Non-convex inverse problems : correction

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Exercice 1

1. First, if x_0 is an eigenvector of M with norm 1, associated to eigenvalue λ_1, it is a feasible point for Problem (Min Eig), with objective value
$$\langle Mx_0, x_0\rangle = \langle \lambda_1 x_0, x_0\rangle = \lambda_1 ||x_0||_2^2 = \lambda_1.$$
Therefore, the optimal value is at most λ_1.
Now, let $x \in \mathbb{R}^n$ be any vector with unit norm. We have
$$\begin{aligned}\langle Mx, x\rangle &= \sum_{i=1}^n \lambda_i x_i^2 \\ &\geq \sum_{i=1}^n \lambda_1 x_i^2 \qquad (1) \\ &= \lambda_1 ||x||_2^2 \\ &= \lambda_1.\end{aligned}$$
This shows that the optimal value is at least λ_1, hence exactly λ_1.
As unit-normed eigenvectors associated with eigenvalue λ_1 reach value λ_1, they are minimizers. Let us show that they are the only minimizers. Let $x \in \mathbb{R}^n$ be a minimizer. Then Inequality (1) must be an equality, that is :
$$\lambda_i x_i^2 = \lambda_1 x_i^2 \text{ for all } i = 1, \dots, n,$$
which is equivalent to
$$x_i = 0 \text{ if } \lambda_i \neq \lambda_1.$$
Therefore,
$$Mx = \begin{pmatrix} \lambda_1 x_1 \\ \lambda_2 x_2 \\ \vdots \\ \lambda_n x_n \end{pmatrix} = \begin{pmatrix} \lambda_1 x_1 \\ \lambda_1 x_2 \\ \vdots \\ \lambda_1 x_n \end{pmatrix} = \lambda_1 x,$$
so x is an eigenvector of M associated to eigenvalue λ_1, and, as it is feasible for (Min Eig), it has norm 1.

2. The objective function is a priori not convex (it is convex if and only if $M \succeq 0$) and the constraint set (the unit sphere) is not convex.

3. Let X_0 be a feasible matrix for Problem (Low Rank). Since it is semidefinite positive with rank 1, it can be written as

$$X_0 = x_0 x_0^*$$

for some $x_0 \in \mathbb{R}^n$. This vector satisfies

$$1 = \mathrm{Tr}(X_0) = \sum_i (x_0)_i^2 = ||x_0||_2^2.$$

Therefore, $||x_0||_2 = 1$ and x_0 is feasible for Problem (Min Eig). It holds

$$\begin{aligned}\mathrm{Tr}(MX_0) &= \mathrm{Tr}(Mx_0x_0^*)\\ &= \langle Mx_0, x_0\rangle\\ &\geq \text{optimal value (Min Eig)}\\ &= \lambda_1.\end{aligned}$$

This shows that the minimal value of (Low Rank) is at least λ_1 and that, if X_0 reaches the objective value λ_1, then X_0 is of the form $X_0 = x_0x_0^*$ for some minimizer $x_0 \in \mathbb{R}^n$ of (Min Eig).
Now, let $x_0 \in \mathbb{R}^n$ be a minimizer of (Min Eig). Let us set $X_0 = x_0x_0^*$. This matrix is feasible for Problem (Low Rank) : we have seen in class that matrices of this form are semidefinite positive and have rank 1. In addition,

$$\mathrm{Tr}(X_0) = \sum_{i=1}^n (X_0)_{ii} = \sum_{i=1}^n (x_0)_i^2 = ||x_0||_2^2 = 1.$$

The objective value associated to X_0 is

$$\mathrm{Tr}(MX_0) = \mathrm{Tr}(Mx_0x_0^*) = \langle Mx_0, x_0\rangle = \lambda_1.$$

Therefore, the optimal value of (Low Rank) is at most λ_1, hence exactly λ_1 and, if x_0 is a minimizer of (Min Eig), $X_0 = x_0x_0^*$ is a minimizer of (Low Rank).

4. a) For any matrix X which is feasible for (Relaxation), it holds $||X||_* = \mathrm{Tr}(X) = 1$, because $X \succeq 0$. Adding a constraint on the nuclear norm would therefore be redundant.

   b) If X_0 is a minimizer of (Low Rank), then X_0 is feasible for (Relaxation) and

   $$\mathrm{Tr}(MX_0) = \lambda_1.$$

As a consequence, the optimal value of (Relaxation) is at most λ_1 and, if we can show that it is exactly λ_1, then we know that minimizers of (Low Rank) are also minimizers of (Relaxation).
Let us show that the optimal value of (Relaxation) is at least λ_1. Let $X \in \text{Sym}_n$ be a feasible matrix for (Relaxation). Then

$$\begin{aligned} \text{Tr}(MX) &= \sum_{i=1}^{n} \lambda_i X_{ii} \\ &\geq \sum_{i=1}^{n} \lambda_1 X_{ii} \qquad (2) \\ &= \lambda_1 \text{Tr}(X) \\ &= \lambda_1. \end{aligned}$$

Equation (2) is true because, for each i, $\lambda_1 \leq \lambda_i$ and $X_{ii} \geq 0$ (since $X \succeq 0$), hence $\lambda_i X_{ii} \geq \lambda_1 X_{ii}$. As a consequence, the optimal value of (Relaxation)is at least λ_1.

5. a) Let $X \in \text{Sym}_n$ be feasible for (Relaxation). Let $H \in \text{Sym}_n, a \in \mathbb{R}$ be such that $H \succeq 0$. Let us show that $\text{Tr}(MX) \geq \mathcal{L}(X, H, a)$.
First, $\text{Tr}(X) = 1$, hence $a(\text{Tr}(X) - 1) = 0$. Second, $X \succeq 0$ and $H \succeq 0$, hence $\text{Tr}(HX) \geq 0$[1] This implies

$$\mathcal{L}(X, H, a) = \text{Tr}(MX) - \text{Tr}(HX) \leq \text{Tr}(MX).$$

b) Let $H \in \text{Sym}_\text{n}, a \in \mathbb{R}$ be fixed. For any $X \in \text{Sym}_n$,

$$\mathcal{L}(X, H, a) = \text{Tr}((M - H + aI_n)X) - a.$$

Therefore, if $M - H + aI_n = 0$, then

$$\inf_{X \in \text{Sym}_n} \mathcal{L}(X, H, a) = \inf_{X \in \text{Sym}_n} -a = -a.$$

If, on the other hand, $M - H + aI_n \neq 0$, then, for any $t \in \mathbb{R}$,

$$\mathcal{L}(t(M - H + aI_n), H, a) = t||M - H + aI_n||_F^2 - a.$$

In particular, for any $t \in \mathbb{R}$,

$$\inf_{X \in \text{Sym}_n} \mathcal{L}(X, H, a) \leq t||M - H + aI_n||_F^2 - a,$$

1. Short proof : if $X, H \succeq 0$, there exist $V, W \in \mathbb{R}^{n \times n}$ such that $H = VV^T, X = WW^T$. Then $\text{Tr}(HX) = \text{Tr}(VV^T WW^T) = \text{Tr}(W^T VV^T W) = ||V^T W||_F^2 \geq 0$.

which, by considering values of t going to $-\infty$, implies that

$$\inf_{X \in \mathrm{Sym}_n} \mathcal{L}(X, H, a) = -\infty.$$

c) We set $H_0 = M - \lambda_1 I_n, a_0 = -\lambda_1$. We indeed have $H_0 = M + a_0 I_n$ and

$$H_0 = \begin{pmatrix} 0 & & & \\ & \lambda_2 - \lambda_1 & & \\ & & \ddots & \\ & & & \lambda_n - \lambda_1 \end{pmatrix},$$

so H_0 is a diagonal matrix with nonnegative entries : it is semidefinite positive. We have shown that (H_0, a_0) is feasible for (Dual). To conclude the question, we simply recall that $\mathrm{Tr}(M x_0 x_0^*) = \langle M x_0, x_0 \rangle = \lambda_1$ since x_0 is a minimizer of (Min Eig), and $\lambda_1 = -a$.

d) We define H_0, a_0 as in the previous question. For any $X \in \mathrm{Sym}_n$ which is feasible for Problem (Relaxation),

$$\begin{aligned} \mathrm{Tr}(MX) &\geq \sup_{\substack{H \in \mathrm{Sym}_n, a \in \mathbb{R} \\ \text{s.t. } H \succeq 0}} \mathcal{L}(X, H, a) \\ &\geq \mathcal{L}(X, H_0, a_0) \\ &\geq \inf_{X \in \mathrm{Sym}_n} \mathcal{L}(X, H_0, a_0) \\ &= -a_0 \\ &= \mathrm{Tr}(M x_0 x_0^*). \end{aligned}$$

Therefore, as $x_0 x_0^*$ is also feasible for Problem (Relaxation), it is a minimizer : no matrix X reaches a lower objective value.

Exercice 2

1. First, we show that the minimum of F is zero. As F is a sum of squares, $\inf_{(\alpha,\theta) \in \mathbb{R}^2} F(\alpha, \theta) \geq 0$. On the other hand,

$$F(1, 0) = \sum_{k=-N}^{N} |1 - 1|^2 = 0.$$

Consequently, $\min_{(\alpha,\theta) \in \mathbb{R}^2} F(\alpha, \theta) = 0$.
Now, let (α, θ) be any global minimizer. We show that $\alpha = 1$ and $\theta \in \mathbb{Z}$. As the minimum is zero, we must have $F(\alpha, \theta) = 0$, hence

$$\alpha e^{-2\pi i k \theta} = \hat{\mu}_*[k] = 1, \quad \forall k = -N, \ldots, N.$$

In particular, for $k = 0$, we get $\alpha = 1$. And for $k = -1$, we have $e^{2\pi i\theta} = 1$, hence θ is an integer.
Conversely, if $\alpha = 1$ and $\theta \in \mathbb{Z}$, then $\alpha e^{-2\pi ik\theta} = 1$ for all $k \in \mathbb{Z}$, hence $F(\alpha, \theta) = 0$, and (α, θ) is a global minimizer.

2. a) It is a *local convergence* result.
   b) For all α, θ,

$$\begin{aligned}
F(\alpha,\theta) &= \sum_{k=-N}^{N} |\alpha e^{-2\pi ik\theta} - 1|^2 \\
&= \sum_{k=-N}^{N} |\alpha(\cos(2\pi k\theta) - i\sin(2\pi k\theta)) - 1|^2 \\
&= \sum_{k=-N}^{N} |\alpha\cos(2\pi k\theta) - 1|^2 + |\alpha\sin(2\pi k\theta)|^2 \\
&= \sum_{k=-N}^{N} \alpha^2\cos^2(2\pi k\theta) - 2\alpha\cos(2\pi k\theta) + 1 + \alpha^2\sin^2(2\pi k\theta) \\
&= \sum_{k=-N}^{N} \alpha^2 - 2\alpha\cos(2\pi k\theta) + 1 \\
&= (2N+1)\alpha^2 - 2\alpha \sum_{k=-N}^{N} \cos(2\pi k\theta) + (2N+1) \\
&= (2N+1)\alpha^2 - 2\alpha D_N(\theta) + (2N+1).
\end{aligned}$$

   c) For all $(\alpha, \theta) \in \mathbb{R}^2$,

$$\frac{\partial F}{\partial \alpha}(\alpha,\theta) = 2(2N+1)\alpha - 2D_N(\theta) = 2\left((2N+1)\alpha - D_N(\theta)\right),$$
$$\frac{\partial F}{\partial \theta}(\alpha,\theta) = -2\alpha D_N'(\theta).$$

   d) For all t,

$$\begin{aligned}
\theta_{t+1} &= \theta_t - \tau\frac{\partial F}{\partial \theta}(\alpha_t,\theta_t) \\
&= \theta_t + 2\tau\alpha_t D_N'(\theta_t).
\end{aligned}$$

   Let us assume that $(\alpha_t, \theta_t) \in]0;2[\times \left]-\frac{1}{2N+1}; \frac{1}{2N+1}\right[$.

If $\theta_t \geq 0$, then $D'_N(\theta_t) \leq 0$. Therefore,

$$\begin{aligned}\theta_t &\geq \theta_{t+1}\\ &\geq \theta_t - 8(2N+1)^3\tau\alpha_t|\theta_t|\\ &\geq \theta_t - 16(2N+1)^3\tau|\theta_t|\\ &\geq \theta_t - 2|\theta_t|\\ &= -\theta_t.\end{aligned}$$

Consequently, $|\theta_{t+1}| \leq |\theta_t|$. The reasoning is almost identical if $\theta_t < 0$.

e) Let us assume that (α_t, θ_t) belongs to $]0;2[\times \left]-\frac{1}{2N+1};\frac{1}{2N+1}\right[$ for some t. Then, from the previous question, θ_{t+1} belongs to $\left]-\frac{1}{2N+1};\frac{1}{2N+1}\right[$. Let us show that $0 < \alpha_{t+1} < 2$. We have

$$\begin{aligned}\alpha_{t+1} &= \alpha_t - \tau\frac{\partial F}{\partial \alpha}(\alpha_t, \theta_t)\\ &= \alpha_t - 2\tau\left((2N+1)\alpha_t - D_N(\theta_t)\right)\\ &= (1 - 2(2N+1)\tau)\alpha_t + 2\tau D_N(\theta_t).\end{aligned}$$

From the condition on τ, $1-2(2N+1)\tau > 0$. Since it also holds $\tau > 0$ and $D_N(\theta_t) > 0$, we have

$$\alpha_{t+1} > 0.$$

On the other hand, $D_N(\theta_t) = \sum_{k=-N}^{N}\cos(2\pi k\theta) \leq \sum_{k=-N}^{N} 1 = 2N+1$, hence

$$\begin{aligned}\alpha_{t+1} &\leq (1 - 2(2N+1)\tau)\alpha_t + 2(2N+1)\tau\\ &\leq (1 - 2(2N+1)\tau) \times 2 + 2(2N+1)\tau\\ &= 2 - 2(2N+1)\tau\\ &< 2.\end{aligned}$$

f) Let us assume that (α_0, θ_0) belongs to $]0;2[\times \left]-\frac{1}{2N+1};\frac{1}{2N+1}\right[$. From the previous question, (α_t, θ_t) belongs to this set for all t. Therefore, from Question d), the sequence $(|\theta_t|)_{t\in\mathbb{N}}$ is non-increasing, so it converges to some limit $\eta < \frac{1}{2N+1}$.

Let us show that $\alpha_t \to \frac{D_N(\eta)}{2N+1}$ when $t \to +\infty$. For any t, it holds

$$\alpha_{t+1} - \frac{D_N(\eta)}{2N+1} = (1 - 2\tau(2N+1))\left(\alpha_t - \frac{D_N(\eta)}{2N+1}\right)$$

$$+ 2\tau \left(D_N(\theta_t) - D_N(\eta) \right).$$

As $D_N(\theta_t) = D_N(|\theta_t|) \overset{t+\infty}{\longrightarrow} D_N(\eta)$, we can say that, for any $\epsilon > 0$, it holds for all t large enough :

$$\left| \alpha_{t+1} - \frac{D_N(\eta)}{2N+1} \right| \leq (1 - 2\tau(2N+1)) \left| \alpha_t - \frac{D_N(\eta)}{2N+1} \right| + \epsilon.$$

Let T be a rank above which this inequality holds true. For any $t \geq T$, we can show iteratively that

$$\left| \alpha_t - \frac{D_N(\eta)}{2N+1} \right| \leq (1 - 2\tau(2N+1))^{t-T} \left| \alpha_T - \frac{D_N(\eta)}{2N+1} \right| + \frac{\epsilon}{2\tau(2N+1)}.$$

Taking the limit $t \to +\infty$ yields

$$\text{limsup}_{t \to +\infty} \left| \alpha_t - \frac{D_N(\eta)}{2N+1} \right| \leq \frac{\epsilon}{2\tau(2N+1)}.$$

This is true for every ϵ, so

$$\alpha_t \overset{t \to +\infty}{\longrightarrow} \frac{D_N(\eta)}{2N+1}.$$

We have seen in Question d) that, for all t,

$$\theta_{t+1} = \theta_t + 2\alpha_t \tau D'_N(\theta_t),$$

and $\theta_t, D'_N(\theta_t)$ have opposite signs, hence

$$\begin{aligned} |\theta_{t+1}| &= \left| |\theta_t| - 2\alpha_t \tau |D'_N(\theta_t)| \right| \\ &= \left| |\theta_t| - 2\alpha_t \tau |D'_N(|\theta_t|)| \right|. \end{aligned}$$

As $|\theta_t| \to \eta$ and $\alpha_t \to \frac{D_N(\eta)}{2N+1}$ when $t \to +\infty$, we must have

$$\left| \eta - 2\tau \frac{D_N(\eta)}{2N+1} |D'_N(\eta)| \right| = \eta.$$

Therefore, either

$$\eta - 2\tau \frac{D_N(\eta)}{2N+1} |D'_N(\eta)| = -\eta$$

or

$$\eta - 2\tau \frac{D_N(\eta)}{2N+1} |D'_N(\eta)| = \eta.$$

If $\eta = 0$, the two equalities are identical. Otherwise, the first equality is not possible :

$$\begin{aligned} 2\tau \frac{D_N(\eta)}{2N+1} |D'_N(\eta)| &\leq 8\tau(2N+1)^3 \eta \frac{D_N(\eta)}{2N+1} \\ &\leq 8\tau(2N+1)^3 \eta \\ &\leq \eta \\ &< 2\eta. \end{aligned}$$

Therefore, the second inequality is necessarily true, which means that

$$2\tau \frac{D_N(\eta)}{2N+1} |D'_N(\eta)| = 0.$$

As τ and $D_N(\eta)$ are positive, we must have $D'_N(\eta) = 0$, implying $\eta = 0$. Therefore, $|\theta_t| \to 0$ when $t \to +\infty$, so $\theta_t \to 0$. And $\alpha_t \to \frac{D_N(\eta)}{2N+1} = 1$.