Differential geometry and differential equations

May 22 2025, 2 hours

You can use any written or printed material.
For each exercise, the number of points is an indication ; it may change.
You can write in either English or French, but don't forget quantifiers !

Exercise 1 *[4 points]*

We define

$$\mathcal{E} = \left\{(x,y,z) \in \mathbb{R}^3, (1+x^2)^2 = y^2 + z^2\right\}.$$

1. *[2 points]* Show that $\mathcal{E}$ is a submanifold of $\mathbb{R}^3$, of dimension 2 and class C^∞.
2. *[2 points]* Show that $\gamma : t \in \mathbb{R} \to (0, \cos(t), \sin(t))$ is a geodesic of $\mathcal{E}$.

Exercise 2 *[4 points]*

Find all maximal solutions of

$$u' = u^2 - 1.$$

[Hint : the map $x \in \mathbb{R} \setminus \{-1, 1\} \to \frac{1}{2} \ln\left(\frac{|x-1|}{|x+1|}\right)$ should be useful.]

Exercise 3 *[4 points]*

We consider the following linear system of equations :

$$\begin{cases} x'(t) &= (1-t)x(t) + e^t y(t), \\ y'(t) &= (1-t^2)e^{-t}x(t) + ty(t). \end{cases}$$

1. *[2.5 points]* Show that the resolvent R satisfies

$$R(t,0) = \begin{pmatrix} e^t & te^t \\ t & 1+t^2 \end{pmatrix}$$

   for all $t \in \mathbb{R}$.
2. *[1.5 points]* Compute the maximal solution such that $(x(1), y(1)) = (0, 1)$.

Exercise 4 *[8 points + 3 bonus points]*

[WARNING : the bonus questions are difficult. You are encouraged to attack them only after you have solved everything else.]
We consider the differential equation over $\mathbb{R}^2$

$$\begin{cases} x' &= xy, \\ y' &= \frac{y^2-1}{x^2+1}. \end{cases}$$

1. a) *[0.5 point]* Compute the equilibria.
   b) *[1.5 point]* Are they unstable ? stable ? asymptotically stable ?

Let $(x,y) : I \to \mathbb{R}^2$ be an arbitrary maximal solution, where I is an open interval.

2. a) *[1.5 point]* Show that, if there exists $t_0 \in I$ such that $y(t_0) = 1$, then $I = \mathbb{R}$ and, for any $t \in \mathbb{R}$,

$$(x(t), y(t)) = (x(t_0)e^{t-t_0}, 1).$$

   b) *[0.5 point]* Show that $\mathbb{R}_+^* \times \{1\}$ and $\mathbb{R}_-^* \times \{1\}$ are orbits of the equation.

The same reasoning would show that, if there exists $t_0 \in I$ such that $y(t_0) = -1$, then $I = \mathbb{R}$ and, for any $t \in \mathbb{R}$,

$$(x(t), y(t)) = (x(t_0)e^{t_0-t}, 1).$$

In addition, $\mathbb{R}_+^* \times \{-1\}$ and $\mathbb{R}_-^* \times \{-1\}$ are orbits of the equation.

3. a) *[Bonus, 1 point]* Show that, if there exists $t_1 \in I$ such that $x(t_1) = 0$, then $x(t) = 0$ for all $t \in I$, and (I, y) is one of the maximal solutions found in Exercise 2.

   b) *[Bonus, 1 point]* Show that $\{0\}\times]-\infty;-1[$, $\{0\}\times]-1;1[$ and $\{0\}\times]1;+\infty[$ are orbits of the equation.

4. In this question, we assume that there exists $t_0 \in I$ such that $|y(t_0)| > 1$.

   a) *[0.5 point]* Show that $|y(t)| > 1$ for all $t \in I$.

   b) *[0.5 point]* From now on, we also assume that $x(t_0) \neq 0$ which, from Question 3, implies that $x(t) \neq 0$, for all $t \in I$. We define

$$\begin{array}{rcccl} f & : & I & \to & \mathbb{R} \\ & & t & \to & \left(1 + \frac{1}{x(t)^2}\right)(y(t)^2 - 1). \end{array}$$

   Show that f is constant. Let $C > 0$ be its value.

   c) *[1 point]* Show that the orbit of $(x(t_0), y(t_0))$ is a subset of

$$\mathcal{O} \stackrel{\text{def}}{=} \left\{ \left(x, \operatorname{sign}(y(t_0))\sqrt{1 + \frac{Cx^2}{x^2+1}} \right), x \in E \right\},$$

   where $E = \mathbb{R}_+^*$ if $x(t_0) > 0$ and $\mathbb{R}_-^*$ otherwise.

   d) *[Bonus, 1 point]* Show that the orbit is exactly $\mathcal{O}$.

5. *[2 points]* Draw a plausible phase portrait.