z_j | a_j^* x_0) - \mathbb{E}(|a_j^* x_0|^2 Z_j).$$

We recall from Equation (45) that

$$|a_j^* x_0|^2 |\mathbb{E}(Z_j | a_j^* x_0)| \leq 2C |a_j^* x_0|^2 \min(1, \gamma^{-k}(1 + |a_j^* x_0|)).$$

As a consequence, because $a_j^* x_0$ is a complex Gaussian random variable with variance $||x_0||^2 = 1$,

$$\begin{aligned}
|\mathbb{E}(|a_j^* x_0|^2 Z_j)| &= |\mathbb{E}\left(|a_j^* x_0|^2 \mathbb{E}(Z_j | a_j^* x_0)\right)| \\
&\leq 2C\mathbb{E}\left(|a_j^* x_0|^2 \min(1, \gamma^{-k}(1 + |a_j^* x_0|))\right) \\
&\leq 2C\gamma^{-k}\mathbb{E}(|a_j^* x_0|^2 (1 + |a_j^* x_0|)) \\
&= 2C\left(1 + \frac{3}{4}\sqrt{\pi}\right)\gamma^{-k}.
\end{aligned}$$

Combining this with Equation (45), we see that there exists a constant $C'' > 0$ such that

$$|\mathcal{Z}_j| = \left| |a_j^* x_0|^2 \mathbb{E}(Z_j | a_j^* x_0) - \mathbb{E}(|a_j^* x_0|^2 Z_j) \right| \leq C''(1 + |a_j^* x_0|)^2 \min(1, \gamma^{-k}(1 + |a_j^* x_0|)).$$

Let us note that, because $\mathbb{E}(\mathcal{Z}_j) = 0$,

$$\begin{aligned}
\log(\mathbb{E}(e^{\lambda \mathrm{Re}\,(\mathcal{Z}_j)})) &\leq \mathbb{E}(e^{\lambda \mathrm{Re}\,(\mathcal{Z}_j)}) - 1 \\
&= \mathbb{E}(e^{\lambda \mathrm{Re}\,(\mathcal{Z}_j)} - \lambda \mathrm{Re}\,(\mathcal{Z}_j) - 1).
\end{aligned}$$

The function $f : x \to e^{\lambda x} - \lambda x - 1$ is non-decreasing over $\mathbb{R}^+$, and satisfies $f(x) \leq f(|x|)$ for any $x \in \mathbb{R}$. Hence,

$$\begin{aligned}
&\log(\mathbb{E}(e^{\lambda \mathrm{Re}\,(\mathcal{Z}_j)})) \\
&\leq \mathbb{E}\left(e^{\lambda C''(1+|a_j^* x_0|)^2 \min(1,\gamma^{-k}(1+|a_j^* x_0|))} - \lambda C''(1 + |a_j^* x_0|)^2 \min(1, \gamma^{-k}(1 + |a_j^* x_0|)) - 1\right) \\
&\quad = \frac{1}{\pi}\int_{\mathbb{C}} \left(e^{\lambda C''(1+|z|)^2 \min(1,\gamma^{-k}(1+|z|))} - \lambda C''(1 + |z|)^2 \min(1, \gamma^{-k}(1 + |z|)) - 1\right) e^{-|z|^2} d^2 z \\
&\quad = 2\int_0^{+\infty} \left(e^{\lambda C''(1+r)^2 \min(1,\gamma^{-k}(1+r))} - \lambda C''(1 + r)^2 \min(1, \gamma^{-k}(1 + r)) - 1\right) r e^{-r^2} dr \\
&\quad = 2\int_1^{+\infty} \left(e^{\lambda C'' r^2 \min(1,\gamma^{-k} r)} - \lambda C'' r^2 \min(1, \gamma^{-k} r) - 1\right) (r - 1) e^{-(r-1)^2} dr \\
&\quad \leq C''' \int_0^{+\infty} \left(e^{\lambda C'' r^2 \min(1,\gamma^{-k} r)} - \lambda C'' r^2 \min(1, \gamma^{-k} r) - 1\right) e^{-r^2/2} dr
\end{aligned}$$

$$= C''' \int_0^{\gamma^k} \left(e^{\lambda C'' r^3 \gamma^{-k}} - \lambda C'' r^3 \gamma^{-k} - 1 \right) e^{-r^2/2} dr \tag{65}$$

$$+ C''' \int_{\gamma^k}^{+\infty} \left(e^{\lambda C'' r^2} - \lambda C'' r^2 - 1 \right) e^{-r^2/2} dr. \tag{66}$$

We need to show that both components (65) and (66) are upper bounded by $C'\lambda^2\gamma^{-2k}$ for some constant $C' > 0$ sufficiently large, provided that $|\lambda| \leq c$ for some constant $c > 0$.

For Term (65), we use the fact that, when $r \leq C''^{-1/3}\gamma^{k/3}\lambda^{-1/3}$,

$$\begin{aligned} & \lambda C'' r^3 \gamma^{-k} \leq 1; \\ \Rightarrow \quad & e^{\lambda C'' r^3 \gamma^{-k}} - \lambda C'' r^3 \gamma^{-k} - 1 \leq (\lambda C'' r^3 \gamma^{-k})^2. \end{aligned}$$

It yields:

$$\begin{aligned} (65) \leq\ & C''' \int_0^{\min(\gamma^k, C''^{-1/3}\gamma^{k/3}\lambda^{-1/3})} (\lambda C'' r^3 \gamma^{-k})^2 e^{-r^2/2} dr \\ & + C''' \int_{\min(\gamma^k, C''^{-1/3}\gamma^{k/3}\lambda^{-1/3})}^{\gamma^k} e^{\lambda C'' r^3 \gamma^{-k}} e^{-r^2/2} dr \\ \leq\ & C''' C''^2 \lambda^2 \gamma^{-2k} \int_0^{+\infty} r^6 e^{-r^2/2} dr + C''' \int_{\min(\gamma^k, C''^{-1/3}\gamma^{k/3}\lambda^{-1/3})}^{\gamma^k} e^{\lambda C'' r^3 \gamma^{-k}} e^{-r^2/2} dr. \end{aligned} \tag{67}$$

For the second term of this sum, if we assume that

$$\lambda < \frac{1}{4C''},$$

we have

$$\begin{aligned} \int_{\min(\gamma^k, C''^{-1/3}\gamma^{k/3}\lambda^{-1/3})}^{\gamma^k} e^{\lambda C'' r^3 \gamma^{-k}} e^{-r^2/2} dr &= \int_{\min(\gamma^k, C''^{-1/3}\gamma^{k/3}\lambda^{-1/3})}^{\gamma^k} e^{r^2\left(\lambda C'' r \gamma^{-k} - \frac{1}{2}\right)} dr \\ &\leq \int_{\min(\gamma^k, C''^{-1/3}\gamma^{k/3}\lambda^{-1/3})}^{\gamma^k} e^{r^2\left(\lambda C'' - \frac{1}{2}\right)} dr \\ &\leq \int_{\min(\gamma^k, C''^{-1/3}\gamma^{k/3}\lambda^{-1/3})}^{\gamma^k} e^{-r^2/4} dr \\ &\leq \int_{C''^{-1/3}\gamma^{k/3}\lambda^{-1/3}}^{+\infty} e^{-r^2/4} dr \\ &\leq C''' \frac{e^{-\left(C''^{-1/3}\gamma^{k/3}\lambda^{-1/3}\right)^2/4}}{C''^{-1/3}\gamma^{k/3}\lambda^{-1/3}} \end{aligned}$$

$$\leq C''''\lambda^2\gamma^{-2k}.$$

For the last inequality, we have used the fact that there exists a constant $D > 0$ such that $e^{-x} \leq Dx^{-5/2}$, for all $x > 0$.

Plugging this into Equation (67), we get

$$(65) \leq C'\lambda^2\gamma^{-2k}. \tag{68}$$

For Term (66), still under the assumption $\lambda < 1/(4C'')$,

$$\begin{aligned}
(66) &\leq C''' \int_{\gamma^k}^{\max(\gamma^k,(\lambda C'')^{-1/2})} (\lambda C'' r^2)^2 e^{-r^2/2} dr + C''' \int_{\max(\gamma^k,(\lambda C'')^{-1/2})}^{+\infty} e^{\lambda C'' r^2} e^{-r^2/2} dr \\
&\leq C''' C''^2 \lambda^2 \int_{\gamma^k}^{+\infty} r^4 e^{-r^2/2} dr + C''' \int_{\max(\gamma^k,(\lambda C'')^{-1/2})}^{+\infty} e^{-r^2/4} dr \\
&\leq C'''' \left(\lambda^2 \gamma^{3k} e^{-\gamma^{2k}/2} + \min\left(\frac{e^{-\gamma^{2k}/4}}{\gamma^k}, \sqrt{\lambda C''} e^{-1/(4\lambda C'')} \right) \right) \\
&\overset{(*)}{\leq} \tilde{C}(\lambda^2\gamma^{-2k} + \min(\gamma^{-4k}, \lambda^4)) \\
&\leq 2\tilde{C}\lambda^2\gamma^{-2k}.
\end{aligned} \tag{69}$$

For Inequality $(*)$, we have used the existence of a constant D such that, for all k, $\gamma^{3k}e^{-\gamma^{2k}/2} \leq D\gamma^{-2k}$ and, for all λ staying in a bounded interval, $\sqrt{\lambda}e^{-1/(4\lambda C'')} \leq D\lambda^4$.

Equations (68) and (69), combined with Equation (66), show that, when $\lambda < 1/(4C'')$,

$$\log(\mathbb{E}(e^{\lambda \mathrm{Re}\,(\mathcal{Z}_j)})) \leq C'\lambda^2\gamma^{-2k},$$

for some constant $C' > 0$ that depends only upon γ. □

C.2.9 Proof of Lemma C.9

Lemme (Lemma C.9)**.** *For any $t \in \mathbb{R}^+$, we set*

$$f(t) = \mathbb{E}\left(\overline{Z_1}|Z_1|\,\mathrm{phase}\,(Z_1 + tZ_2)\right).$$

The function f is real-valued. For any $\gamma > 0$, there exist $\delta > 0$ such that

$$\forall t \in [\gamma; +\infty[, \qquad f(t) \geq \frac{1+\delta}{\sqrt{1+t^2}}.$$

Proof of Lemma C.9. As $(\overline{Z}_1, \overline{Z}_2)$ has the same distribution as (Z_1, Z_2),

$$\forall t \in \mathbb{R}^+, \qquad f(t) = \mathbb{E}(Z_1|Z_1| \operatorname{phase}(\overline{Z}_1 + t\overline{Z}_2)) = \overline{f(t)},$$

so $f(t)$ is a real number, for any $t \geq 0$.

Let us now show the second part of the result. We have

$$\begin{aligned}
f(t) &= \frac{1}{\pi^2}\int_{\mathbb{C}^2} \overline{z_1}|z_1| \operatorname{phase}(z_1 + tz_2) e^{-|z_1|^2} e^{-|z_2|^2} d^2z_1 d^2z_2 \\
&= \frac{1}{\pi^2}\int_{\mathbb{C}^2} \overline{y_1}|y_1| \operatorname{phase}(y_2) e^{-|y_1|^2} e^{-|y_2 - y_1/t|^2} d^2y_1 d^2y_2 \\
&= \frac{1}{\pi^2}\int_{\mathbb{C}^2} \overline{y_1}|y_1| \operatorname{phase}(y_2) e^{-|y_1|^2} e^{-|y_2|^2} \left(\sum_{k\geq 0} \frac{1}{k!}\left(y_2\overline{y_1}/t + y_1\overline{y_2}/t - |y_1|^2/t^2\right)^k\right) d^2y_1 d^2y_2 \\
&= \frac{1}{\pi^2}\sum_k \sum_{k_1+k_2\leq k} \frac{(-1)^{k-(k_1+k_2)}}{k_1!k_2!(k-k_1-k_2)!}\frac{1}{t^{2k-(k_1+k_2)}} \times \\
&\qquad\qquad \int_{\mathbb{C}^2} y_1^{k-k_1}\overline{y_1}^{k-k_2+1}|y_1| y_2^{k_1}\overline{y_2}^{k_2} \operatorname{phase}(y_2) e^{-|y_1|^2} e^{-|y_2|^2} d^2y_1 d^2y_2 \\
&\overset{(*)}{=} \frac{1}{\pi^2}\sum_k \sum_{2k_1+1\leq k} \frac{(-1)^{k-2k_1-1}}{k_1!(k_1+1)!(k-2k_1-1)!}\frac{1}{t^{2k-2k_1-1}} \int_{\mathbb{C}^2} |y_1|^{2(k-k_1)+1}|y_2|^{2k_1+1} e^{-|y_1|^2} e^{-|y_2|^2} d^2y_1 d^2y_2 \\
&\overset{(**)}{=} \sum_l \frac{1}{t^{2l+1}}\left(\frac{1}{\pi}\int_{\mathbb{C}^2} |y|^{2l+3} e^{-|y|^2} d^2y\right) \sum_{k_1\leq l} \frac{(-1)^{k_1+l}}{k_1!(k_1+1)!(l-k_1)!}\left(\frac{1}{\pi}\int_{\mathbb{C}^2} |y|^{2k_1+1} e^{-|y|^2} d^2y\right) \\
&\overset{(***)}{=} \sum_l \frac{\pi}{t^{2l+1}}(l+1)(l+2)\binom{2(l+2)}{l+2}\sum_{k_1\leq l}(-1)^{k_1+l}\binom{2(k_1+1)}{k_1+1}\binom{l}{k_1} 2^{-2(l+k_1+3)}.
\end{aligned}$$

Equality $(*)$ is true because the integral is zero if $k_2 \neq k_1 + 1$, as can be seen with a change of variable $y_1 \to uy_1$ for u a complex number of modulus 1. Equality $(**)$ is obtained by setting $l = k - k_1 - 1$. Equality $(***)$ is a consequence of the following inequality, valid for all odd K:

$$\frac{1}{\pi}\int_{\mathbb{C}} |y|^K e^{-|y|^2} d^2y = \sqrt{\pi}2^{-K}\frac{K!}{(\frac{K-1}{2})!}.$$

This reasoning is valid only for t large enough; for small values of t, the series may not converge. We see that, in order for all the involved series to be absolutely convergent, it is enough that the following one is absolutely convergent:

$$\sum_k \sum_{k_1+k_2\leq k} \frac{1}{k_1!k_2!(k-k_1-k_2)!}\frac{1}{t^{2k-(k_1+k_2)}} \times$$

$$\int_{\mathbb{C}^2} \left| y_1^{k-k_1} \overline{y_1}^{k-k_2+1} |y_1| y_2^{k_1} \overline{y_2}^{k_2} \,\mathrm{phase}(y_2) e^{-|y_1|^2} e^{-|y_2|^2} \right| d^2y_1 d^2y_2.$$

When $t \geq 2$, for example, this series can be upper bounded by

$$\begin{aligned}
&\sum_k \sum_{k_1+k_2 \leq k} \frac{1}{k_1!k_2!(k-k_1-k_2)!} \frac{1}{t^{2k-(k_1+k_2)}} \int_{\mathbb{C}^2} |y_1|^{2k-(k_1+k_2)+2} |y_2|^{k_1+k_2} e^{-|y_1|^2} e^{-|y_2|^2} d^2y_1 d^2y_2 \\
&= \int_{\mathbb{C}} \left(\sum_k \frac{1}{k!} |y_1|^2 \left(\frac{|y_1||y_2|}{t} + \frac{|y_1||y_2|}{t} + \frac{|y_1|^2}{t^2} \right)^k e^{-|y_1|^2} e^{-|y_2|^2} \right) d^2y_1 d^2y_2 \\
&= \int_{\mathbb{C}} |y_1|^2 \exp\left(-|y_1|^2 - |y_2|^2 + 2\frac{|y_1||y_2|}{t} + \frac{|y_1|^2}{t^2} \right) d^2y_1 d^2y_2 \\
&\leq \int_{\mathbb{C}} |y_1|^2 \exp\left(-\left(1 - \frac{1}{t} - \frac{1}{t^2}\right) |y_1|^2 - \left(1 - \frac{1}{t}\right) |y_2|^2 \right) d^2y_1 d^2y_2 \\
&\leq \int_{\mathbb{C}} |y_1|^2 \exp\left(-\frac{1}{4}|y_1|^2 - \frac{1}{2}|y_2|^2 \right) d^2y_1 d^2y_2 < +\infty.
\end{aligned}$$

So the series converge.

For any $l \in \mathbb{N}, k_1 \in \{0, \ldots, l\}$, we set

$$\begin{aligned}
c_{l,k_1} &= \binom{2(k_1+1)}{k_1+1} \binom{l}{k_1} 2^{-2(l+k_1+3)}; \\
C_l &= (l+1)(l+2) \binom{2(l+2)}{l+2} \sum_{k_1 \leq l} (-1)^{k_1+l} c_{l,k_1}.
\end{aligned}$$

The series $\sum_{k_1 \leq l} (-1)^{k_1+l} c_{l,k_1}$ is alternating, and we can check that

$$\max_{k_1 \leq l} |c_{l,k_1}| = |c_{l,[l/2]}|.$$

This allows us to see that

$$\begin{aligned}
\left| \sum_{k_1 \leq l} (-1)^{k_1+l} c_{l,k_1} \right| &\leq \max_{k_1 \leq l} |c_{l,k_1}| \\
&= c_{l,[l/2]} \\
&\leq \frac{1}{8\pi} \frac{1}{l 2^l},
\end{aligned}$$

We do not derive the second inequality in full detail: the principle is to compute the upper limit of the sequence $(c_{l,[l/2]} l 2^l)_{l \in \mathbb{N}}$ with Sterling's formula, then to study the variations of this sequence, to show that it is bounded by its upper limit. Hence, using this inequality and the fact that, for any s, $\binom{2s}{s} \leq 2^{2s}/\sqrt{\pi s}$, we see that

$$|C_l| \leq \frac{l+1}{l} \cdot \sqrt{\frac{l+2}{\pi}} \frac{2^{l+1}}{\pi}.$$

So for any $l \geq 3$,

$$|C_l| \leq \frac{l 2^{l+1}}{\pi^{3/2}}.$$

We explicitly compute C_0, C_1, C_2:

$$C_0 = \frac{3}{8}; \qquad C_1 = -\frac{15}{64}; \qquad C_2 = \frac{105}{512}.$$

Hence, combining the previous results, for any $t \geq 2$,

$$\begin{aligned}
f(t) &= \pi \sum_{l \geq 0} \frac{C_l}{t^{2l+1}} \\
&\geq \pi \sum_{l=0}^{2} \frac{C_l}{t^{2l+1}} - -\frac{1}{\sqrt{\pi}} \sum_{l=3}^{+\infty} \frac{l 2^{l+1}}{t^{2l+1}} \\
&= \pi \left(\frac{3}{8}\frac{1}{t} - \frac{15}{64}\frac{1}{t^3} + \frac{105}{512}\frac{1}{t^5} \right) - \frac{16}{\sqrt{\pi} t^5} \frac{3t^2 - 4}{t^4 - 4}.
\end{aligned}$$

From here, we can easily verify with a computer that, for any $t > 2.5$,

$$f(t) > \frac{1.05}{\sqrt{1+t^2}}. \tag{70}$$

Let us now show that $f(t) > (1+t^2)^{-1/2}$ for any $t \in]0; 2.5]$. If we set

$$Y_1 = \frac{-tZ_1 + Z_2}{\sqrt{1+t^2}} \quad \text{and} \quad Y_2 = \frac{Z_1 + tZ_2}{\sqrt{1+t^2}},$$

we see that Y_1 and Y_2 are independent Gaussian random variables, with variance 1, and that

$$f(t) = \frac{1}{1+t^2} \mathbb{E}\left((\overline{Y_2 - tY_1}) |Y_2 - tY_1| \, \mathrm{phase}(Y_2) \right).$$

We set

$$g(t) = \mathbb{E}\left((\overline{Y_2 - tY_1}) |Y_2 - tY_1| \, \mathrm{phase}(Y_2) \right).$$

A straight computation yields

$$g'(t) = \mathbb{E}\left(\left(-\frac{3}{2}\overline{Y}_1|Y_2 - tY_1| - \frac{1}{2}Y_1\frac{(\overline{Y_2 - tY_1})^2}{|Y_2 - tY_1|}\right)\text{phase}(Y_2)\right); \tag{71}$$

$$g''(t) = \mathbb{E}\left(\left(\frac{3}{2}|Y_1|^2\,\text{phase}(\overline{Y_2 - tY_1}) + \frac{3}{4}\overline{Y}_1^2\,\text{phase}(Y_2 - tY_1) - \frac{1}{4}Y_1^2\,\text{phase}(\overline{Y_2 - tY_1})^3\right)\text{phase}(Y_2)\right). \tag{72}$$

For any $u, t > 0$, we see by triangular inequality that

$$\begin{aligned}
&|\text{phase}(Y_2 - tY_1) - \text{phase}(Y_2 - uY_1)| \\
&\quad \le \left|\frac{Y_2 - tY_1}{|Y_2 - tY_1|} - \frac{Y_2 - uY_1}{|Y_2 - tY_1|}\right| + \left|\frac{Y_2 - uY_1}{|Y_2 - tY_1|} - \frac{Y_2 - uY_1}{|Y_2 - uY_1|}\right| \\
&\quad \le 2|t - u|\frac{|Y_1|}{|Y_2 - tY_1|},
\end{aligned}$$

which also implies

$$\left|\text{phase}(Y_2 - tY_1)^3 - \text{phase}(Y_2 - uY_1)^3\right| \le 6|t - u|\frac{|Y_1|}{|Y_2 - tY_1|}.$$

Plugging this into Equation (72):

$$\begin{aligned}
|g''(t) - g''(u)| &\le 6|t - u|\mathbb{E}\left(\frac{|Y_1|^3}{|Y_2 - tY_1|}\right) \\
&\le 6|t - u|\mathbb{E}\left(\frac{|Y_1|^3}{|Y_2|}\right) \\
&= \frac{9}{2}\pi|t - u|.
\end{aligned}$$

We deduce from here that, for any u, t such that $0 \le u \le t$,

$$\begin{aligned}
g(t) &= g(u) + (t - u)g'(u) + \frac{(t - u)^2}{2}g''(u) + \int_u^t (t - s)(g''(s) - g''(u))ds \\
&\ge g(u) + (t - u)g'(u) + \frac{(t - u)^2}{2}g''(u) - \frac{9}{2}\pi\int_u^t (t - s)(s - u)ds \\
&= g(u) + (t - u)g'(u) + \frac{(t - u)^2}{2}g''(u) - \frac{9}{2}\pi\frac{(t - u)^3}{6}.
\end{aligned}$$

In $u = 0$, Equations (71) and (72) allow us to compute $g'(0)$ and $g''(0)$: we have $g'(0) = 0$ and $g''(0) = \frac{3}{2}$. Thus, from the last equation, for any $t \geq 0$,

$$g(t) \geq 1 + \frac{3}{4}t^2 - \frac{3}{4}\pi t^3,$$

which allows us to verify (with a computer) that, for any $t \in]0; 0.1]$,

$$f(t) = \frac{g(t)}{1+t^2} \geq \frac{1 + \frac{3}{4}t^2 - \frac{3}{4}\pi t^3}{1+t^2} > \frac{1}{\sqrt{1+t^2}}.$$

We can apply the same reasoning to values of u that are different from 0. Equations (71) and (72) do not appear to have a simple analytic expression when $u \neq 0$. They can however be computed with a computer. We do so for $u = 0.1, 0.2, 0.3, 0.4, \dots, 2.4$, and successively show that the previous inequality also holds on the intervals $[0.1; 0.2], [0.2, 0.3], \dots, [2.7, 2.5]$.

We have thus proven that $f(t) > (1+t^2)^{-1/2}$ for any $t \in]0; 2.5]$. By compacity (as f is continuous), it means that there exists $\delta > 0$ such that

$$\forall t \in [\gamma; 2.5], \qquad f(t) \geq \frac{1+\delta}{\sqrt{1+t^2}}.$$

Together with Equation (70), this implies the lemma.

□

C.2.10 Proof of Proposition C.10

Proposition (Proposition C.10)**.** *Let us define the function*

$$\begin{array}{rccc} G: & \mathbb{C}^2 & \to & \mathbb{C} \\ & (a,b) & \to & 1 - \frac{1}{\pi}\mathrm{Re} \int_{\mathbb{C}} \mathrm{phase}(\overline{z+a})\, \mathrm{phase}(z+b) e^{-|z|^2} d^2 z. \end{array}$$

For some constant $c_1 > 0$, the following inequalities are true:

$$\begin{aligned} \forall a, b \in \mathbb{C}, \qquad & |\mathrm{Re}\, G(a,b)| \leq c_1 |a-b|^2 \max\left(1, \log\left(|a-b|^{-1}\right)\right), \\ & |\mathrm{Im}\, G(a,b)| \leq c_1 |a-b|. \end{aligned}$$

Proof of Proposition C.10.

$$\begin{aligned} |\mathrm{Re}\, G(a,b)| &= \frac{1}{\pi}\left|\mathrm{Re} \int_{\mathbb{C}} \left(1 - \mathrm{phase}(\overline{z+a})\, \mathrm{phase}(z+b)\right) e^{-|z|^2} d^2 z\right| \\ &= \frac{1}{\pi}\left|\mathrm{Re} \int_{\mathbb{C}} \left(1 - \mathrm{phase}\left(1 + \frac{b-a}{z+a}\right)\right) e^{-|z|^2} d^2 z\right| \end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{\pi}\left|\mathrm{Re}\int_{|z+a|>2|b-a|}\left(1-\mathrm{phase}\left(1+\frac{b-a}{z+a}\right)\right)e^{-|z|^2}d^2z\right| \\
&\quad+\frac{2}{\pi}\left|\int_{|z+a|\leq 2|b-a|}e^{-|z|^2}d^2z\right| \\
&\overset{(a)}{\leq}\frac{c_2}{\pi}\left|\int_{|z+a|>2|b-a|}\left|\frac{b-a}{z+a}\right|^2e^{-|z|^2}d^2z\right|+\frac{2}{\pi}\left|\int_{|z+a|\leq 2|b-a|}1d^2z\right| \\
&\leq\frac{c_2}{\pi}\left|\int_{1\geq|z+a|>2|b-a|}\left|\frac{b-a}{z+a}\right|^2e^{-|z|^2}d^2z\right|+\frac{c_2}{\pi}\left|\int_{|z+a|>1}\left|\frac{b-a}{z+a}\right|^2e^{-|z|^2}d^2z\right|+8|b-a|^2 \\
&\leq\frac{c_2}{\pi}\left|\int_{1\geq|z+a|>2|b-a|}\left|\frac{b-a}{z+a}\right|^2d^2z\right|+\frac{c_2}{\pi}|b-a|^2\left|\int_{\mathbb{C}}e^{-|z|^2}d^2z\right|+8|b-a|^2 \\
&\leq c_1|b-a|^2\max\left(1,\log\left(|b-a|^{-1}\right)\right).
\end{aligned}
$$

Inequality (a) comes from the fact that $z\in\mathbb{C}\to\mathrm{Re}\,(1-\mathrm{phase}(1+z))\in\mathbb{R}$ is a $\mathcal{C}^\infty$ function on $\{z\in\mathbb{C},|z|<1/2\}$, and its derivative in 0 is 0 (because the function reaches a local minimum at this point). So by compacity, there exists a constant $c_2>0$ such that, for any z verifying $|z|<1/2$,

$$
|\mathrm{Re}\,(1-\mathrm{phase}(1+z))|\leq c_2|z|^2.
$$

The proof of the second inequality is identical, except that we bound $\left|\mathrm{Im}\,\left(1-\mathrm{phase}\left(1+\frac{b-a}{z-a}\right)\right)\right|$ by $c_2\left|\frac{b-a}{z+a}\right|$ on the set $\{z,|z+a|>2|b-a|\}$. ☐

C.3 Proof of Lemma 4.5

Lemme (Lemma 4.5)**.** *For any $c>0$, there exist $C_1,C_2,C_3>0$ such that, with probability at least*

$$
1-C_1\exp(-C_2m^{1/8}),
$$

the following property holds for any unit-normed $x,y\in\mathbb{C}^n$, when $m\geq 2n^2$:

$$
|\langle Ax_0,b\odot\mathrm{phase}(Ax)\rangle-\langle Ax_0,b\odot\mathrm{phase}(Ay)\rangle|\leq C_3||x_0||^2nm^{1/4}\quad \textit{if } ||x-y||\leq cm^{-7/2}.
$$

Proof of Lemma 4.5. We write

$$
\begin{aligned}
&|\langle Ax_0,b\odot\mathrm{phase}(Ax)\rangle-\langle Ax_0,b\odot\mathrm{phase}(Ay)\rangle| \\
&\quad=\left|\sum_{i=1}^m\overline{(Ax_0)_i}|(Ax_0)_i|\left(\mathrm{phase}((Ax)_i)-\mathrm{phase}((Ay)_i)\right)\right|
\end{aligned}
$$

$$
\begin{aligned}
&\le \sum_{i=1}^{m} |(Ax_0)_i|^2 \, |\text{phase}((Ax)_i) - \text{phase}((Ay)_i)| \\
&\le 2 \sum_{i=1}^{m} |(Ax_0)_i|^2 \min\left(1, \frac{|(Ax)_i - (Ay)_i|}{|(Ax)_i|}\right) \\
&\le 2 \sum_{|(Ax)_i| \le 1/m^2} |(Ax_0)_i|^2 + 2 \sum_{|(Ax)_i| > 1/m^2} |(Ax_0)_i|^2 \frac{|(Ax)_i - (Ay)_i|}{|(Ax)_i|} \\
&\le 2 \sum_{|(Ax)_i| \le 1/m^2} |(Ax_0)_i|^2 + 2m^2 |||A|||^3 \, ||x-y|| \, ||x_0||^2.
\end{aligned}
$$

From Proposition 3.5, if $m \ge 2n^2 \ge 2n$, $|||A||| \le 3\sqrt{m}$ with probability at least

$$1 - 2\exp(-m).$$

On this event, we can deduce from the previous inequality that, for any x, y such that $||x-y|| \le cm^{-7/2}$,

$$
\begin{aligned}
&| \langle Ax_0, b \odot \text{phase}(Ax) \rangle - \langle Ax_0, b \odot \text{phase}(Ay) \rangle | \\
&\qquad \le 2 \sum_{|(Ax)_i| \le 1/m^2} |(Ax_0)_i|^2 + 54 m^{7/2} ||x-y|| \, ||x_0||^2 \\
&\qquad \le 2 \sum_{|(Ax)_i| \le 1/m^2} |(Ax_0)_i|^2 + 54c ||x_0||^2.
\end{aligned}
$$

To upper bound the first term of the right-hand side, we use two auxiliary lemmas, proven in Paragraphs C.3.1 and C.3.2.

Lemme C.11. *For any unit-normed $x \in \mathbb{C}^n$, we define $I_x = \left\{ i \in \{1, \dots, m\}, |(Ax)_i| \le \frac{1}{m^2} \right\}$. There exist $C_1, C_2 > 0$ such that, when $m \ge n^2$, the event*

$$\left(\forall x, \operatorname{Card} I_x < n m^{1/8} \right)$$

has probability at least

$$1 - C_1 \exp(-C_2 m^{1/8}).$$

Lemme C.12. *There exist $C > 0$ such that, with probability at least*

$$1 - \exp(-n m^{1/4}),$$

for any $I \subset \{1, \dots, m\}$ such that $\operatorname{Card} I \le n m^{1/8}$,

$$\sum_{i \in I} |(Ax_0)_i|^2 \le C ||x_0||^2 n m^{1/4}.$$

We combine these lemmas with the last inequality. This proves that, with probability at least

$$1 - C_1 \exp(-C_2 m^{1/8}),$$

(for some constants $C_1, C_2 > 0$ possibly different from the ones introduced in Lemma C.11),

$$\begin{aligned} |\langle Ax_0, b \odot \mathrm{phase}(Ax)\rangle - \langle Ax_0, b \odot \mathrm{phase}(Ay)\rangle| &\leq ||x_0||^2 \left(2Cnm^{1/4} + 54c\right), \\ &\leq C_3 ||x_0||^2 nm^{1/4}, \end{aligned}$$

for all x, y verifying $||x - y|| \leq cm^{-7/2}$.

□

C.3.1 Proof of Lemma C.11

Lemme (Lemma C.11)**.** *For any unit-normed $x \in \mathbb{C}^n$, we define $I_x = \left\{i \in \{1, \dots, m\}, |(Ax)_i| \leq \frac{1}{m^2}\right\}$. There exist $C_1, C_2 > 0$ such that, when $m \geq n^2$, the event*

$$\left(\forall x, \mathrm{Card}\, I_x < nm^{1/8}\right)$$

has probability at least

$$1 - C_1 \exp(-C_2 m^{1/8}).$$

Proof of Lemma C.11. Let $\mathcal{M} \geq 1$ be temporarily fixed.

For any n, m, let $\mathcal{N}_{n,m}$ be a $\frac{1}{\mathcal{M}m^2}$-net of the unit sphere of $\mathbb{C}^n$. From [Vershynin, 2012, Lemma 5.2], there is one of cardinality at most

$$\left(1 + 4\mathcal{M}m^2\right)^{2n} \leq (5\mathcal{M}m^2)^{2n}.$$

We define two events:

$$\mathcal{E}_1 = \left\{\forall x \in \mathcal{N}_{n,m}, \mathrm{Card}\left\{i, |(Ax)_i| \leq \frac{2}{m^2}\right\} < nm^{1/8}\right\};$$
$$\mathcal{E}_2 = \{\forall i \in \{1, \dots, m\}, ||a_i^*|| \leq \mathcal{M}\}.$$

(We recall that a_i^* is the i-th line of A.)

On the intersection of these two elements, we have $\mathrm{Card}\, I_x < nm^{1/8}$ for any unit-normed $x \in \mathbb{C}^n$. Indeed, for any such x, there exists $x' \in \mathcal{N}_{n,m}$ such that $||x - x'|| \leq 1/(\mathcal{M}m^2)$. For any $i \in I_x$,

$$|(Ax')_i| \leq |(Ax)_i| + |a_i^*(x - x')|$$

$$\leq \frac{1}{m^2} + ||a_i^*||\,||x - x'||$$
$$\leq \frac{2}{m^2}.$$

As a consequence, $I_x \subset \left\{i, |(Ax')_i| \leq \frac{2}{m^2}\right\}$, whose cardinality is strictly less than $nm^{1/8}$ because we are on event $\mathcal{E}_1$.

Let us find lower bounds on the probabilities of $\mathcal{E}_1$ and $\mathcal{E}_2$.

For any $x \in \mathcal{N}_{n,m}$, for any $i = 1, \dots, m$,

$$P\left(|(Ax)_i| \leq \frac{2}{m^2}\right) = 1 - e^{-\frac{4}{m^4}} \leq \frac{4}{m^4},$$

because $(Ax)_i$ is a complex Gaussian random variable with variance 1. So by Hoeffding's inequality, for x fixed,

$$\begin{aligned}
P\left(\text{Card}\left\{i, |(Ax)_i| \leq \frac{2}{m^2}\right\} \geq nm^{1/8}\right) &= P\left(\sum_{i=1}^m 1_{|(Ax)_i| \leq 2/m^2} \geq nm^{1/8}\right) \\
&\leq P\left(\sum_{i=1}^m 1_{|(Ax)_i| \leq 2/m^2} \geq m\mathbb{E}\left(1_{|(Ax)_1| \leq 2/m^2}\right) + \left(nm^{1/8} - \frac{4}{m^3}\right)\right) \\
&\leq \exp\left(-\frac{4}{m^3} h\left(\frac{m^{3+1/8}n}{4} - 1\right)\right),
\end{aligned}$$

where h is the function $t \to (1+t)\log(1+t) - t$.

We simplify:

$$\begin{aligned}
P\left(\text{Card}\left\{i, |(Ax)_i| \leq \frac{2}{m^2}\right\} \geq 2n\right) &\leq \exp\left(-nm^{1/8}\log(m^{3+1/8}n/4) + nm^{1/8} - \frac{4}{m^3}\right) \\
&\leq \exp\left(-nm^{1/8}\left(\log(m^{3+1/8}n) - 3\right)\right).
\end{aligned}$$

Finally, as the cardinality of $\mathcal{N}_{n,m}$ is at most $(5\mathcal{M}m^2)^{2n}$,

$$\begin{aligned}
P(\mathcal{E}_1) &\geq 1 - (5\mathcal{M}m^2)^{2n} e^{-nm^{1/8}\left(\log(m^{3+1/8}n) - 3\right)} \\
&= 1 - \exp\left(-nm^{1/8}\left(\log(m^{3+1/8}n) - 3\right) + 2n\log(5\mathcal{M}m^2)\right)\right). \qquad (73)
\end{aligned}$$

Let us now consider $\mathcal{E}_2$. For any i, a_i^* is a random vector with n independent random complex Gaussian coordinates, of variance 1. Gaussian measure concentration results (see for example [Barvinok, 2005, Proposition 2.2]) imply that, for any $\delta > 0$,

$$P\left(||a_i^*|| > \sqrt{n + \delta}\right) \leq \left(1 + \frac{\delta}{n}\right)^n e^{-\delta}.$$

For $\delta = \mathcal{M}^2 - n$, we get

$$P\left(||a_i^*|| > \mathcal{M}\right) \leq \left(\frac{\mathcal{M}^2}{n}\right)^n e^{-\left(\mathcal{M}^2 - n\right)}$$
$$\leq 3\mathcal{M}^{2n} e^{-\mathcal{M}^2}.$$

As a consequence,

$$P(\mathcal{E}_2) \geq 1 - 3m\mathcal{M}^{2n} e^{-\mathcal{M}^2}. \tag{74}$$

We can take, for example, $\mathcal{M} = \sqrt{m}$. We evaluate Equations (73) and (74) for this value of $\mathcal{M}$ and get, when $m \geq n^2$,

$$P(\mathcal{E}_1 \cap \mathcal{E}_2) \geq 1 - C_1 e^{-C_2 m^{1/8}}.$$

□

C.3.2 Proof of Lemma C.12

Lemme (C.12)**.** *There exist $C > 0$ such that, with probability at least*

$$1 - \exp(-nm^{1/4}),$$

for any $I \subset \{1, \dots, m\}$ such that $\mathrm{Card}\, I \leq nm^{1/8}$,

$$\sum_{i \in I} |(Ax_0)_i|^2 \leq C||x_0||^2 nm^{1/4}.$$

Proof of Lemma C.12. By homogeneity, we can assume $||x_0|| = 1$.

The random variables $(Ax_0)_1, \dots, (Ax_0)_m$ are independent and (complex) Gaussian with variance 1. Hence, by Bernstein's inequality for subexponential variables, there exist a constant $c > 0$ such that, for any $t > 0$, and for any fixed $I \subset \{1, \dots, m\}$,

$$P\left(\sum_{i \in I} |(Ax_0)_i|^2 \geq \mathrm{Card}\, I + t\right) \leq \exp\left(-c \min\left(t, \frac{t^2}{\mathrm{Card}\, I}\right)\right).$$

In particular, if $\mathrm{Card}\, I = nm^{1/8}$,

$$P\left(\sum_{i \in I} |(Ax_0)_i|^2 \geq nm^{1/8} + \frac{2}{c} nm^{1/4}\right) \leq \exp\left(-2nm^{1/4} \min\left(1, \frac{2}{c} m^{1/8}\right)\right).$$

So as soon as m is large enough,

$$P\left(\sum_{i \in I} |(Ax_0)_i|^2 \geq \frac{3}{c} nm^{1/4}\right) \leq \exp(-2nm^{1/8}).$$

There are less than $m^{nm^{1/8}} = e^{nm^{1/8}\log m}$ subsets of $\{1,\dots,m\}$ with cardinality $nm^{1/8}$, so

$$P\left(\exists I \text{ s.t. } \operatorname{Card} I \leq nm^{1/8}, \sum_{i\in I} |(Ax_0)_i|^2 \geq \frac{3}{c} nm^{1/4}\right) \leq \exp(-nm^{1/4}).$$

□

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