Devoir : le théorème de Cauchy-Lipschitz

Corrigé

1.a) Let $\epsilon > 0$ be such that $[t_0; t_0 + \epsilon] \subset H_I$ and
— for all $t \in [t_0; t_0 + \epsilon] \cap J_1$, $u_1(t) \in H_U$;
— for all $t \in [t_0; t_0 + \epsilon] \cap J_2$, $u_2(t) \in H_U$.
Such ϵ exists because H_U is a neighborhood of $u_1(t_0) = u_2(t_0) = u_0$ and u_1, u_2 are continuous.
For all $t \in [t_0; t_0 + \epsilon] \cap J_1 \cap J_2$,

$$\begin{aligned} ||u_1'(t) - u_2'(t)||_2 &= ||f(t, u_1(t)) - f(t, u_2(t))||_2 \\ &\leq C||u_1(t) - u_2(t)||_2. \end{aligned}$$

For the inequality, we used the fact that $t \in H_I$ and $u_1(t), u_2(t) \in H_U$.

b) Let $t \in [t_0; t_0 + \epsilon] \cap J_1 \cap J_2$ be arbitrary. Notice that, since $[t_0; t_0 + \epsilon]$, J_1, and J_2 are intervals, $[t_0; t] \subset [t_0; t_0 + \epsilon] \cap J_1 \cap J_2$.
The fundamental theorem of calculus and the triangle inequality for integrals allow us to write

$$\begin{aligned} &||u_1(t) - u_2(t)||_2 \\ &\qquad = \left|\left| u_1(t_0) - u_2(t_0) + \int_{t_0}^{t} (u_1'(s) - u_2'(s)) ds \right|\right|_2 \\ &\qquad = \left|\left| \int_{t_0}^{t} (u_1'(s) - u_2'(s)) ds \right|\right|_2 \quad \text{(since } u_1(t_0) = u_2(t_0) = u_0\text{)} \\ &\qquad \leq \int_{t_0}^{t} ||u_1'(s) - u_2'(s)||_2 ds. \end{aligned}$$

For all $s \in [t_0; t]$, since $s \in [t_0; t_0 + \epsilon] \cap J_1 \cap J_2$, we can use the previous question to say that $||u_1'(s) - u_2'(s)||_2 \leq C||u_1(s) - u_2(s)||_2$. Consequently,

$$||u_1(t) - u_2(t)||_2 \leq C \int_{t_0}^{t} ||u_1(s) - u_2(s)||_2 ds.$$

c) Let $\phi : t \in [t_0; t_0+\epsilon] \cap J_1 \cap J_2 \to ||u_1(t) - u_2(t)||_2$. According to the previous question, we have, for all t,

$$\phi(t) \leq C \int_{t_0}^{t} \phi(s) ds,$$

which means that ϕ satisfies the hypothesis of Gronwall's lemma, where c is the zero function and a is the constant function with value C. Therefore, by the lemma, for all $t \in [t_0; t_0+\epsilon] \cap J_1 \cap J_2$,

$$\phi(t) \leq 0 + \int_{t_0}^{t} e^{\int_s^t C d\tau} C \times 0 ds = 0.$$

Thus, ϕ is zero, which implies that $u_1 - u_2$ is zero (i.e., $u_1 = u_2$) on $[t_0; t_0 + \epsilon] \cap J_1 \cap J_2$.

d) Similarly to Question a), we can show that there exists $\tilde{\epsilon} > 0$ such that, for all $t \in [t_0 - \tilde{\epsilon}; t_0] \cap J_1 \cap J_2$, $||u_1'(t) - u_2'(t)||_2 \leq C||u_1(t) - u_2(t)||_2$. With a reasoning similar to Question b), we deduce that, for all $t \in [t_0 - \tilde{\epsilon}; t_0] \cap J_1 \cap J_2$,

$$||u_1(t) - u_2(t)||_2 \leq C \int_{t}^{t_0} ||u_1(s) - u_2(s)||_2 ds.$$

We can then apply Gronwall's lemma, which implies that $u_1 = u_2$ on $[t_0 - \tilde{\epsilon}; t_0] \cap J_1 \cap J_2$.
By setting $\epsilon' = \min(\epsilon, \tilde{\epsilon})$, we have the desired result.

2. Let $\eta > 0$ such that $[t_0 - \eta; t_0 + \eta] \subset H_I$ and $\bar{B}(u_0, \eta) \subset H_U$. The map $||f||$ is continuous on $[t_0 - \eta; t_0 + \eta] \times \bar{B}(u_0, \eta)$, which is a compact set. Therefore, it is bounded. Let M be an upper bound and define $H_I' = [t_0 - \eta; t_0 + \eta]$ and $H_U' = \bar{B}(u_0, \eta)$.

3.a) Let $n \in \mathbb{N}^*$. We will prove by induction on k that, for all $k = 0, \ldots, n$, u_n is well-defined, M-Lipschitz and piecewise C^1 with values in $\bar{B}(u_0, M\epsilon)$ on $\left[t_0 - \frac{k}{n}\epsilon; t_0 + \frac{k}{n}\epsilon\right]$.
For $k = 0$, it is true : u_0 is a fixed element of U so the definition "$u_n(t_0) = u_0$" is valid. Moreover, any function defined on a singleton set is M-Lipschitz and piecewise C^1 ; we also have $u_0 \in \bar{B}(u_0, M\epsilon)$.
Let us assume the property is true for some $k \in \{0, \ldots, n-1\}$ and prove it for $k + 1$.
By the induction hypothesis, u_n is well-defined on $\left[t_0 - \frac{k}{n}\epsilon; t_0 + \frac{k}{n}\epsilon\right]$. Let's show that it is also well-defined on $\left]t_0 + \frac{k}{n}\epsilon; t_0 + \frac{k+1}{n}\epsilon\right]$. A similar reasoning would show that it is well-defined on $\left[t_0 - \frac{k+1}{n}\epsilon; t_0 - \frac{k}{n}\epsilon\right[$.

According to the induction hypothesis, $u_n\left(t_0+\frac{k}{n}\epsilon\right)\in\bar{B}(u_0,M\epsilon)\subset H'_U\subset U$. Furthermore, $\left[t_0+\frac{k}{n}\epsilon;t_0+\frac{k+1}{n}\epsilon\right]\subset[t_0-\epsilon;t_0+\epsilon]\subset I$. So the function

$$s\in\left[t_0+\frac{k}{n}\epsilon;t_0+\frac{k+1}{n}\epsilon\right]\quad\rightarrow\quad f\left(s,u_n\left(t_0+\frac{k}{n}\epsilon\right)\right)\in U$$

is well-defined. Moreover, it is continuous (since f is continuous). Consequently, the definition

$$u_n(t)=u_n\left(t_0+\frac{k}{n}\epsilon\right)+\int_{t_0+\frac{k}{n}\epsilon}^{t}f\left(s,u_n\left(t_0+\frac{k}{n}\epsilon\right)\right)ds$$

is valid for all $t\in\left]t_0+\frac{k}{n}\epsilon;t_0+\frac{k+1}{n}\epsilon\right]$. Thus, we have shown that u_n is well-defined on $\left]t_0+\frac{k}{n}\epsilon;t_0+\frac{k+1}{n}\epsilon\right]$.

Now let's prove that u_n is M-Lipschitz, piecewise C^1, and with values in $\bar{B}(u_0,M\epsilon)$ on $\left[t_0-\frac{k+1}{n}\epsilon;t_0+\frac{k+1}{n}\epsilon\right]$.

It is piecewise C^1 because it is defined, piecewise, as the integral of a continuous function. Furthermore, it is continuous. Indeed,

— it is continuous (since it is M-Lipschitz) on $\left[t_0-\frac{k}{n}\epsilon;t_0+\frac{k}{n}\epsilon\right]$;

— it is continuous at $t_0+\frac{k}{n}\epsilon$: its right limit is $u_n\left(t_0+\frac{k}{n}\epsilon\right)$ according to the properties of the integral, and its left limit is the same (due to the continuity of u_n on $\left[t_0-\frac{k}{n}\epsilon;t_0+\frac{k}{n}\epsilon\right]$);

— it is continuous at $t_0-\frac{k}{n}\epsilon$ for the same reason;

— it is continuous on $\left[t_0-\frac{k+1}{n}\epsilon;t_0-\frac{k}{n}\epsilon\right[$ and $\left]t_0+\frac{k}{n}\epsilon;t_0+\frac{k+1}{n}\epsilon\right]$ as the integral of a continuous function.

Moreover, at any point of $\left[t_0-\frac{k+1}{n}\epsilon;t_0+\frac{k+1}{n}\epsilon\right]$ where u_n is differentiable, its derivative is of the form

$$f\left(t,u_n\left(t_0\pm\frac{k'}{n}\right)\right)$$

for some $t\in\left[t_0-\frac{k+1}{n}\epsilon;t_0+\frac{k+1}{n}\epsilon\right]$ and some $k'\leq k$. We have already seen that, for such values of t and k',

$$\left(t,u_n\left(t_0\pm\frac{k'}{n}\right)\right)\in H'_I\times H'_U.$$

It follows that, for any point $t\in\left[t_0-\frac{k+1}{n}\epsilon;t_0+\frac{k+1}{n}\epsilon\right]$ where u_n is differentiable,

$$||u'_n(t)||_2\leq M. \tag{1}$$

As u_n is continuous and piecewise C^1, this inequality suffices to guarantee that it is M-Lipschitz on $\left[t_0 - \frac{k+1}{n}\epsilon; t_0 + \frac{k+1}{n}\epsilon\right]$. [1]
Finally, for any $t \in \left[t_0 - \frac{k+1}{n}\epsilon; t_0 + \frac{k+1}{n}\epsilon\right]$,

$$||u_n(t) - u_0||_2 = ||u_n(t) - u_n(t_0)||_2 \leq M||t - t_0|| \leq M\epsilon,$$

meaning that $u_n(t) \in \bar{B}(u_0, M\epsilon)$.

b) According to the definition of u_n and the fundamental theorem of calculus, u_n is differentiable on

$$[t_0 - \epsilon; t_0 + \epsilon] \setminus \left\{t_0 - \epsilon, t_0 - \frac{n-1}{n}\epsilon, \ldots, t_0 + \epsilon\right\}.$$

and, for all t in this set,

$$u_n'(t) = f\left(t, u_n\left(t_0 + \frac{m_t}{n}\epsilon\right)\right), \tag{2}$$

where $m_t = E\left(\frac{n(t-t_0)}{\epsilon}\right)$ if $t > t_0$ and $m_t = E\left(\frac{n(t-t_0)}{\epsilon}\right) + 1$ otherwise. For any t, $\left|m_t - \frac{n(t-t_0)}{\epsilon}\right| \leq 1$ so

$$\left|\left(t_0 + \frac{m_t}{n}\epsilon\right) - t\right| \leq \frac{\epsilon}{n}.$$

Since u_n is M-Lipschitz,

$$\left|u_n\left(t_0 + \frac{m_t}{n}\epsilon\right) - u_n(t)\right| \leq \frac{M\epsilon}{n}.$$

Furthermore, $u_n\left([t_0 - \epsilon; t_0 + \epsilon]\right) \subset \bar{B}(u_0, M\epsilon) \subset H_U' \subset H_U$. Using the assumption that f is C-Lipschitz with respect to its second variable on $H_I \times H_U$, we can assert that, for any t,

$$\left|f\left(t, u_n\left(t_0 + \frac{m_t}{n}\epsilon\right)\right) - f\left(t, u_n(t)\right)\right| \leq \frac{CM\epsilon}{n}.$$

According to Equation (2), this is exactly the desired result.

1. Since u_n is continuous and piecewise C^1, it holds for any $a, b \in \left[t_0 - \frac{k+1}{n}; t_0 + \frac{k+1}{n}\right]$ that $u_n(b) - u_n(a) = \int_a^b u_n'(t)dt$. From the triangular inequality and Equation (1), this implies, for any a, b such that $a < b : |u_n(b) - u_n(a)| \leq \int_a^b ||u_n'(t)||dt \leq M(b-a)$.

c) Let $n_1, n_2 \in \mathbb{N}^*$ and $t \in [t_0 - \epsilon; t_0 + \epsilon]$ be fixed. If the inequality from the previous question holds for $n = n_1$ and $n = n_2$ (which happens for all t but a finite number of values), then, by the triangle inequality,

$$\begin{aligned}||u'_{n_1}(t) - u'_{n_2}(t)||_2 &\leq ||u'_{n_1}(t) - f(t, u_{n_1}(t))||_2 + ||f(t, u_{n_1}(t)) - f(t, u_{n_2}(t))||_2 \\ &\quad + ||f(t, u_{n_2}(t)) - u'_{n_2}(t)||_2 \\ &\leq \frac{CM\epsilon}{n_1} + ||f(t, u_{n_1}(t)) - f(t, u_{n_2}(t))||_2 + \frac{CM\epsilon}{n_2}.\end{aligned}$$

Now, as previously seen, t belongs to H_I and $u_{n_1}(t), u_{n_2}(t)$ belong to H_U, so

$$||u'_{n_1}(t) - u'_{n_2}(t)||_2 \leq \frac{CM\epsilon}{n_1} + C||u_{n_1}(t) - u_{n_2}(t)||_2 + \frac{CM\epsilon}{n_2}.$$

d) Let $n_1, n_2 \in \mathbb{N}^*$ be fixed. We will prove the requested inequality for all $t \in [t_0; t_0 + \epsilon]$; a similar reasoning can be used to prove it for $t \in [t_0 - \epsilon; t_0[$ (as in Question 1.d)).
For any $t \in [t_0; t_0 + \epsilon]$,

$$\begin{aligned}&||u_{n_1}(t) - u_{n_2}(t)||_2 \\ &\quad = \left|\left| u_{n_1}(t_0) - u_{n_2}(t_0) + \int_{t_0}^{t} \left(u'_{n_1}(s) - u'_{n_2}(s)\right) ds \right|\right|_2 \\ &\quad = \left|\left| \int_{t_0}^{t} \left(u'_{n_1}(s) - u'_{n_2}(s)\right) ds \right|\right|_2 \\ &\quad \leq \int_{t_0}^{t} \left|\left|u'_{n_1}(s) - u'_{n_2}(s)\right|\right| ds \\ &\quad \leq \int_{t_0}^{t} \left(CM\epsilon \left(\frac{1}{n_1} + \frac{1}{n_2}\right) + C||u_{n_1}(s) - u_{n_2}(s)||_2 \right) ds \\ &\quad = CM\epsilon \left(\frac{1}{n_1} + \frac{1}{n_2}\right)(t - t_0) + \int_{t_0}^{t} C||u_{n_1}(s) - u_{n_2}(s)||_2 ds.\end{aligned}$$

We apply Gronwall's lemma with

$$\begin{gathered} u : t \in [t_0; t_0 + \epsilon] \to ||u_{n_1}(t) - u_{n_2}(t)||_2, \\ a : t \in [t_0; t_0 + \epsilon] \to C, \\ c : t \in [t_0; t_0 + \epsilon] \to CM\epsilon \left(\frac{1}{n_1} + \frac{1}{n_2}\right)(t - t_0). \end{gathered}$$

It tells us that, for any $t \in [t_0; t_0 + \epsilon]$,

$$\begin{aligned}||u_{n_1}(t) - u_{n_2}(t)||_2 &\leq CM\epsilon\left(\frac{1}{n_1} + \frac{1}{n_2}\right)(t - t_0) \\ &\qquad + \int_{t_0}^{t} C^2 M\epsilon \left(\frac{1}{n_1} + \frac{1}{n_2}\right) e^{C(t-s)}(s - t_0)ds \\ &= CM\epsilon\left(\frac{1}{n_1} + \frac{1}{n_2}\right)(t - t_0) \\ &\qquad + M\epsilon\left(\frac{1}{n_1} + \frac{1}{n_2}\right)\left[-Ce^{C(t-s)}(s - t_0) - e^{C(t-s)}\right]_{t_0}^{t} \\ &= CM\epsilon\left(\frac{1}{n_1} + \frac{1}{n_2}\right)(t - t_0) \\ &\qquad + M\epsilon\left(\frac{1}{n_1} + \frac{1}{n_2}\right)\left(e^{C(t-t_0)} - C(t - t_0) - 1\right) \\ &= M\epsilon\left(\frac{1}{n_1} + \frac{1}{n_2}\right)\left(e^{C(t-t_0)} - 1\right).\end{aligned}$$

e) According to the previous question, for all $n, m \in \mathbb{N}^*$,

$$\begin{aligned}d_{sup}(u_n, u_m) &= \sup_{t\in[t_0-\epsilon;t_0+\epsilon]} ||u_n(t) - u_m(t)||_2 \\ &\leq \sup_{t\in[t_0-\epsilon;t_0+\epsilon]} M\epsilon\left(\frac{1}{n} + \frac{1}{m}\right)\left(e^{C|t-t_0|} - 1\right) \\ &= M\epsilon\left(\frac{1}{n} + \frac{1}{m}\right)\left(e^{C\epsilon} - 1\right).\end{aligned}$$

In particular, for any n,

$$\sup_{m\geq n} d_{sup}(u_n, u_m) \leq \frac{2M\epsilon}{n}\left(e^{C\epsilon} - 1\right),$$

which goes to 0 as $n \to +\infty$.

f) The set $\bar{B}(u_0, M\epsilon)$ is compact, hence complete. The set $C_b^0([t_0 - \epsilon; t_0 + \epsilon], \bar{B}(u_0, M\epsilon))$ of continuous and bounded functions from $[t_0 - \epsilon; t_0 + \epsilon]$ to $\bar{B}(u_0, M\epsilon)$ is also complete. This set is equal to $C^0([t_0-\epsilon; t_0+\epsilon], \bar{B}(u_0, M\epsilon))$ (since $[t_0 - \epsilon; t_0 + \epsilon]$ is compact and a continuous function on a compact set is always bounded). Therefore, $C^0([t_0 - \epsilon; t_0 + \epsilon], \bar{B}(u_0, M\epsilon))$ is complete. As $(u_n)_{n\in\mathbb{N}^*}$ is Cauchy, it has a limit in this set (for the uniform distance).

g)

$$\begin{aligned}\left|\left|u_n(t) - u_n(t_0) - \int_{t_0}^{t} f(s, u_n(s))ds\right|\right|_2 &= \left|\left|\int_{t_0}^{t} u_n'(s)ds - \int_{t_0}^{t} f(s, u_n(s))ds\right|\right|_2 \\ &\leq \int_{[t_0;t]} ||u_n'(s) - f(s, u_n(s))||_2 ds \\ &\leq \int_{[t_0;t]} \frac{CM\epsilon}{n} ds \quad \text{(by question b))} \\ &= \frac{CM\epsilon}{n}|t - t_0| \\ &\leq \frac{CM\epsilon^2}{n}.\end{aligned}$$

h) For any $s \in [t_0 - \epsilon; t_0 + \epsilon]$,

$$\begin{aligned}|f(s, u_n(s)) - f(s, u_\infty(s))| &\leq C||u_n(s) - u_\infty(s)||_2 \\ &\leq C d_{sup}(u_n, u_\infty).\end{aligned}$$

So, for any t,

$$\begin{aligned}&\left|\left|\int_{t_0}^{t} f(s, u_n(s))ds - \int_{t_0}^{t} f(s, u_\infty(s))ds\right|\right|_2 \\ &\leq \int_{[t_0;t]} ||f(s, u_n(s)) - f(s, u_\infty(s))||_2 ds \\ &\leq C d_{sup}(u_n, u_\infty)|t - t_0| \\ &\to 0 \quad \text{as } n \to +\infty,\end{aligned}$$

which implies that $\int_{t_0}^{t} f(s, u_n(s))ds \overset{n\to+\infty}{\longrightarrow} \int_{t_0}^{t} f(s, u_\infty(s))ds$.
For any t, $u_n(t) \to u_\infty(t)$ and $u_n(t_0) \to u_\infty(t_0)$ as $n \to +\infty$, so

$$\begin{aligned}&u_n(t) - u_n(t_0) - \int_{t_0}^{t} f(s, u_n(s))ds \\ &\qquad \overset{n\to+\infty}{\longrightarrow} u_\infty(t) - u_\infty(t_0) - \int_{t_0}^{t} f(s, u_\infty(s))ds.\end{aligned}$$

Now, as seen in the previous question,

$$u_n(t) - u_n(t_0) - \int_{t_0}^{t} f(s, u_n(s))ds \overset{n\to+\infty}{\longrightarrow} 0.$$

So, for any t, by the uniqueness of the limit,

$$u_\infty(t) - u_\infty(t_0) - \int_{t_0}^{t} f(s, u_\infty(s))ds = 0.$$

Therefore u_∞ is a primitive of $(t \to f(t, u_\infty(t)))$, which is continuous. As a consequence, u_∞ is differentiable and, for all $t \in [t_0 - \epsilon; t_0 + \epsilon]$,

$$u'_\infty(t) = f(t, u_\infty(t)).$$

Moreover, $u_\infty(t_0) = \lim_{n \to +\infty} u_n(t_0) = u_0$, so u_∞ is a solution of the Cauchy problem.