Discussion on "Approximate Bayesian inference for latent Gaussian models by using integrated nested Laplace approximations" by H. Rue, S. Martino, and N. Chopin, written jointly by Roberto Casarin and Christian P. Robert, ANR 2005-2008 ADAP'MC, CEREMADE, Université Paris Dauphine. In order to evaluate the impact of the Gaussian approximation on the marginal posterior on θ, we consider here a slightly different albeit standard stochastic volatility model

$$\mathbf{x}, \mathbf{y}|\theta \sim \frac{\sigma^{-T-1}}{\sqrt{1-\varrho^2}} \exp\left\{-\frac{1}{2}\left[\frac{x_0^2}{1-\varrho^2} + \sum_{t=1}^{T}(x_t - \varrho x_{t-1})^2 + \sum_{t=0}^{T}\left(y_t^2 e^{-x_t}\sigma^{-2} + x_t\right)\right]\right\}.$$

(The difference with Rue et al. (2008) is that the variance of the x_t's is set to 1 and that we use the notations ϱ instead of ϕ and σ^2 instead of $\exp\mu$.) If we look at the second order approximation of the non-linear term, we have

$$y_t^2 e^{-x_t}\sigma^{-2} + x_t \approx y_t^2 e^{-x_t^\star}\sigma^{-2} + x_t^\star + \frac{y_t^2 e^{-x_t^\star}}{2\sigma^2}(x_t - x_t^\star)^2 = 1 + x_t^\star + \frac{1}{2}(x_t - x_t^\star)^2$$

where $x_t^\star = \log(y_t^2/\sigma^2)$. A Gaussian approximation to the stochastic volatility model is thus

$$\mathbf{x}|\mathbf{y}, \theta \sim |\mathbf{Q}(\theta)|^{-1/2} \exp\left\{-\frac{1}{2}\left[\frac{x_0^2}{1-\varrho^2} + \sum_{t=1}^{T}(x_t - \varrho x_{t-1})^2 + \frac{1}{2}\sum_{t=0}^{T}\sigma^{-2}(x_t - x_t^\star)^2\right]\right\},$$

where the Gaussian precision matrix $\mathbf{Q}(\theta)^{-1}$ has $3/2 + \varrho^2$ on its diagonal, $-\varrho$ on its first sub- and sup-diagonals, and zero elsewhere. Therefore the approximation (3) of the marginal posterior of θ is equal to

$$\begin{aligned}\tilde{\pi}(\theta|\mathbf{y}) &\propto \frac{\pi(\mathbf{x}, \theta|\mathbf{y})}{\pi_G(\mathbf{x}|\theta, \mathbf{y})} \\ &\propto \frac{\sigma^{-T-1}|\mathbf{Q}(\theta)|^{1/2}}{\sqrt{1-\varrho^2}} \exp\left\{-\frac{1}{2}\sum_{t=0}^{T}\left[y_t^2 e^{-x_t}\sigma^{-2} + x_t - \frac{1}{2}(x_t - x_t^\star)^2\right]\right\}\pi(\theta),\end{aligned}$$

for a specific pluggin value of $\mathbf{x}$.

Using for this pluggin value the mode (and mean) $\mathbf{x}^M$ of the Gaussian approximation, as it is readily available, contrary to the mode of the full conditional of $\mathbf{x}$ given $\mathbf{y}$ and θ suggested in Rue et al. (2008), we obtain a straightforward recurrence relation on the components of $\mathbf{x}^M$

$$-\varrho(x_{t+1}^M - \varrho x_t^M) + (x_t^M - \varrho x_{t-1}^M) + \frac{1}{2}(x_t^M - x_t^\star) = 0,$$

with appropriate modifications for $t = 0, T$. We thus get the recurrence ($t > 0$)

$$x_t^M = \alpha_t x_{t-1}^M + \beta_t,$$

with

$$\begin{cases} \alpha_T = 2\varrho/3,\ \beta_T = x_T^\star/3, \\ \alpha_t = \dfrac{\varrho}{3/2 + \varrho^2 - \varrho\alpha_{t+1}}, \\ \beta_t = \dfrac{\varrho\beta_{t+1} + x_t^\star/2}{3/2 + \varrho^2 - \varrho\alpha_{t+1}}, \quad 1 \le t < T \end{cases}$$

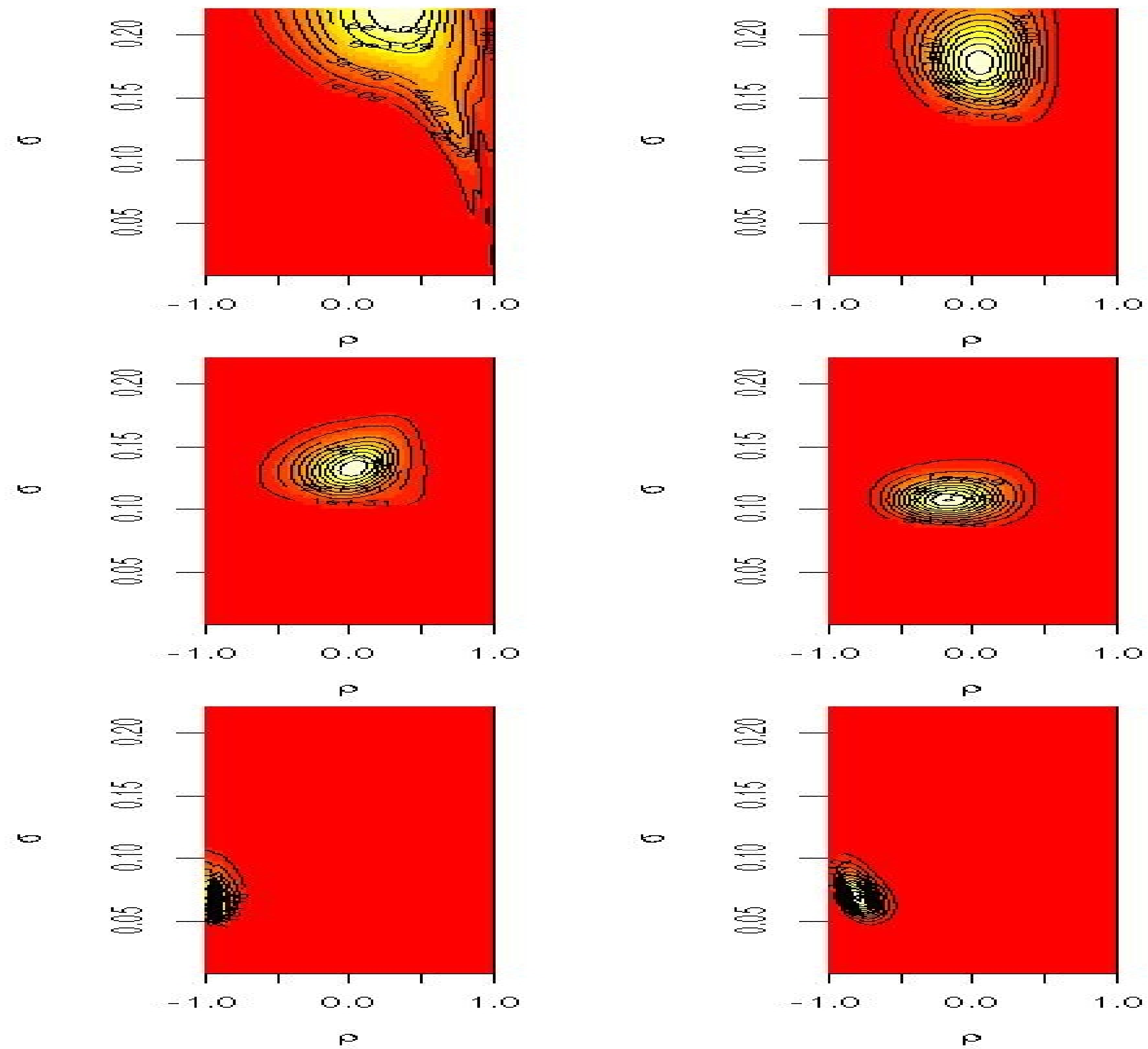

Figure 1: Comparison of an importance sampling approximation (with 10^3 simulations) to the likelihood of a stochastic volatility model *(left)* with the approximation based on the Gaussian approximation of Rue et al. (2008) *(right)* when centred at $\mathbf{x}^M$, mode of the Gaussian approximation. This likelihood is associated with *(top)* 25 simulated values with $\sigma = 0.1$ and $\varrho = 0.9$ *(middle)* 50 simulated values with $\sigma = 0.1$ and $\varrho = -0.3$ and *(bottom)* 20 simulated values with $\sigma = 0.1$ and $\varrho = -0.9$.

and

$$x_0^M = \frac{\varrho\beta_1 + x_t^\star/2}{(1-\varrho^2)^{-1} + \varrho^2 - \alpha_1\varrho + 1/2}\,.$$

This choice of $\mathbf{x}^M$ as a pluggin value for the approximation to $\pi(\theta|y)$ gives rather accurate results, when compared with the "true" likelihood obtained by a regular (and unrealistic) importance sampling approximation. Figure 1 shows the correspondence between both approximations, indicating that the Gaussian approximation (3) can be used as a good proxy to the true marginal.