Homework : Cauchy-Lipschitz

Before starting, you should read the reminders on the last page.

Let $d \in \mathbb{N}^*$. Let $I \subset \mathbb{R}$ be an open interval, $U \subset \mathbb{R}^d$ an open set, and $f : I \times U \to \mathbb{R}^d$ a function. Let $t_0 \in I, u_0 \in U$.
We consider the following Cauchy problem :

$$\begin{cases} u'(t) & = & f(t, u(t)), \\ u(t_0) & = & u_0. \end{cases} \tag{Cauchy}$$

We will prove the Cauchy-Lipschitz theorem. We assume that

— f is continuous ;

— there exist neighborhoods $H_I \subset I$ and $H_U \subset U$ of t_0 and u_0, and $C > 0$ such that

$$||f(t, v) - f(t, v')||_2 \leq C||v - v'||_2, \quad \forall t \in H_I, \forall v, v' \in H_U.$$

We fix such H_I, H_U and C for the whole homework.

1. We first prove the local uniqueness of the solution to (Cauchy). Let $u_1 : J_1 \to U$ and $u_2 : J_2 \to U$ be two solutions.
   a) Show that there exists $\epsilon > 0$ such that for all $t \in [t_0; t_0 + \epsilon] \cap J_1 \cap J_2$, $||u_1'(t) - u_2'(t)||_2 \leq C||u_1(t) - u_2(t)||_2$.
   b) For this ϵ, show that for all $t \in [t_0; t_0 + \epsilon] \cap J_1 \cap J_2$,
   $$||u_1(t) - u_2(t)||_2 \leq C \int_{t_0}^{t} ||u_1(s) - u_2(s)||_2 ds.$$
   [Hint : Use the equality $u_1(t) - u_2(t) = u_1(t_0) - u_2(t_0) + \int_{t_0}^{t} (u_1'(s) - u_2'(s))ds$, as well as the triangle inequality for integrals.]
   c) Using Gronwall's lemma, show that $u_1 = u_2$ on $[t_0; t_0 + \epsilon] \cap J_1 \cap J_2$.
   d) Show that there exists $\epsilon' > 0$ such that $u_1 = u_2$ on $[t_0 - \epsilon'; t_0 + \epsilon'] \cap J_1 \cap J_2$.
2. Show that there exists a neighborhood $H_I' \subset H_I$ of t_0, a neighborhood $H_U' \subset H_U$ of u_0, and $M > 0$ such that $||f(t, v)||_2 \leq M$ for all $t \in H_I', v \in H_U'$.
3. Let's prove existence. Let $\epsilon > 0$ be such that $[t_0 - \epsilon; t_0 + \epsilon] \subset H_I'$ and $\bar{B}(u_0, M\epsilon) \subset H_U'$. For all $n \in \mathbb{N}^*$, we define $u_n : [t_0 - \epsilon; t_0 + \epsilon] \to U$ as follows.
   — We set $u_n(t_0) = u_0$.

— For all $k \in \{0, \dots, n-1\}$, if u_n is defined on $\left[t_0 - \frac{k}{n}\epsilon; t_0 + \frac{k}{n}\epsilon\right]$, we set, for all $t \in \left]t_0 + \frac{k}{n}\epsilon; t_0 + \frac{k+1}{n}\epsilon\right]$,

$$u_n(t) = u_n\left(t_0 + \frac{k}{n}\epsilon\right) + \int_{t_0+\frac{k}{n}\epsilon}^{t} f\left(s, u_n\left(t_0 + \frac{k}{n}\epsilon\right)\right) ds$$

and, for all $t \in \left[t_0 - \frac{k+1}{n}\epsilon; t_0 - \frac{k}{n}\epsilon\right[$,

$$u_n(t) = u_n\left(t_0 - \frac{k}{n}\epsilon\right) + \int_{t_0-\frac{k}{n}\epsilon}^{t} f\left(s, u_n\left(t_0 - \frac{k}{n}\epsilon\right)\right) ds.$$

a) For all n, show that u_n is well-defined, M-Lipschitz, piecewise C^1, and takes values in $\bar{B}(u_0, M\epsilon)$.

b) Show that for all $t \in [t_0 - \epsilon; t_0 + \epsilon]$ except for a finite number of values, u_n is differentiable at t and

$$||u_n'(t) - f(t, u_n(t))||_2 \leq \frac{CM\epsilon}{n}.$$

c) Deduce that for all $n_1, n_2 \in \mathbb{N}^*$ and for all $t \in [t_0 - \epsilon; t_0 + \epsilon]$ except for a finite number of values,

$$||u_{n_1}'(t) - u_{n_2}'(t)||_2 \leq CM\epsilon\left(\frac{1}{n_1} + \frac{1}{n_2}\right) + C||u_{n_1}(t) - u_{n_2}(t)||_2.$$

d) Show that for all $n_1, n_2 \in \mathbb{N}^*$ and $t \in [t_0 - \epsilon; t_0 + \epsilon]$,

$$||u_{n_1}(t) - u_{n_2}(t)||_2 \leq M\epsilon\left(\frac{1}{n_1} + \frac{1}{n_2}\right)\left(e^{C|t-t_0|} - 1\right).$$

[Hint : Use Gronwall, reasoning as in Question 1.]

e) Show that $(u_n)_{n\in\mathbb{N}^*}$ is a Cauchy sequence in $C^0\left([t_0 - \epsilon; t_0 + \epsilon], \bar{B}(u_0, M\epsilon)\right)$ equipped with the uniform distance.

f) Deduce that $(u_n)_{n\in\mathbb{N}^*}$ converges uniformly to some map u_∞, which belongs to $C^0\left([t_0 - \epsilon; t_0 + \epsilon], \bar{B}(u_0, M\epsilon)\right)$.

g) Show that for all $n \in \mathbb{N}^*$ and all $t \in [t_0 - \epsilon; t_0 + \epsilon]$,

$$\left|\left|u_n(t) - u_n(t_0) - \int_{t_0}^{t} f(s, u_n(s))ds\right|\right|_2 \leq \frac{CM\epsilon^2}{n}.$$

h) Deduce that for all $t \in [t_0 - \epsilon; t_0 + \epsilon]$,

$$u_\infty(t) = u_\infty(t_0) + \int_{t_0}^{t} f(s, u_\infty(s))ds,$$

then that u_∞ is a solution of (Cauchy).

Gronwall's lemma
Let $t_0, T \in \mathbb{R}$, with $t_0 \leq T$. Let $a, c, u \in C^0([t_0;T], \mathbb{R})$ be such that $a \geq 0$ and

$$u(t) \leq c(t) + \int_{t_0}^{t} a(s)u(s)ds, \quad \forall t \in [t_0;T].$$

Then, for all $t \in [t_0;T]$,

$$u(t) \leq c(t) + \int_{t_0}^{t} e^{\int_s^t a(\tau)d\tau} a(s)c(s)ds.$$

The lemma is also true if $T < t_0$, provided the segment "$[t_0;T]$" is replaced by "$[T;t_0]$" and the bounds are exchanged in each integral.

Cauchy sequence
A sequence $(x_n)_{n\in\mathbb{N}}$ in a metric space (X,d) is a *Cauchy sequence* if

$$\sup_{m\geq n} d(x_n, x_m) \overset{n\to+\infty}{\longrightarrow} 0.$$

Complete spaces
A metric space (X,d) is *complete* if every Cauchy sequence in X is convergent.
Important examples of complete spaces are :

- all compact metric spaces,
- the set $C_b^0(X,Y)$ of bounded continuous functions between an arbitrary metric space (X, d_X) and a complete metric space (Y, d_Y), equipped with the sup norm d_{sup} :

$$d_{sup}(f,g) = \sup_{x\in X} d_Y(f(x), g(x)), \quad \forall f,g \in C_b^0(X,Y).$$