Partiel de géométrie différentielle : corrigé

4 mars 2024

Answer of exercise 1

1. We define

$$\begin{aligned}\pi_1 :&(x_1, \ldots, x_{2n}) \in \mathbb{R}^{2n} \to (x_1, \ldots, x_n) \in \mathbb{R}^n,\\ \pi_2 :&(x_1, \ldots, x_{2n}) \in \mathbb{R}^{2n} \to (x_{n+1}, \ldots, x_{2n}) \in \mathbb{R}^n.\end{aligned}$$

   These maps are C^1 (they are linear). Therefore, $f \circ \pi_1$ and $g \circ \pi_2$ are C^1, as compositions of C^1 maps, and $\phi = f \circ \pi_1 + g \circ \pi_2$ is also C^1, as the sum of C^1 maps.
   We have, for any $h = (h_1, \ldots, h_{2n}) \in \mathbb{R}^{2n}$,

$$\begin{aligned}d\phi(0)(h) &= d(f \circ \pi_1)(0)(h) + d(g \circ \pi_2)(0)(h)\\ &= df(\pi_1(0)) \circ d\pi_1(0)(h) + dg(\pi_2(0)) \circ d\pi_2(0)(h)\\ &= df(0) \circ \pi_1(h) + dg(0) \circ \pi_2(h)\\ &= df(0)(h_1, \ldots, h_n) + dg(0)(h_{n+1}, \ldots, h_{2n}).\end{aligned}$$

2. First, we show that $d\phi(0) : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is bijective. As it is a linear map between two spaces with the same dimension, it suffices to show that $d\phi(0)$ is injective.
   Let $h = (h_1, \ldots, h_{2n})$ be any element in $\mathrm{Ker}(d\phi(0))$. Then

$$df(0)(h_1, \ldots, h_n) + dg(0)(h_{n+1}, \ldots, h_{2n}) = 0.$$

   As a consequence, $df(0)(h_1, \ldots, h_n) = dg(0)(-h_{n+1}, \ldots, -h_{2n})$ belongs to $\mathrm{Im}(df(0)) \cap \mathrm{Im}(dg(0)) = \{0\}$. This implies that

$$df(0)(h_1, \ldots, h_n) = 0 = dg(0)(h_{n+1}, \ldots, h_{2n}).$$

   As $df(0), dg(0)$ are injective (since f, g, are immersions at 0), this means that $(h_1, \ldots, h_n) = 0$ and $(h_{n+1}, \ldots, h_{2n}) = 0$. Therefore, $h = 0$. This shows that $\mathrm{Ker}(d\phi(0)) = \{0\}$, which is to say that $d\phi(0)$ is injective, hence bijective.
   We apply the local inversion theorem : there exists two neighborhoods of 0, V_1 and V_2, such that ϕ is a C^1-diffeomorphism between V_1 and V_2.

Answer of exercise 2

We define

$$\begin{array}{cccc}\phi : & \mathbb{R} & \to & \mathbb{R}^2\\ & t & \to & (e^{t^2}, te^{t^2}).\end{array}$$

This map is C^∞ (as composition of C^∞ maps). For any $t \in \mathbb{R}$, its derivative at t is

$$\phi'(t) = (2te^{t^2}, (1+2t^2)e^{t^2}).$$

This is always different from zero, since the second coordinate never cancels. This means that ϕ is an immersion on $\mathbb{R}$.
Let us define

$$\begin{array}{rccc} \psi : & \mathbb{R}_+^* \times \mathbb{R} & \to & \mathbb{R} \\ & (x,y) & \to & \frac{y}{x}. \end{array}$$

It is a continuous map (C^∞, actually). We observe that, for any $t \in \mathbb{R}$, $\phi(t)$ belongs to $\mathbb{R}_+^* \times \mathbb{R}$ and

$$\psi \circ \phi(t) = \frac{te^{t^2}}{e^{t^2}} = t.$$

From this equality, we deduce that ϕ is injective on $\mathbb{R}$. As it is surjective onto its image, it is a bijection between $\mathbb{R}$ and $\phi(\mathbb{R})$. From the equality again, its reciprocal is ψ (more precisely, the restriction of ψ to $\phi(\mathbb{R})$), which is a continuous map. Therefore, ϕ is a homeomorphism between $\mathbb{R}$ and $\phi(\mathbb{R})$.
We have shown that ϕ is an immersion, of class C^∞, which is a homeomorphism between $\mathbb{R}$ and its image. From a property seen in class (which is essentially the "immersion" definition of submanifolds), its image is a submanifold of $\mathbb{R}^2$, of class C^∞ and dimension 1.

Answer of exercise 3

1.
$$\begin{aligned} M \cap (\mathbb{R}^2 \times \{0\}) &= \{(x,y,0) \in \mathbb{R}^3, F(x,y) = 0\} \\ &= \{(x,y,0) \in \mathbb{R}^3, 16 - x^2 - y^2 = 0\} \\ &\quad \cup \{(x,y,0) \in \mathbb{R}^3, (x+2)^2 + y^2 - 1 = 0\} \\ &\quad \cup \{(x,y,0) \in \mathbb{R}^3, (x-2)^2 + y^2 - 1 = 0\} \\ &= \{(x,y,0) \in \mathbb{R}^3, x^2 + y^2 = 4^2\} \\ &\quad \cup \{(x,y,0) \in \mathbb{R}^3, (x+2)^2 + y^2 = 1^2\} \\ &\quad \cup \{(x,y,0) \in \mathbb{R}^3, (x-2)^2 + y^2 = 1^2\} \\ &= (C((0,0),4) \times \{0\}) \cup (C((-2,0),1) \times \{0\}) \cup (C((2,0),1) \times \{0\}), \end{aligned}$$

   where, for any $(x_0, y_0) \in \mathbb{R}^2$ and any $R > 0$, $C((x_0,y_0),R)$ is the circle of $\mathbb{R}^2$ with center (x_0,y_0) and radius R.
2. For any $(x,y) \in \mathbb{R}^2$,

$$Jf(x,y) = \begin{pmatrix} \frac{\partial f}{\partial x}(x,y) & \frac{\partial f}{\partial y}(x,y) \end{pmatrix},$$

   with

$$\frac{\partial f}{\partial x}(x,y) = -2x((x+2)^2 + y^2 - 1)((x-2)^2 + y^2 - 1)$$

$$
\begin{aligned}
&+ 2(x+2)(16-x^2-y^2)((x-2)^2+y^2-1) \\
&+ 2(x-2)(16-x^2-y^2)((x+2)^2+y^2-1)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial f}{\partial y}(x,y) = &-2y((x+2)^2+y^2-1)((x-2)^2+y^2-1) \\
&+ 2y(16-x^2-y^2)((x-2)^2+y^2-1) \\
&+ 2y(16-x^2-y^2)((x+2)^2+y^2-1).
\end{aligned}
$$

3. We define

$$
\begin{array}{cccc}
f: & \mathbb{R}^3 & \to & \mathbb{R} \\
 & (x,y,z) & \to & F(x,y)-z^2.
\end{array}
$$

This map is polynomial, therefore C^∞ and $M = \{(x,y,z), f(x,y,z) = 0\}$. We show that f is a submersion on M; this will imply that M is a submanifold of $\mathbb{R}^3$ with dimension 2 and class C^∞.
Let $(x,y,z) \in M$ be fixed. We show that f is a submersion at (x,y,z), that is $df(x,y,z)$ is surjective. Since $df(x,y,z)$ is linear, with images in $\mathbb{R}$, it suffices to show that it is non-zero.
If $z \neq 0$, then $\frac{\partial f}{\partial z}(x,y,z) = -2z \neq 0$, so that $df(x,y,z) \neq 0$.
Now, let us consider the case where $z = 0$. In this case, we must have $F(x,y) = 0$ and, from Question 1., (x,y) belongs to either $C((0,0),4)$ or $C((-2,0),1)$ or $C((2,0),1)$. In the first case ($(x,y) \in C((0,0),4)$),

$$
\begin{aligned}
\frac{\partial f}{\partial x}(x,y) &= -2x((x+2)^2+y^2-1)((x-2)^2+y^2-1) \\
\frac{\partial f}{\partial y}(x,y) &= -2y((x+2)^2+y^2-1)((x-2)^2+y^2-1).
\end{aligned}
$$

As the three circles are disjoint, $((x+2)^2+y^2-1)((x-2)^2+y^2-1) \neq 0$. Therefore, $\frac{\partial f}{\partial x}(x,y) = 0$ if and only if $x = 0$ and $\frac{\partial f}{\partial y}(x,y) = 0$ if and only if $y = 0$. Since $(0,0)$ does not belong to the circle $C((0,0),4)$, at least one of the two partial derivatives is non-zero, hence $df(x,y,z) \neq 0$.
In the second case ($(x,y) \in C((-2,0),1)$),

$$
\begin{aligned}
\frac{\partial f}{\partial x}(x,y) &= 2(x+2)(16-x^2-y^2)((x-2)^2+y^2-1) \\
\frac{\partial f}{\partial y}(x,y) &= 2y(16-x^2-y^2)((x-2)^2+y^2-1).
\end{aligned}
$$

As before, since the circles are disjoint, $(16-x^2-y^2)((x-2)^2+y^2-1) \neq 0$, so $\frac{\partial f}{\partial x}(x,y) = 0$ if and only if $x+2 = 0$, that is $x = -2$, and $\frac{\partial f}{\partial y}(x,y) = 0$ if and only if $y = 0$. As $(-2,0)$ does not belong to the circle $C((-2,0),1)$, at least one of the partial derivatives is non-zero, which implies that $df(x,y,z) \neq 0$.
The third case is identical.

4. We define f as in the previous question.

$$T_{(2,1,0)}M = \{(h_x, h_y, h_z), df(2,1,0)(h_x, h_y, h_z) = 0\}.$$

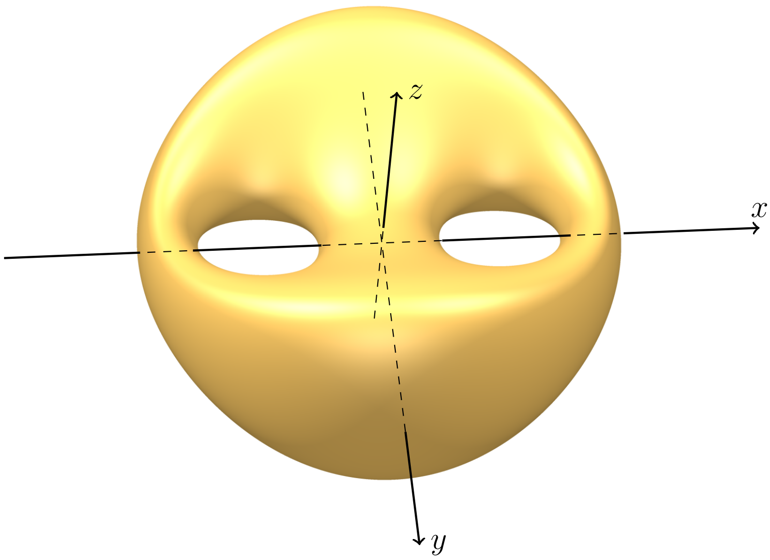
In addition,

$$\begin{aligned} Jf(2,1,0) &= \begin{pmatrix} \frac{\partial F}{\partial x}(2,1,0) & \frac{\partial F}{\partial y}(2,1,0) & -2 \times 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 352 & 0 \end{pmatrix}. \end{aligned}$$

As a consequence,

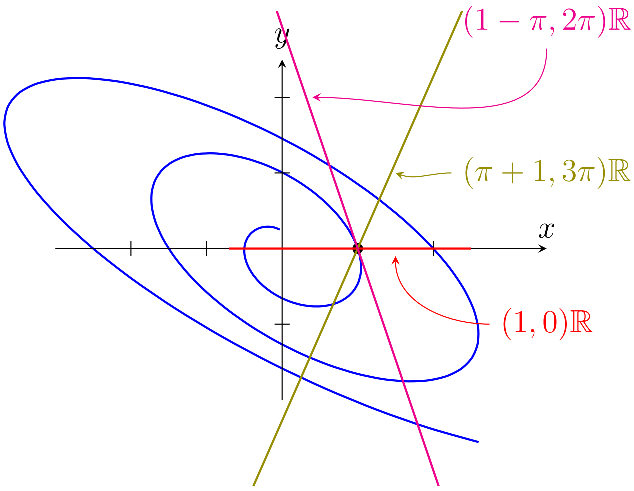
$$T_{(2,1,0)}M = \{(h_x, 0, h_z) \text{ for all } h_x, h_z \in \mathbb{R}\}.$$

5.



Answer of exercise 4

For Figure (a), the correct expression is $(1-\pi, 2\pi)\mathbb{R}$.



For Figure (b), the correct expression is $\{(0, 2t, t), t \in \mathbb{R}\}$. The other two subspaces have dimension 2, hence cannot be tangent spaces of a 1-dimensional manifold.

Answer of exercise 5

1. It is the product of a submanifold of $\mathbb{R}^2$ with dimension 1, and a submanifold of $\mathbb{R}$ with dimension 0, both of class C^∞.
2. a) For any $((x, y), \epsilon) \in \mathbb{S}^1 \times \{-1, 1\}$,

$$\begin{pmatrix} x & y \\ -\epsilon y & \epsilon x \end{pmatrix} \begin{pmatrix} x & y \\ -\epsilon y & \epsilon x \end{pmatrix}^T = \begin{pmatrix} x^2 + y^2 & 0 \\ 0 & \epsilon^2(x^2 + y^2) \end{pmatrix} = I_2.$$

   The last inequality is true because $(x, y) \in \mathbb{S}^1$, hence $x^2 + y^2 = 1$, and $\epsilon = \pm 1$, hence $\epsilon^2 = 1$. Therefore, $\phi_1((x, y), \epsilon)$ belongs to $O_2(\mathbb{R})$.

   b) We see ϕ_1 as a map between $\mathbb{S}^1 \times \{-1, 1\}$ and $\mathbb{R}^4$:

$$\begin{array}{rccc} \tilde{\phi}_1 : & \mathbb{S}^1 \times \{-1, 1\} & \to & \mathbb{R}^4 \\ & ((x, y), \epsilon) & \to & (x, y, -\epsilon y, \epsilon x). \end{array}$$

   This map is C^∞. Indeed, each of its coordinates is polynomial in x, y, ϵ, and the maps $((x, y), \epsilon) \to x$, $((x, y), \epsilon) \to y$, $((x, y), \epsilon) \to \epsilon$ are C^∞ on $\mathbb{S}^1 \times \{-1, 1\}$ (they are the projections onto the coordinates). Consequently, $\tilde{\phi}_1$ is a composition of C^∞ maps, so it is C^∞. This implies, from the definition of C^∞ maps between submanifolds, that ϕ_1 is C^∞.
3. a) Let $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ belong to $O_2(\mathbb{R})$. From the definition of $O_2(\mathbb{R})$,

$$\begin{aligned} a^2 + b^2 &= 1 \\ ac + bd &= 0 \\ c^2 + d^2 &= 1. \end{aligned}$$

   As a consequence, (a, b) belongs to $\mathbb{S}^1$. In addition,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}^T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

   so

$$\left(\det \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)^2 = \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \det \begin{pmatrix} a & b \\ c & d \end{pmatrix}^T = 1,$$

   Therefore, $ad - bc = 1$ or -1.

   b) We see ϕ_2 as a map from $O_2(\mathbb{R})$ to $\mathbb{R}^3$. Each of its three components is polynomial in a, b, c, d. As the projections onto the coordinates are C^∞ on $O_2(\mathbb{R})$, ϕ_2 is a composition of C^∞ maps, therefore C^∞ itself.

c) For any $((x,y),\epsilon) \in \mathbb{S}^1 \times \{-1,1\}$,

$$\begin{aligned}
\phi_2 \circ \phi_1((x,y),\epsilon) &= \phi_2\left(\left(\begin{smallmatrix} x & y \\ -\epsilon y & \epsilon x \end{smallmatrix}\right)\right) \\
&= ((x,y), x(\epsilon x) - y(-\epsilon y)) \\
&= ((x,y), \epsilon(x^2+y^2)) \\
&= ((x,y),\epsilon).
\end{aligned}$$

d) For any $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in O_2(\mathbb{R})$,

$$\begin{aligned}
\phi_1 \circ \phi_2\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right) &= \phi_1((a,b), ad-bc) \\
&= \left(\begin{smallmatrix} a & b \\ -(ad-bc)b & (ad-bc)a \end{smallmatrix}\right) \\
&= \left(\begin{smallmatrix} a & b \\ b^2c-a(bd) & a^2d-b(ac) \end{smallmatrix}\right) \\
&= \left(\begin{smallmatrix} a & b \\ b^2c-a(-ac) & a^2d-b(-bd) \end{smallmatrix}\right) \\
&= \left(\begin{smallmatrix} a & b \\ (a^2+b^2)c & (a^2+b^2)d \end{smallmatrix}\right) \\
&= \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right).
\end{aligned}$$

We have used the fact that $ac + bd = 0$.

4. We show that ϕ_1 is a C^∞-diffeomorphism between $\mathbb{S}^1 \times \{-1,1\}$ and $O_2(\mathbb{R})$. From Question 3.c), it is injective (otherwise, its composition with ϕ_2 would not be injective). From Question 3.d), it is surjective (otherwise, its composition with ϕ_2 would not be surjective). Therefore, it is a bijection between $\mathbb{S}^1 \times \{-1,1\}$ and $O_2(\mathbb{R})$.
From Question 2.b), it is C^∞.
From Question 3.c) (or Question 3.d)), $\phi_1^{-1} = \phi_2$. From Question 3.b), ϕ_1^{-1} is C^∞.