# Non-convex inverse problems: project
# Super-resolution

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## General instructions

- You can use any programming language.
- You can use optimization toolboxes.
- If you reuse code written by someone else, you must cite your source.
- You must provide your code (and it must run without errors!). You should notably include the instructions you used to generate the pictures, so that I can reproduce them.
- The most important thing to me is that you describe, analyze and interpret correctly your numerical results, and relate them to the theory seen in class when relevant.
- If you are stuck, do not hesitate to send me an email describing your problem; I will try to help.
- You can write in either French or English.

We consider the problem discussed in class of recovering a signed measure on $[0;1[$, which is a sum of a few Diracs, from its low-frequency Fourier coefficients:

$$\begin{aligned} &\text{find } \mu \in \mathcal{M}_S \\ &\text{such that } \hat{\mu}[k] = y_k, \forall k = -f_c, \dots, f_c. \end{aligned} \tag{SR}$$

Here, $f_c$ is the *cutoff frequency*, $S$ is the (known) number of Diracs, and $y_{-f_c}, \dots, y_{f_c}$ are the available measurements. The set $\mathcal{M}_S$ contains all measures which are sums of $S$ Diracs:

$$\mathcal{M}_S = \left\{ \sum_{s=1}^{S} a_s \delta_{\tau_s}, a_1, \dots, a_S \in \mathbb{R}, \tau_1, \dots, \tau_S \in [0;1[ \right\}.$$

1. Implement a function which, given $S$, returns a random measure in $\mathcal{M}_S$. The positions of diracs must be uniformly and independently sampled in $[0;1[$, and the weights independently chosen according to identical normal distributions.
   (The measure may simply be represented as a pair containing the list of positions and the list of weights.)
2. Implement a function which, given $S, f_c$, returns a random measure $\mu_{sol}$, as described in the previous question, and its low-frequency Fourier coefficients $(\hat{\mu}_{sol}[k])_{k=-f_c}^{f_c}$.

We propose to solve Problem (SR) through the total variation minimization approach described in class:

$$\begin{aligned} &\text{minimize } ||\mu||_{TV}, \\ &\text{over all } \mu \in \mathcal{M}([0;1[), \\ &\text{such that } \hat{\mu}[k] = y_k, \forall k = -f_c, \dots, f_c. \end{aligned}$$

To overcome the infinite-dimensionality of the search space, we approximate the unknown measure $\mu$ over $[0;1[$ by a measure over $\left\{ \frac{0}{M}, \frac{1}{M}, \dots, \frac{M-1}{M} \right\}$, for some integer $M \in \mathbb{N}^*$ which must be adequately chosen. We denote the corresponding search space by

$$\mathcal{M}_{grid,M} = \left\{ \sum_{m=0}^{M-1} a_m \delta_{\frac{m}{M}}, a_0, \dots, a_{M-1} \in \mathbb{R} \right\}.$$

Concretely, we have to solve the following optimization problem.

$$\begin{aligned}
&\text{minimize } \sum_{m=0}^{M-1} \left| \mu\left(\left\{\frac{m}{M}\right\}\right) \right|, \\
&\text{over all } \mu(0), \mu\left(\frac{1}{M}\right), \ldots, \mu\left(\frac{M-1}{M}\right) \in \mathbb{R}, \qquad \text{(SR-TV-grid)} \\
&\text{such that } \sum_{m=0}^{M-1} \mu\left(\left\{\frac{m}{M}\right\}\right) e^{-2\pi i k \frac{m}{M}} = y_k, \forall k = -f_c, \ldots, f_c.
\end{aligned}$$

3. Implement a solver for this problem, which takes $M$ as an input parameter.
4. For $S = 1, f_c = 2$, select a random instance of Problem (SR) using the function from Question 2.
   a) Test your solver with several values of $M$ going from 1 to 20. For a selection of values of $M$ plot on the same figure the true measure, and the reconstruction returned by your solver.
   b) Describe and comment your results.

From now on, we set $M = 100$.

5. a) Propose a notion of distance between elements of $\mathcal{M}_S$ and $\mathcal{M}_{disc,M}$, which will allow to quantify the performance of (SR-TV-grid) for solving Problem (SR). Describe it and implement it.
   b) Check your implementation: choose a few simple elements of $\mathcal{M}_S$ and $\mathcal{M}_{disc,M}$. Compute the distance between them analytically. Check that your code returns the same result.
6. For $S = 1, ..., 5$ and $f_c \in \{1, 2, 4, 6, 8, 10, 13, 16, 20\}$, approximately compute the probability that, given a random instance of Problem (SR), with ground true measure $\mu_{sol}$, its reconstruction $\mu_{rec}$ by your solver satisfies

$$\frac{\text{dist}(\mu_{sol}, \mu_{rec})}{\text{dist}(\mu_{sol}, 0)} < 0.05.$$

   Here, dist is the distance that you have defined in the previous question, and "0" is the zero measure on $[0; 1[$.
   (You can replace the threshold 0.05, which is somewhat arbitrary, by a different value if it seems more interesting.)
   For each $S$, plot the probability as a function of $f_c$. Comment the results.

7. For $S = 4$ and $f_c = 8$, run the solver on several random instances. Try to formulate conjecture(s) about the properties of $\mu_{sol}$ which make the solver more likely to succeed or more likely to fail. Support your conjectures with well-chosen plots.
8. Problem (SR-TV-grid) can be seen as an instance of the $\ell_1$ norm minimization problem used as a convex relaxation in compressed sensing. Do the results seen in class about the tightness of this convex relaxation say anything meaningful on (SR-TV-grid)?