Differential geometry, mid-term

March 4 2024, 2 hours

You can use any written or printed material.
For each exercise, the number of points is an indication; it may change. Observe that the subject is long, but the total number of points is above 20.

Exercise 1

Let $n \in \mathbb{N}^*$ be a fixed integer. Let $f, g : \mathbb{R}^n \to \mathbb{R}^{2n}$ be C^1 maps such that

— $f(0) = g(0) = 0$;

— f, g are immersions at 0.

We define

$$\begin{array}{rccc} \phi : & \mathbb{R}^{2n} & \to & \mathbb{R}^{2n} \\ & (x_1, \dots, x_{2n}) & \to & f(x_1, \dots, x_n) + g(x_{n+1}, \dots, x_{2n}). \end{array}$$

1. Justify that ϕ is C^1 and compute $d\phi(0)$ as a function of $df(0)$ and $dg(0)$.
2. We assume that $\mathrm{Im}(df(0)) \cap \mathrm{Im}(dg(0)) = \{0\}$. Show that there exists two neighborhoods of 0 in $\mathbb{R}^{2n}$, V_1 and V_2, such that ϕ is a C^1-difféomorphism between V_1 and V_2.

[3.5 points]

Exercise 2

Show that $\{(e^{t^2}, te^{t^2}), t \in \mathbb{R}\}$ is a submanifold of $\mathbb{R}^2$, of class C^∞ and dimension 1.
[3 points]

Exercise 3: double torus

We consider the function

$$\begin{array}{rccc} F : & \mathbb{R}^2 & \to & \mathbb{R} \\ & (x, y) & \to & (16 - x^2 - y^2) \times ((x+2)^2 + y^2 - 1) \times ((x-2)^2 + y^2 - 1) \end{array}$$

and define the set

$$M = \left\{(x, y, z) \in \mathbb{R}^3, F(x, y) = z^2\right\}.$$

1. Compute $M \cap (\mathbb{R}^2 \times \{0\})$ (that is, write this set as the union of a small number of simple geometric shapes).
2. Compute the Jacobian matrix of F at any $(x, y) \in \mathbb{R}^2$.
3. Show that M is a submanifold of $\mathbb{R}^3$, of class C^∞; give its dimension.
4. Give an explicit formula for the tangent space to M at $(2, 1, 0)$.
5. Draw M (approximately).

[7 points]

Exercise 4

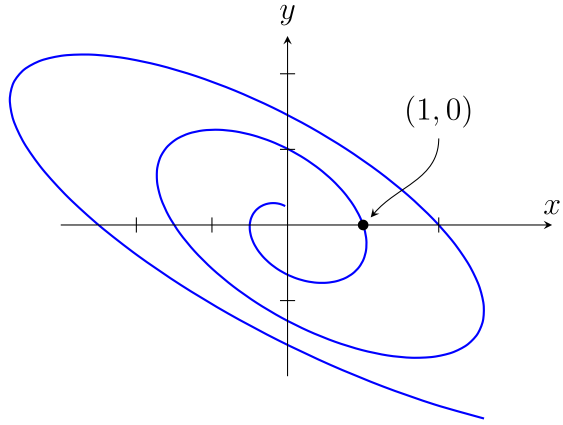
Each figure represents a submanifold in solid blue, and specifies a point. Among the proposed expressions, which one is plausible for the tangent space at this point ? (Only one answer is correct. No justification is expected.)

Figure (a), point $(1,0)$:

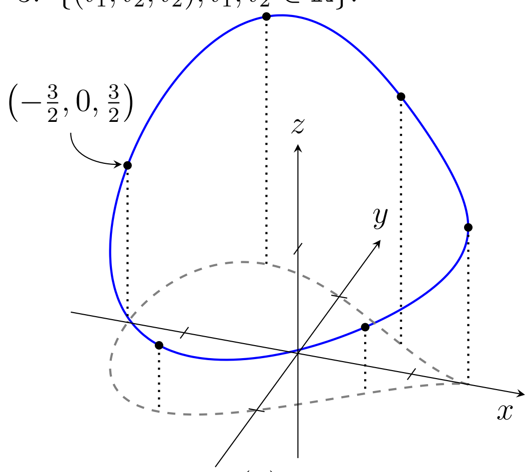
1. $(1,0)\mathbb{R}$;
2. $(\pi+1,3\pi)\mathbb{R}$;
3. $(1-\pi,2\pi)\mathbb{R}$.

Figure (b), point $\left(-\frac{3}{2},0,\frac{3}{2}\right)$:

1. $\{(x,y,z)\in\mathbb{R}^3, 2x-2y+z=0\}$;
2. $\{(0,2t,t), t\in\mathbb{R}\}$;
3. $\{(t_1,t_2,t_2), t_1,t_2\in\mathbb{R}\}$.



(a)



(b)

[2.5 points]

Exercise 5: $O_2(\mathbb{R}) \approx \mathbb{S}^1 \times \{-1,1\}$

1. Justify that $\mathbb{S}^1\times\{-1,1\}$ is a submanifold of $\mathbb{R}^3$ of class C^∞. What is its dimension ?
2. We define
$$\begin{array}{rccc} \phi_1: & \mathbb{S}^1\times\{-1,1\} & \to & O_2(\mathbb{R}) \\ & ((x,y),\epsilon) & \to & \begin{pmatrix} x & y \\ -\epsilon y & \epsilon x \end{pmatrix}. \end{array}$$
   a) Show that the definition is correct (i.e. $\phi_1((x,y),\epsilon)$ indeed belongs to $O_2(\mathbb{R})$ for all $((x,y),\epsilon)\in\mathbb{S}^1\times\{-1,1\}$).
   b) Show that ϕ_1 is C^∞.
3. We define
$$\begin{array}{rccc} \phi_2: & O_2(\mathbb{R}) & \to & \mathbb{S}^1\times\{-1,1\} \\ & \begin{pmatrix} a & b \\ c & d \end{pmatrix} & \to & ((a,b),ad-bc). \end{array}$$
   a) Show that the definition is correct.
   b) Show that ϕ_2 is C^∞.
   c) Show that $\phi_2\circ\phi_1:\mathbb{S}^1\times\{-1,1\}\to\mathbb{S}^1\times\{-1,1\}$ is the identity map.
   d) Show that $\phi_1\circ\phi_2: O_2(\mathbb{R})\to O_2(\mathbb{R})$ is the identity map.
4. Show that $O_2(\mathbb{R})$ is C^∞-diffeomorphic to $\mathbb{S}^1\times\{-1,1\}$.

[7 points]