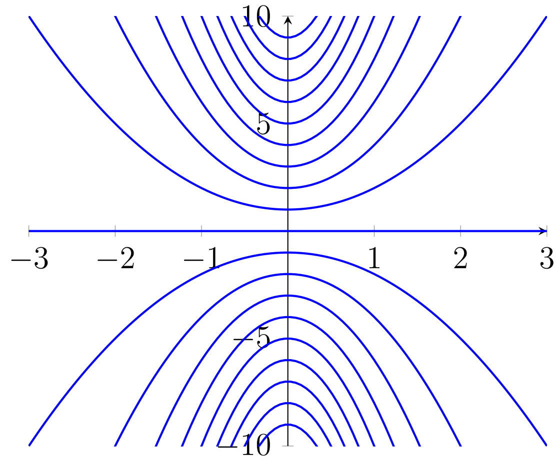
Differential geometry - homework

1.

- (a) represents a submanifold of $\mathbb{R}^2$.
- (b) does not represent a submanifold of $\mathbb{R}^2$: the definition is not valid in the neighborhood of $(0, 0)$.
- (c) does not represent a submanifold of $\mathbb{R}^2$: the definition is not valid in the neighborhood of $(-1, 0)$ and $(0, 1)$.
- (d) does not represent a submanifold of $\mathbb{R}^2$: the definition for dimension 1 is not valid in the neighborhood of $(0, 0)$, while the definition for dimension 0 is not valid in the neighborhood of any other point.
- (e) represents a submanifold of $\mathbb{R}^3$.
- (f) does not represent a submanifold of $\mathbb{R}^3$: the definition is not valid in the neighborhood of the points which belong to the lower boundary, $\{(x, y, 0), x^2 + y^2 = 1\}$.
- (g) represents a submanifold of $\mathbb{R}^3$.

2. (a) Let us define

$$M = \{(x, n(x^2 + 1)), x \in \mathbb{R}, n \in \mathbb{Z}\}.$$



We show that it is a submanifold of $\mathbb{R}^2$, of class C^∞ and dimension 1, using the graph definition.

Let $(x_0, n_0(x_0^2+1))$ be any point of M. We show that there exist U a neighborhood of $(x_0, n_0(x_0^2+1))$, V an open set of $\mathbb{R}$, and $h : V \to \mathbb{R}$ a C^∞ function such that

$$M \cap U = \{(x, h(x)), x \in V\}. \tag{1}$$

Many possibilities exist. Let us for instance define

$$U = \{(x, y), n_0(x^2+1) - 1 < y < n_0(x^2+1) + 1\}.$$

It is an open set, which contains $(x_0, n_0(x_0^2+1))$.

We set $V = \mathbb{R}$ and

$$\begin{array}{rccc} h: & \mathbb{R} & \to & \mathbb{R} \\ & x & \to & n_0(x^2+1). \end{array}$$

As h is polynomial, it is C^∞. It thus suffices to show Equation (1) to conclude.

For any $x \in V$, $(x, h(x)) = (x, n_0(x^2+1)) \in M$ and $(x, h(x)) \in U$, since $n_0(x^2+1) - 1 < n_0(x^2+1) < n_0(x^2+1) + 1$. This already shows $\{(x, h(x)), x \in V\} \subset M \cap U$.

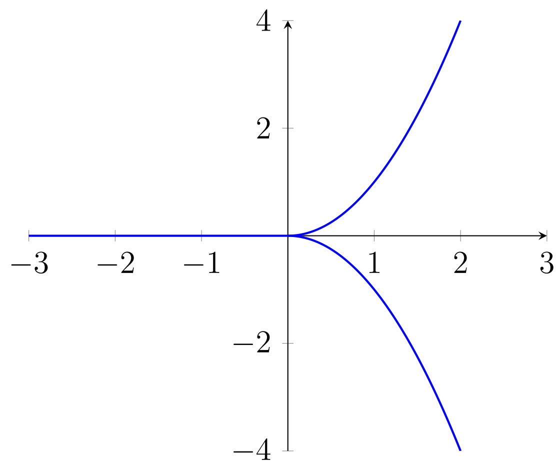
On the other hand, for any $(x_1, n_1(x_1^2+1)) \in M \cap U$, it holds $n_0(x_1^2+1) - 1 < n_1(x_1^2+1) < n_0(x_1^2+1) + 1$, hence

$$n_0 - 1 \leq n_0 - \frac{1}{x_1^2+1} < n_1 < n_0 + \frac{1}{x_1^2+1} \leq n_0 + 1.$$

As n_0, n_1 are integers, we must have $n_0 = n_1$. In particular, $(x_1, n_1(x_1^2+1)) = (x_1, h(x_1)) \in \{(x, h(x)), x \in V\}$. This shows $M \cap U \subset \{(x, h(x)), x \in V\}$, which concludes the proof of Equation (1).

(b) We define

$$M = \{(x, 0), x \in \mathbb{R}^-\} \cup \{(x, x^2), x \in \mathbb{R}^+\} \cup \{(x, -x^2), x \in \mathbb{R}^-\}.$$



We show that M is not a C^1-submanifold of $\mathbb{R}^2$. Intuitively, the reason is that it has a "trifurcation" at $(0, 0)$. To make this formal, we proceed by contradiction, and assume that M is a submanifold.

It must have dimension 1, since it does not have dimension 0 (submanifolds of dimension 0 are discrete sets of points) nor dimension 2 (submanifolds of $\mathbb{R}^2$ with dimension 2 are open subsets of $\mathbb{R}^2$). Therefore, from the "submersion" definition applied at point $(0, 0)$, there exist

- U a neighborhood of $(0, 0)$ in $\mathbb{R}^2$,
- $g : U \to \mathbb{R}$ a C^1 function, submersive at $(0, 0)$,

such that $M \cap U = g^{-1}(\{0\})$.

We fix U, g as above. As $\mathrm{Im}(dg(0, 0))$ is a subspace of $\mathbb{R}$, it is $\mathbb{R}$ itself if and only if $dg(0, 0) \neq 0$: the condition that g is a submersion at $(0, 0)$ simply means that $dg(0, 0) \neq 0$.

For any $x \leq 0$ close enough to 0, $(x, 0)$ belongs to $M \cap U$, hence $g(x, 0) = 0$. We differentiate this equality and deduce

$$\frac{\partial g}{\partial x}(0, 0) = 0. \tag{2}$$

Also, for any $x > 0$ close enough to 0, $g(x, -x^2) = g(x, x^2) = 0$. From Rolle's lemma applied to the function $y \to g(x, y)$, there exists $y_x \in]-x^2; x^2[$ such that

$$\frac{\partial g}{\partial y}(x, y_x) = 0.$$

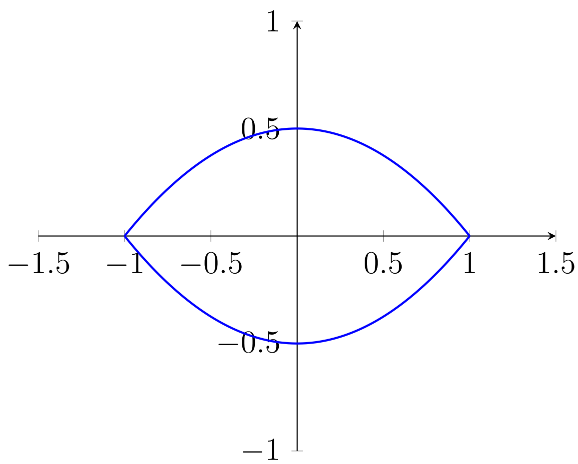
When $x \to 0$, $y_x \to 0$. Therefore, since $\frac{\partial g}{\partial y}$ is continuous,

$$\frac{\partial g}{\partial y}(0, 0) = 0. \tag{3}$$

Together, Equations (2) and (3) imply that $dg(0, 0) = 0$, which contradicts the fact that g is a submersion at $(0, 0)$.

(c) We define

$$M = \left\{ \left(x, \frac{1 - x^2}{2} \right), x \in [-1; 1] \right\} \cup \left\{ \left(x, \frac{x^2 - 1}{2} \right), x \in [-1; 1] \right\}.$$



We show that M is not a C^1-submanifold of $\mathbb{R}^2$. Intuitively, the reason is that it has "non-regular" points at $(-1, 0)$ and $(1, 0)$. To make it formal, let us proceed by contradiction, and assume that it is a C^1-submanifold.

This submanifold neither has dimension 0 (it would be a discrete set of points) nor 2 (it would be an open set of $\mathbb{R}^2$). Therefore, its dimension is 1.

From the "submersion" definition of submanifolds applied at $(1, 0)$, there exist

- U a neighborhood of $(1,0)$ in $\mathbb{R}^2$,
- $g : U \to \mathbb{R}$ a C^1 function, submersive at $(1,0)$,

such that $M \cap U = g^{-1}(\{0\})$. Let such U, g be fixed.

For any $x \in [-1;1]$ close enough to 1,

$$g\left(x, \frac{1-x^2}{2}\right) = 0.$$

We differentiate this equality at $x = 1$, on the left:

$$\frac{\partial g}{\partial x}(1,0) - \frac{\partial g}{\partial y}(1,0) = 0.$$

In the same way, for any $x \in [-1;1]$ close enough to 1,

$$g\left(x, \frac{x^2-1}{2}\right) = 0.$$

We differentiate this equality at $x = 1$, on the left:

$$\frac{\partial g}{\partial x}(1,0) + \frac{\partial g}{\partial y}(1,0) = 0.$$

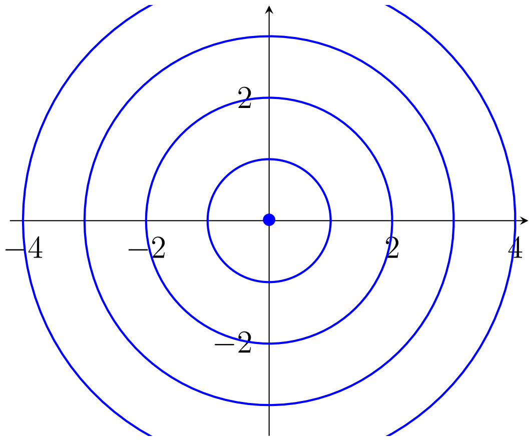
Combining the two equalities, we reach

$$\frac{\partial g}{\partial x}(1,0) = \frac{\partial g}{\partial y}(1,0) = 0,$$

which means that $dg(1,0) = 0$ and contradicts the fact that g is submersive at $(1,0)$.

(d) We define

$$M = \left\{(x,y) \in \mathbb{R}^2, \sqrt{x^2+y^2} \in \mathbb{N}\right\}.$$



We show that M is not a C^1-submanifold of $\mathbb{R}^2$. Intuitively, the reason is that it is a submanifold of dimension 1 in the neighborhood of any point different from $(0,0)$, but a submanifold of dimension 0 in the neighbordhood of $(0,0)$.

By contradiction, let us assume that M is a submanifold of $\mathbb{R}^2$. It cannot have dimension 0 (it would be a discrete set of points) or 2 (it would be an open subset of $\mathbb{R}^2$). Therefore, it has dimension 1.

In particular, using the "diffeomorphism" definition of submanifolds, we know that there exist U, V two neighborhoods of 0 and $\phi : U \to V$ a C^1-diffeomorphism such that

$$\phi(M \cap U) = (\mathbb{R} \times \{0\}) \cap V.$$

If we replace U and V by smaller neighborhoods, we can assume that $M \cap U$ is a singleton:

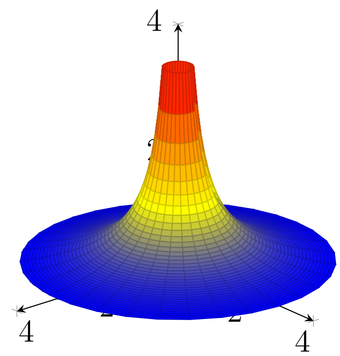
$$M \cap U = \{(0,0)\}.$$

Therefore, $(\mathbb{R} \times \{0\}) \cap V = \phi(\{(0,0)\})$ is also a singleton. This is impossible, since this set contains $(t,0)$ for all t close enough to 0.

(e) We set

$$M = \left\{ \left(x, y, \frac{1}{\sqrt{x^2+y^2}} \right), (x,y) \in \mathbb{R}^2 \setminus \{(0,0)\} \right\}.$$

Let us show that it is a submanifold of $\mathbb{R}^3$, of class C^∞ and dimension 2. We use the definition by graph.



We define $U = \mathbb{R}^3, V = \mathbb{R}^2 \setminus \{(0,0)\}$, and

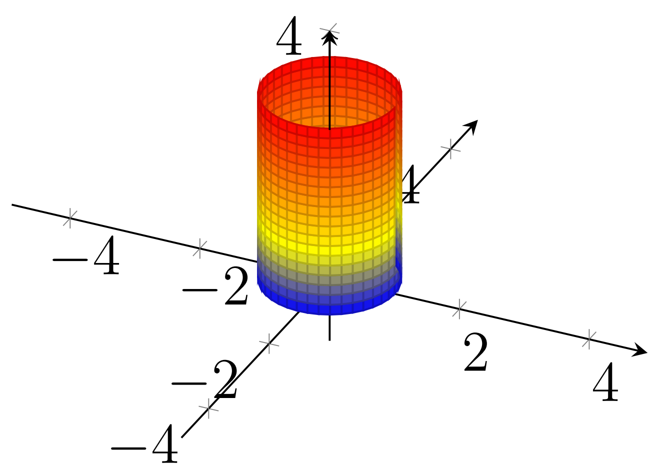
$$h : \begin{array}{ccc} V & \to & \mathbb{R} \\ (x,y) & \to & \frac{1}{\sqrt{x^2+y^2}}. \end{array}$$

The map h is C^∞ (as $\sqrt{.}$ is C^∞ on $\mathbb{R}_+^*$ and x^2+y^2 belongs to $\mathbb{R}_+^*$ for all $(x,y) \in V$).

In addition, $M \cap U = M = \{(x,y,h(x,y)), (x,y) \in V\}$.

(f) We define

$$M = \{(x,y,z) \in \mathbb{R}^3 \text{ s.t. } x^2+y^2 = 1, z \geq 0\}.$$



This set is a cylinder, extending to infinity in one direction, but bounded in the other direction. It is not a C^1-submanifold of $\mathbb{R}^3$. The reason is that it contains its boundary, $\{(x,y,0) \text{ s.t. } x^2+y^2 = 1\}$.

Let us prove rigorously that it is not a C^1-submanifold. By contradiction, let us assume that it is.

Proposition. *The submanifold M has dimension* 2.

Proof. Indeed, let us consider $U = \{(x, y, z) \in \mathbb{R}^3 \text{ s.t. } x > 0 \text{ and } z > 0\}$. It is an open set of $\mathbb{R}^3$. Therefore, $M \cap U$ is a submanifold of $\mathbb{R}^3$, with the same dimension as M (since, close to any point of $M \cap U$, the two sets coincide; therefore, if M satisfies the definition of a submanifold, for some dimension d, then $M \cap U$ also does).

We observe that $M \cap U = f(V)$, where $V = \left]-\frac{\pi}{2}; \frac{\pi}{2}\right[\times \mathbb{R}_+^* \subset \mathbb{R}^2$ and f is defined as

$$\begin{array}{rccc} f: & \left]-\frac{\pi}{2}; \frac{\pi}{2}\right[\times \mathbb{R}_+^* & \to & \mathbb{R}^3 \\ & (\theta, r) & \to & (\cos(\theta), \sin(\theta), r). \end{array}$$

The function f is an immersion on V : for each $(\theta, r) \in \left]-\frac{\pi}{2}; \frac{\pi}{2}\right[\times \mathbb{R}_+^*$,

$$df(\theta, r) = \left((h_1, h_2) \in \mathbb{R}^2 \to (-\sin(\theta)h_1, \cos(\theta)h_1, h_2) \in \mathbb{R}^3\right),$$

which is an injective linear map (since $\cos(\theta) > 0$). In addition, f is a homeomorphism from V to $f(V)$. Indeed, it is injective: for any $\theta_1, r_1, \theta_2, r_2$, if $f(\theta_1, r_1) = f(\theta_2, r_2)$, then $\sin(\theta_1) = \sin(\theta_2)$, hence $\theta_1 = \theta_2$, as sin is injective on $\left]-\frac{\pi}{2}; \frac{\pi}{2}\right[$; looking at the third coordinate, $r_1 = r_2$. Therefore, f is a bijection from V to $f(V)$. It is continuous. Its inverse is

$$\begin{array}{rccc} f^{-1}: & f(V) & \to & V \\ & (x, y, z) & \to & (\arcsin(y), z), \end{array}$$

which is also continuous.

This means that, from the "immersion" definition of submanifolds, $M \cap U$ is a submanifold of $\mathbb{R}^3$ with dimension 2. This proves that M also has dimension 2. □

To obtain a contradiction, we apply the "immersion" definition at point $(1, 0, 0)$. Since M is a submanifold of dimension 2, there exist U an open neighborhood of $(0, 0)$ in $\mathbb{R}^2$, V an open neighborhood of $(1, 0, 0)$ in $\mathbb{R}^3$ and $f : U \to V$ a C^1 map such that $f(U) = M \cap V$, f is a homeomorphism between U and $f(U)$, and f is an immersion at $f^{-1}(1, 0, 0)$. Let such U, V, f be fixed.

We denote $a = f^{-1}(1,0,0) \in U$. Let us show that $df(a)(\mathbb{R}^2) \subset \{0\} \times \mathbb{R} \times \{0\}$. Let $h \in \mathbb{R}^2$ be any vector; we define

$$(y_1, y_2, y_3) = df(a)(h).$$

It is impossible that $y_3 \neq 0$: otherwise, for $t \in \mathbb{R}$ small enough, $f(a + th)_3 = f(a)_3 + t df(a)(h)_3 + o(t) = ty_3 + o(t)$ can have negative values, which contradicts the fact that $f(a+th) \in M \subset \mathbb{R} \times \mathbb{R} \times \mathbb{R}^+$ for any t. Therefore, $y_3 = 0$.

In addition, for all $t \in \mathbb{R}$ close to 0,

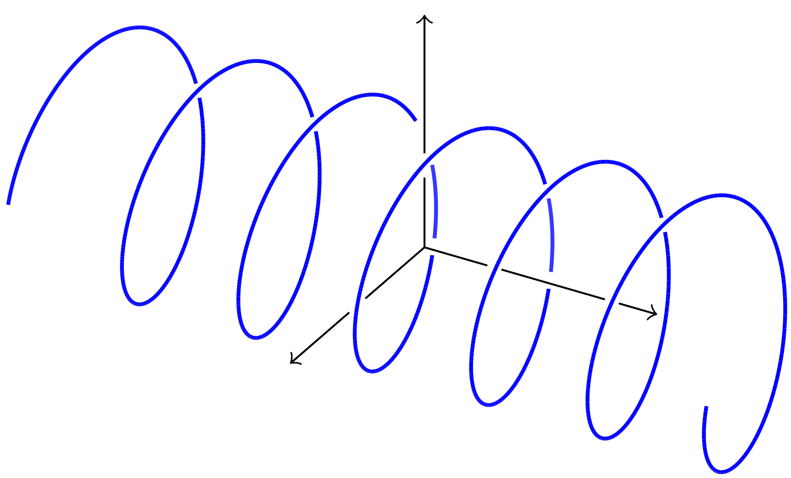
$$\begin{aligned}
1 &= f(a+th)_1^2 + f(a+th)_2^2 \\
&= f(a)_1^2 + f(a)_2^2 + 2tf(a)_1 df(a)(h)_1 + 2tf(a)_2 df(a)(h)_2 + o(t) \\
&= 1 + 2ty_1 + o(t),
\end{aligned}$$

so that $y_1 = 0$. This concludes the proof that $df(a)(\mathbb{R}^2) \subset \{0\} \times \mathbb{R} \times \{0\}$.

As a consequence, $df(a)(\mathbb{R}^2)$, which is a vector space of dimension 2 (because $df(a)$ is injective) is included in a vector space of dimension 1. This is a contradiction.

(g) We define

$$M = \{(x, \cos(2\pi x), \sin(2\pi x)), x \in \mathbb{R}\}.$$



This set is the graph of $(x \in \mathbb{R} \to (\cos(2\pi x), \sin(2\pi x)) \in \mathbb{R}^2)$, which is a C^∞ map. It is therefore a submanifold of $\mathbb{R}^3$ with dimension 1.