Non-convex optimization: exercise

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Exercise

This exercise is about phase retrieval problems. During the lecture, we have given the definition of general *complex* phase retrieval problems, where the goal is to recover a vector with *complex* coordinates from the *modulus* of linear measurements. In the exercise, for simplicity, we restrict ourselves to *real* phase retrieval problems:

$$\text{recover } x_* \in \mathbb{R}^n \text{ from } |\langle x_*, v_1\rangle|, \dots, |\langle x_*, v_m\rangle| \ ?$$

Here, $v_1, \dots, v_m$ are known vectors in $\mathbb{R}^n$, "$\langle ., . \rangle$" denotes the usual Euclidean scalar product and "$|.|$" is the absolue value.
Observe that $|\langle x_*, v_k\rangle| = |\langle -x_*, v_k\rangle|$ for any $k = 1, \dots, m$, hence recovery of x_* is at best possible up to sign.

1. We define $y_k = |\langle x_*, v_k\rangle|$ for any $k = 1, \dots, m$ and

$$\begin{array}{rcccc} \mathcal{L} & : & \mathbb{R}^n & \to & \mathbb{R} \\ & & x & \to & \sum_{k=1}^m (\langle x, v_k\rangle^2 - y_k^2)^2. \end{array}$$

   Show that a vector $x \in \mathbb{R}^n$ is a global minimizer of $\mathcal{L}$ if and only if

$$|\langle x, v_k\rangle| = |\langle x_*, v_k\rangle|, \quad \forall k = 1, \dots, m.$$

2. Show that $\mathcal{L}$ is C^∞ and that, for all $x, h \in \mathbb{R}^n$,

$$\nabla \mathcal{L}(x) = 4 \sum_{k=1}^m (\langle x, v_k\rangle^2 - y_k^2) \langle x, v_k\rangle v_k,$$

$$\nabla^2 f(x) \cdot (h, h) = 4 \sum_{k=1}^m (3 \langle x, v_k\rangle^2 - y_k^2) \langle h, v_k\rangle^2 .$$

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3. From now on, we assume that $v_1, \ldots, v_m$ are chosen at random according to independent normal distributions (that is, each coordinate of each v_k is independently chosen according to the law $\mathcal{N}(0,1)$).

   Show that, for any $x, h \in \mathbb{R}^n$,

$$\begin{aligned}
\mathbb{E}(\nabla\mathcal{L}(x)) &= 4m \left(\left(3||x||^2 - ||x_*||^2 \right) x - 2 \langle x_*, x \rangle x_* \right), \\
\mathbb{E}(\nabla^2\mathcal{L}(x) \cdot (h,h)) &= 4m \left(6 \langle x, h \rangle^2 - 2 \langle x_*, h \rangle^2 \right. \\
&\qquad \left. + (3||x||^2 - ||x_*||^2)||h||^2 \right).
\end{aligned}$$

   [Hint: you can admit that, for arbitrary $a, b \in \mathbb{R}^n$ and any k,

$$\begin{aligned}
\mathbb{E}(\langle a, v_k \rangle^2 \langle b, v_k \rangle v_k) &= 2 \langle a, b \rangle a + ||a||^2 b, \\
\mathbb{E}(\langle a, v_k \rangle^2 \langle b, v_k \rangle^2) &= 2 \langle a, b \rangle^2 + ||a||^2 ||b||^2.
\end{aligned}$$

   Do not treat $y_1, \ldots, y_m$ as constants: they depend on $v_1, \ldots, v_m$.]

4. Assuming, for simplicity, $x_* \neq 0$, compute the first and second order stationary points of $\mathbb{E}\mathcal{L}$.

   [Remark : for any $x, h \in \mathbb{R}^n$, it holds $\nabla(\mathbb{E}\mathcal{L})(x) = \mathbb{E}(\nabla\mathcal{L}(x))$ and $\nabla^2(\mathbb{E}\mathcal{L})(x) \cdot (h,h) = \mathbb{E}(\nabla^2\mathcal{L}(x) \cdot (h,h))$.]