Differential geometry and differential equations

May 22 2025, 2 hours

Exercise 1

1. Let us define
$$\begin{array}{rccc} F & : & \mathbb{R}^3 & \to & \mathbb{R} \\ & & (x,y,z) & \to & (1+x^2)^2 - y^2 - z^2. \end{array}$$
This map is C^∞. In order to show that $\mathcal{E} = F^{-1}(\{0\})$ is a C^∞ submanifold of $\mathbb{R}^3$ with dimension $\dim(\mathbb{R}^3) - \dim(\mathbb{R}) = 2$, it suffices to show that F is a submersion at every point of $\mathcal{E}$.
Let $(x,y,z) \in \mathcal{E}$ be arbitrary. We show that the linear map
$$dF(x,y,z) = \big((h_x,h_y,h_z) \in \mathbb{R}^3 \to 4x(1+x^2)h_x - 2yh_y - 2zh_z \in \mathbb{R}\big)$$
is surjective. As its codomain is $\mathbb{R}$, it is enough to show that it is non-zero. We observe that $y^2 + z^2 = (1+x^2)^2 \geq 1$, so $y \neq 0$ or $z \neq 0$. In the first case,
$$dF(x,y,z)(0,1,0) = -2y \neq 0.$$
In the second case,
$$dF(x,y,z)(0,0,1) = -2z \neq 0.$$
This shows that $dF(x,y,z) \neq 0$.
2. First, we observe that γ is C^2 (actually, C^∞) and $\gamma(t) \in \mathcal{E}$ for any $t \in \mathbb{R}$, as
$$(1+0^2)^2 = 1 = \cos^2(t) + \sin^2(t).$$
Second, we compute the tangent spaces to $\mathcal{E}$. For any $(x,y,z) \in \mathcal{E}$, $T_{(x,y,z)}\mathcal{E} = \operatorname{Ker} dF(x,y,z)$, where F is the map introduced at the previous question, i.e.
$$\begin{aligned} T_{(x,y,z)}\mathcal{E} &= \{(h_x,h_y,h_z), 4x(1+x^2)h_x - 2yh_y - 2zh_z = 0\} \\ &= \{(2x(1+x^2), -y, -z)\}^\perp, \end{aligned}$$
hence $\big(T_{(x,y,z)}\mathcal{E}\big)^\perp = \operatorname{Vect}\{(2x(1+x^2), -y, -z)\}$.
Finally, we check that $\gamma''(t) \in \big(T_{\gamma(t)}\mathcal{E}\big)^\perp$ for any $t \in \mathbb{R}$.
Let us fix $t \in \mathbb{R}$. From the last equality, it holds
$$\big(T_{\gamma(t)}\mathcal{E}\big)^\perp = \operatorname{Vect}\{(0, -\cos(t), -\sin(t))\}.$$
As $\gamma''(t) = (0, -\cos(t), -\sin(t))$, it is true that $\gamma''(t) \in \big(T_{\gamma(t)}\mathcal{E}\big)^\perp$.

Exercise 2

We want to solve the scalar autonomous equation $u' = F(u)$, where $F = (x \in \mathbb{R} \to x^2 - 1)$. The map F cancels at -1 and 1. Therefore, the constant maps
$$u_1 : t \in \mathbb{R} \to 1, \tag{1}$$
$$u_2 : t \in \mathbb{R} \to -1, \tag{2}$$
are solutions. Since they are defined over all $\mathbb{R}$, they are maximal.

Let us now find the non-constant solutions. The three maximal intervals over which F does not cancel are $]-\infty;-1[,]-1;1[,]1;+\infty[$. If $u : I \to \mathbb{R}$ is a solution of the equation, we know from the class that $u(I)$ is a subset of one of these intervals. Therefore, there are three families of non-constant maximal solutions.

Family 1 : maximal solutions with image in $]1;+\infty[$.

We denote $\Phi : x \in \mathbb{R}\setminus\{-1,1\} \to \frac{1}{2}\ln\left(\frac{|x-1|}{|x+1|}\right)$ the map suggested in the hint. It is differentiable, with derivative

$$\Phi'(x) = \frac{1}{2}\left(\frac{1}{x-1} - \frac{1}{x+1}\right) = \frac{1}{x^2-1} = \frac{1}{F(x)}, \quad \forall x \in \mathbb{R}\setminus\{-1,1\}.$$

Therefore, $\Phi_{|]1;+\infty[} :]1;+\infty[\to \mathbb{R}$ is a primitive of $\frac{1}{F}$ over $]1;+\infty[$.

As Φ has a positive derivative, it is strictly increasing. It goes to $-\infty$ at 1 and to 0 at $+\infty$, meaning that its image is $]-\infty;0[$. It is therefore a homeomorphism between $]1;+\infty[$ and $]-\infty;0[$.

We compute $\Phi^{-1}_{|]1;+\infty[} :]-\infty;0[\to]1;+\infty[$. Let $t \in]-\infty;0[$ be arbitrary. Let us denote $y = \Phi^{-1}_{|]1;+\infty[}(t)$. It must hold

$$\begin{aligned} t &= \Phi(y) \\ &= \frac{1}{2}\ln\left(\frac{|y-1|}{|y+1|}\right) \\ &= \frac{1}{2}\ln\left(\frac{y-1}{y+1}\right) \\ &\quad \text{(as } y > 1\text{)}, \end{aligned}$$

from which we deduce

$$\Phi^{-1}_{|]1;+\infty[}(t) = y = \frac{1+e^{2t}}{1-e^{2t}}.$$

From Theorem 5.4 from the lecture notes, the maximal solutions with image in $]1;+\infty[$ are all maps of the form

$$t \in]-\infty;D[\to \Phi^{-1}_{|]1;+\infty[}(t-D) = \frac{1+e^{2(t-D)}}{1-e^{2(t-D)}}, \tag{3}$$

for all $D \in \mathbb{R}$.

Family 2 : maximal solutions with image in $]-1;1[$.

The reasoning is very similar to the previous one. A primitive of $\frac{1}{F}$ over $]-1;1[$ is $\Phi_{|]-1;1[}$. This (decreasing) map is a homeomorphism from $]-1;1[$ to $\mathbb{R}$, with reciprocal

$$\Phi^{-1}_{|]-1;1[}(t) = \frac{1-e^{2t}}{1+e^{2t}}, \quad \forall t \in \mathbb{R}.$$

Therefore, the maximal solutions with image in $]-1;1[$ are all maps of the form

$$t \in \mathbb{R} \to \Phi^{-1}_{|]-1;1[}(t-D) = \frac{1-e^{2(t-D)}}{1+e^{2(t-D)}}, \tag{4}$$

for any $D \in \mathbb{R}$.

Family 3 : maximal solutions with image in $]-\infty;-1[$.

The reasoning is the same as for Family 1. The map $\Phi_{|]-\infty;-1[}$ is a homeomorphism from $]-\infty;-1[$ to $]0;+\infty[$, with reciprocal

$$\Phi_{|]-\infty;-1[}^{-1}(t) = \frac{1+e^{2t}}{1-e^{2t}}, \quad \forall t \in]0;+\infty[.$$

Therefore, the maximal solutions with image in $]-\infty;-1[$ are all maps of the form

$$t \in]D;+\infty[\to \Phi_{|]-\infty;-1[}^{-1}(t-D) = \frac{1+e^{2(t-D)}}{1-e^{2(t-D)}}, \tag{5}$$

for all $D \in \mathbb{R}$.
Summary : the maximal solutions are all maps from Equations (1), (2), (3), (4), (5).

Exercise 3

1. The considered equation can be written as

$$\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = A(t) \begin{pmatrix} x(t) \\ y(t) \end{pmatrix},$$

   where, for any $t \in \mathbb{R}$, $A(t) = \left(\begin{smallmatrix} 1-t & e^t \\ (1-t^2)e^{-t} & t \end{smallmatrix} \right)$. (Observe that A is continuous, so we are in the setting considered in class.)
   We define $r : t \to \left(\begin{smallmatrix} e^t & te^t \\ t & 1+t^2 \end{smallmatrix} \right)$ and show that r is the maximal solution of

$$\left\{ \begin{array}{rcl} r'(t) & = & A(t)r(t) \\ r(0) & = & I_2. \end{array} \right.$$

   From Theorem 5.9 in the lecture notes, this ensures that $r(t) = R(t,0)$ for any $t \in \mathbb{R}$.
   The initial condition is satisfied :

$$r(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

   For the differential equation, we note that r is differentiable. For any $t \in \mathbb{R}$,

$$\begin{aligned} A(t)r(t) &= \begin{pmatrix} 1-t & e^t \\ (1-t^2)e^{-t} & t \end{pmatrix} \begin{pmatrix} e^t & te^t \\ t & 1+t^2 \end{pmatrix} \\ &= \begin{pmatrix} e^t & (1+t)e^t \\ 1 & 2t \end{pmatrix} \\ &= r'(t). \end{aligned}$$

2. Maximal solutions are all maps of the form $t \in \mathbb{R} \to R(t,0) \left(\begin{smallmatrix} \alpha \\ \beta \end{smallmatrix} \right)$, for some $\alpha, \beta \in \mathbb{R}$.
   Let us find α, β such that $R(1,0) \left(\begin{smallmatrix} \alpha \\ \beta \end{smallmatrix} \right) = \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix} \right)$.
   This equation rewrites

$$\begin{pmatrix} e & e \\ 1 & 2 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

   Its only solution is $\alpha = -1, \beta = 1$. The corresponding maximal solution is

$$t \in \mathbb{R} \to \begin{pmatrix} (t-1)e^t \\ 1-t+t^2 \end{pmatrix}.$$

Exercise 4 The equation can be written $(x,y)' = F(x,y)$, where F is defined as

$$\begin{array}{cccc} F & : & \mathbb{R}^2 & \to & \mathbb{R}^2 \\ & & (x,y) & \to & \left(xy, \frac{y^2-1}{x^2+1}\right). \end{array}$$

Note that F is C^1. We are therefore in the setting of autonomous equations considered in class.

1. a) Equilibria are points (x_0, y_0) such that $F(x_0, y_0) = 0$. This is equivalent to

$$x_0 y_0 = 0,$$
$$\frac{y_0^2 - 1}{x_0^2 + 1} = 0.$$

The second equation is equivalent to $y_0 \in \{-1, 1\}$ and, when $y_0 \neq 0$, the first equation holds true if and only if $x_0 = 0$. Therefore, the only solutions are

$$(x_0, y_0) = (0, -1) \text{ and } (x_0, y_0) = (0, 1).$$

These are the equilibria.

b) The Jacobian matrix of F at any point $(x,y) \in \mathbb{R}^2$ is

$$JF(x,y) = \begin{pmatrix} y & x \\ -\frac{2x(y^2-1)}{(x^2+1)^2} & \frac{2y}{x^2+1} \end{pmatrix}.$$

In particular,

$$JF(0,-1) = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix} \text{ and } JF(0,-1) = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}.$$

The eigenvalues of the first matrix are -1 and -2 : all have a negative real part. Therefore, from Theorem 6.11 from the lecture notes, $(0,-1)$ is asymptotically stable (and, consequently, also stable).
The eigenvalues of the first matrix are 1 and 2 : one of them (both, actually) has a positive real part. Therefore, $(0,1)$ is unstable.

2. a) Let us assume that there exists $t_0 \in I$ such that $y(t_0) = 1$. Let us fix such t_0. We define

$$\begin{array}{cccc} (\tilde{x}, \tilde{y}) & : & \mathbb{R} & \to & \mathbb{R}^2, \\ & & t & \to & (x(t_0)e^{t-t_0}, 1). \end{array}$$

The solution (x,y) is the maximal solution to the following Cauchy problem :

$$\left\{ \begin{array}{rcl} (x,y)' & = & F(x,y) \\ (x(t_0), y(t_0)) & = & (x(t_0), 1). \end{array} \right.$$

The map $(\tilde{x}, \tilde{y})$ is also a solution to this Cauchy problem. Indeed,

$$(\tilde{x}(t_0), \tilde{y}(t_0)) = (x(t_0), 1)$$

and, for any $t \in \mathbb{R}$,

$$\tilde{x}'(t) = x(t_0)e^{t-t_0} = \tilde{x}(t)\tilde{y}(t),$$
$$\tilde{y}'(t) = 0 = \frac{\tilde{y}(t)^2 - 1}{\tilde{x}(t)^2 + 1}.$$

As it is global, it can only be a maximal solution. The maximal solution is unique, from the Cauchy-Lipschitz theorem (which applies, because F is C^1, hence locally Lipschitz). Therefore, $(x,y) = (\tilde{x}, \tilde{y})$ and $I = \mathbb{R}$.

b) We use the previous subquestion to compute the orbit of $(1,1)$. From the above, the maximal solution (x,y) such that $(x(0),y(0))=(1,1)$ is

$$t\in\mathbb{R}\to(e^t,1).$$

Therefore, the orbit of $(1,1)$ is $\{(e^t,1),t\in\mathbb{R}\}=\mathbb{R}_+^*\times\{1\}$.
Similarly, the maximal solution (x,y) such that $(x(0),y(0))=(-1,1)$ is

$$t\in\mathbb{R}\to(-e^t,1).$$

Therefore, the orbit of $(-1,1)$ is $\{(-e^t,1),t\in\mathbb{R}\}=\mathbb{R}_-^*\times\{1\}$.

3. a) Let us fix $t_1\in I$ such that $x(t_1)=0$. Let $u:\tilde{I}\to\mathbb{R}$ be the maximal solution of the following Cauchy problem :

$$\begin{cases} u' &= u^2-1 \\ u(t_1) &= y(t_1). \end{cases}$$

It is one of the solutions found at Exercise 2.
We define

$$\begin{array}{rcl} (\tilde{x},\tilde{y}) : \tilde{I} & \to & \mathbb{R}^2, \\ t & \to & (0,u(t)). \end{array}$$

This maps satisfies the equation $(\tilde{x},\tilde{y})'=F(\tilde{x},\tilde{y})$, since, for any $t\in\tilde{I}$,

$$\tilde{x}'(t)=0=\tilde{x}(t)\tilde{y}(t),$$
$$\tilde{y}'(t)=u'(t)=u(t)^2-1=\frac{\tilde{y}(t)^2-1}{\tilde{x}(t)^2+1}.$$

As $(\tilde{x}(t_1),\tilde{y}(t_1))=(0,u(t_1))=(0,y(t_1))$, it is a solution of the Cauchy problem

$$\begin{cases} (\tilde{x},\tilde{y})' &= F(\tilde{x},\tilde{y}) \\ (\tilde{x}(t_1),\tilde{y}(t_1)) &= (0,y(t_1)). \end{cases}$$

Since (x,y) is the maximal solution of the same Cauchy problem, we must have $\tilde{J}\subset J$ and $(x,y)=(\tilde{x},\tilde{y})$ on $\tilde{J}$ (from the Cauchy-Lipschitz theorem).
The last thing we have to show is that $\tilde{J}=J$, i.e. the inclusion is an equality. First, we show that $\sup\tilde{J}=\sup J$. We proceed by contradiction, and assume $\sup\tilde{J}<\sup J$. In this case, $\sup\tilde{J}<+\infty$. In addition, since $u=y$ on $\tilde{J}$ and y is continuous at $\sup\tilde{J}$, it must hold that u goes to $y(\sup\tilde{J})$ at $\sup\tilde{J}$. This contradicts the théorème des bouts.
An identical result shows that $\inf\tilde{J}=\inf J$, hence $\tilde{J}=J$.

b) Let a be an arbitrary element from $\mathbb{R}\setminus\{-1,1\}$. From the reasoning done for the previous subquestion, the orbit of $(0,a)$ is the image of

$$\begin{array}{rcl} (\tilde{x},\tilde{y}) : \tilde{I} & \to & \mathbb{R}^2, \\ t & \to & (0,u(t)), \end{array}$$

where $u:\tilde{I}\to\mathbb{R}$ is the maximal solution of

$$\begin{cases} u' &= u^2-1 \\ u(t_1) &= a. \end{cases}$$

Since $a\neq\pm1$, u is not one of the constant solutions of Exercise 2. We have the following three possibilities :

1. $a > 1$: in this case, the image of u is the image of the map Φ^{-1} appearing at Equation (3), which is $]1;+\infty[$. The image of $(\tilde{x}, \tilde{y})$ is then

$$\{0\}\times]1;+\infty[,$$

which means that this set is an orbit.

2. $-1 < a < 1$: the image of u is the image of Φ^{-1} from Equation (4). The orbit is then

$$\{0\}\times]-1;1[.$$

3. $a < -1$: the image of u is the image of Φ^{-1} from Equation (5). The orbit is then

$$\{0\}\times]-\infty;-1[.$$

4. a) From Question 2., if there exists $t_1 \in I$ such that $|y(t_1)| = 1$ (i.e. $y(t_1) = 1$ or $y(t_1) = -1$), then y is constant, so that $|y(t)| = 1$ for any $t \in I$. This is in contradiction with the assumption that $|y(t_0)| > 1$ for some $t_0 \in I$. This shows that $|y(t)| \neq 1$, for all $t \in I$.
The image of $|y|$ is an interval, as $|y|$ is continuous (from the intermediate values theorem). It intersects $]1;+\infty[$ and does not contain 1. Therefore, it is a subset of $]1;+\infty[$, meaning that $|y(t)| > 1$ for all $t \in I$.

b) The map f is differentiable, as it is a sum of products of differentiable maps. For each $t \in I$,

$$\begin{aligned} f'(t) &= -2\frac{x'(t)}{x(t)^3}\left(y(t)^2 - 1\right) + 2\left(1 + \frac{1}{x(t)^2}\right) y'(t)y(t) \\ &= -2\frac{x(t)y(t)}{x(t)^3}\left(y(t)^2 - 1\right) + 2\left(1 + \frac{1}{x(t)^2}\right)\left(\frac{y(t)^2 - 1}{x(t)^2 + 1}\right) y(t) \\ &= 2\left(-\frac{y(t)}{x(t)^2} + \frac{y(t)}{x(t)^2}\right)\left(y(t)^2 - 1\right) \\ &= 0. \end{aligned}$$

Therefore, f is constant.

c) The orbit of $(x(t_0), y(t_0))$ is

$$\{(x(t), y(t)), t \in I\}.$$

Let us show that this set is included in $\mathcal{O}$.
For any $t \in I$, $x(t)$ is non-zero and has the same sign as $x(t_0)$ (since x is continuous and does not cancel on I). Therefore, $x(t) \in E$.
In addition, still for any t,

$$\begin{aligned} C = f(t) &= \left(1 + \frac{1}{x(t)^2}\right)(y(t)^2 - 1), \\ \Rightarrow y(t)^2 &= 1 + \frac{Cx(t)^2}{x(t)^2 + 1}, \\ \Rightarrow y(t) &= \pm\sqrt{1 + \frac{Cx(t)^2}{x(t)^2 + 1}}. \end{aligned}$$

Since $|y(t)| > 1 > 0$ for any $t \in I$, y cannot cancel, hence the sign of y is constant, equal to $\text{sign}(y(t_0))$. This implies that, for any $t \in I$,

$$y(t) = \text{sign}(y(t_0))\sqrt{1 + \frac{Cx(t)^2}{x(t)^2+1}},$$

which shows that $(x(t), y(t)) \in \mathcal{O}$.

d) Let us discuss the case where $E = \mathbb{R}_+^*$ and $y(t_0) > 0$. The other three cases are almost identical.
The map x is continuous, and strictly increasing (as $x' = xy > 0$ on I). Its domain is an open interval, hence its image is also an open interval, which we call $]\alpha;\beta[\subset \mathbb{R}_+^*$ (with β possibly equal to $+\infty$). To show that

$$\{(x(t), y(t)), t \in I\} = \mathcal{O},$$

it is enough to show that $]\alpha;\beta[= E$, that is to say $\alpha = 0$ and $\beta = +\infty$.
First, we show that $\alpha = 0$. For this, we observe that (x, y) is bounded in the neighborhood of $\inf I$, because

$$x(t) \overset{t\to\inf I}{\longrightarrow} \alpha,$$
$$y(t) = \sqrt{1 + \frac{Cx(t)^2}{x(t)^2+1}} \overset{t\to\inf I}{\longrightarrow} \sqrt{1 + \frac{C\alpha^2}{\alpha^2+1}}.$$

From the thoérème des bouts, it holds $\inf I = -\infty$. In addition, for any $t \in I$,

$$x'(t) = x(t)y(t) > x(t),$$

hence $[\ln(x)]'(t) > 1$, which implies that, for all $t \in]-\infty; t_0]$,

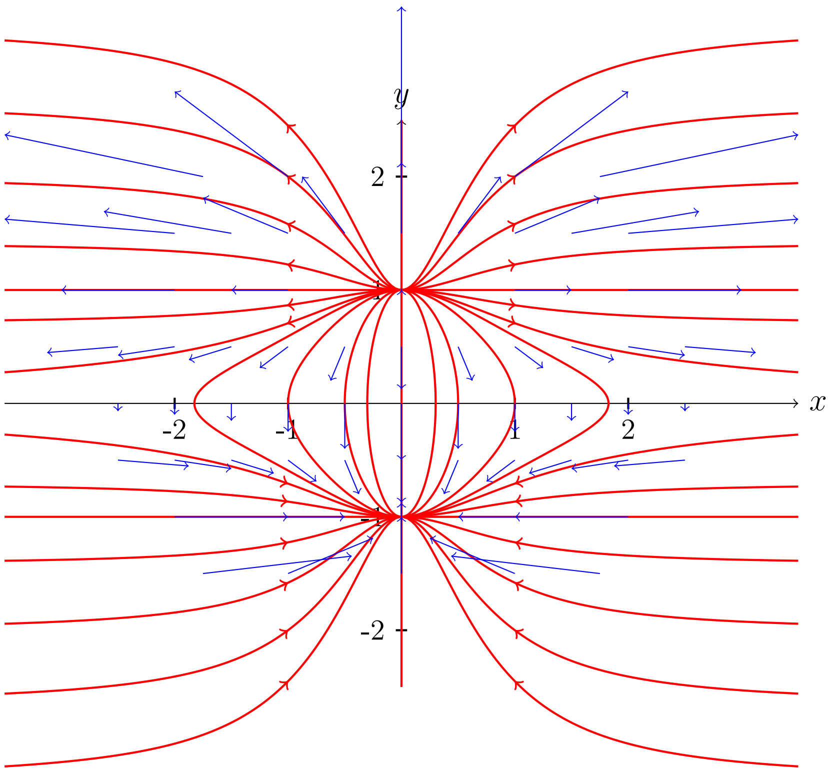
$$\begin{aligned}\ln(x(t)) &= \ln(x(t_0)) - \int_t^{t_0} [\ln(x)]'(s)ds \\ &< \ln(x(t_0)) - (t_0 - t).\end{aligned}$$

Therefore, $\ln(x(t)) \to -\infty$ when $t \to -\infty$, which means that $\alpha = \lim_{-\infty} x = 0$.
Second, we show that $\beta = +\infty$. The reasoning is similar as before, but it is easier to proceed by contradiction : we assume that $\beta < +\infty$. Then (x, y) is bounded in the neighborhood of $\sup I$, hence $\sup I = +\infty$, from the théorème des bouts. For any $t \in [t_0; +\infty[$,

$$\begin{aligned}\ln(x(t)) &= \ln(x(t_0)) + \int_{t_0}^{t} [\ln(x)]'(s)ds \\ &> \ln(x(t_0)) + (t - t_0).\end{aligned}$$

Therefore, $\ln(x(t)) \overset{t\to+\infty}{\longrightarrow} +\infty$, hence $x(t) \overset{t\to+\infty}{\longrightarrow} +\infty$, meaning that $\beta = +\infty$, which is a contradiction.

5. The size of the arrows has been divided by 2 for better lisibility.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